Chapter VII
Decidability and Quantifier-Elimination

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The decidability of the elementary theory for a given class K of structures reflects a certain low expressive power of the elementary language with respect to that class. Therefore, it is natural to look for stronger logics L such that K has a decidable L-theory. The rigorous establishment of decidability for the L-theory of K often provides results about the L-definable properties and L-equivalence of structures in K. This means, then, that investigations into the decidability of the L-theory of K are closely related to the L-model theory of K.

In this chapter we will investigate the decidability of such logics. We will concentrate on Malitz quantifiers (particularly on cardinality quantifiers) and Härtig quantifiers as well as on stationary logic. The first result in this direction was the decidability of the theory of unary predicates without equality in the logic with the quantifier "there are $\aleph_\alpha$ many". This result was proven in a fundamental paper by Mostowski [1957]. Topological and monadic second-order logics are treated in other chapters of this volume; and, we therefore, will not consider them here. However, we wish to emphasize at this point that results concerning the latter do have important consequences for the material that will be presented in our discussions.

Our chapter is basically organized along the lines sketched below. First, with respect to three main methods of proving decidability, there is a division into three sections which are respectively entitled *Quantifier-Elimination*, *Interpretations*, and *Dense Systems*. In each of these the general method is introduced and then clarified with respect to several concrete classes of structures. These classes are: the class of modules and abelian groups (Section 1), the class of well-orderings (Section 2), and the classes of linear orderings and boolean algebras (Section 3). At the end of each subsection we refer to some further results without making any claims that the discussions given present a complete picture of the material. However, the reader will find references to most of the corresponding investigations in the bibliography given at the end of the volume.

Much of the material of this chapter is related to our text (see Baudisch–Seese–Tuschik–Weese [1980]), in which the reader can find more detailed proofs as well as some similar investigations on the class of trees.

We wish to express our gratitude to Philipp Rothmaler who contributed so many of his ideas and so much of his time and energy to the creation of this chapter that we can justly say that he is a co-author of this study.

1. Quantifier-Elimination

1.1. The Framework

In general an extended language has a more expressive power than the original language. However, in many cases there are model classes which cannot be further distinguished in the extended language. Such model classes often have interesting properties, and it is this very fact that leads us to the following

Definition. Let L be a sublanguage of a language L' and K a class of L'-structures. We say that *L' is reducible to L with respect to the class K* if for every formula $\varphi(\bar{x})$ of L' there is a formula $\psi(\bar{x})$ of L such that $K \models \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$. L' is said to be *effectively reducible to L with respect to K* if ψ can be found effectively (depending on φ).

An important special case arises from extensions obtained by adding certain quantifiers. And this case we will examine more closely in the

Definition. Suppose L' arises from L by adding (in the canonical way) an arbitrary quantifier Q to it. If L' is reducible to L with respect to a class K, we say that *K admits the L-elimination of Q*, or *Q is L-eliminable in K*. If K is an L'-elementary class, that is, if $K = \mathrm{Mod}(\mathrm{Th}_{L'}(K))$, then we also say that $\mathrm{Th}_{L'}(K)$ *admits the elimination of Q*, or *Q is eliminable in* $\mathrm{Th}_{L'}(K)$.

The examples below show that important model-theoretic properties are reflected by the notion of reducibility.

Example. Let L' be a first-order language, K an L'-elementary class.

(1) Let L be the set of all open L'-formulas. Clearly, L' is the extension of L by adding the quantifier $\exists$.
   (a) $\mathrm{Th}_{L'}(K)$ is substructure complete iff $\exists$ is eliminable in $\mathrm{Th}_{L'}(K)$.
   (b) If, in addition, L' is the language of fields, then ACF—the class of algebraically closed fields (or $\mathrm{Th}_{L'}(\mathrm{ACF})$ since ACF is L'-elementary)—admits the elimination of the quantifier $\exists$ (see Sacks [1972]).

(2) Let L be the set of all existential L'-formulas. Then $\mathrm{Th}_{L'}(K)$ is model-complete iff L' is reducible to L with respect to K (see Sacks [1972]).

For quantifier-elimination there are two ways to look at the problem. On the one hand, we can regard the existence of a quantifier-elimination as a certain model-theoretic property, this being reflected, for instance, in Example (1a) above. On the other, we might be interested more in the manner of elimination itself. This is especially true when decidability is under consideration. If we take the first position, then we will speak of "eliminability". The case in which L is an elementary language and L' is $L(Q)$, the language obtained from L by adding a certain generalized quantifier Q (or even a set of them), is then of particular interest. In the second subsection, we will consider precisely this situation, admitting the following abuse of language.

Let K be an elementary class axiomatized by the theory T in a first-order language L, and let Q be a certain generalized quantifier. We say *Q is eliminable in* K (or *in* T) if $L(Q)$ is reducible to L with respect to the class $\mathrm{Mod}(T \cup \{Q\bar{x}(\bar{x} = \bar{x})\}$ or, equivalently, if Q is L-eliminable in the class $\{\mathfrak{M} \in K : \mathfrak{M} \models Q\bar{x}(\bar{x} = \bar{x})\}$. Notice then that for eliminability of Q in T it will suffice to eliminate Q in expressions of the form $Q\bar{x}\,\varphi(\bar{x}, \bar{z})$, where φ is first-order.

If we take the second of the positions we have noted, we will speak of "elimination procedures". Observe that by an "elimination procedure for a class K" we do not mean a procedure providing a complete elimination of a given quantifier in K, but rather one that is applicable only up to a certain set of sentences (and, in some cases, formulas also)—the so-called core sentences—which should be easy to survey. Thus, finding an elimination procedure will, in most cases, include finding an appropriate set of core sentences (and definable predicates); and, of course, it will yield eliminability results for those subclasses on which the truth values of the core sentences are constant. In the third subsection we will consider this problem for the class of modules as well as the class of abelian groups. Finally, we emphasize that throughout this section we will be mainly concerned with the Malitz quantifiers Q_α^m ($m < \omega$, α an ordinal), where in the next subsection we will concentrate on the cardinality quantifiers Q_0 ($= Q_0^1$) and Q_1 ($= Q_1^1$) and the Ramsey quantifiers Q_0^m ($m < \omega$).

As concerns other generalized quantifiers, we would like to draw the readers' attention to the results of Steinhorn [1980], results which once again fortify our conviction that the method of generalized quantifiers can be an excellent tool for investigations into first-order model theory.

Convention. Throughout this section the length of the sequence $\mathbf{x}$ is assumed to be equal to the arity of the given quantifier and, if not stated otherwise, this to be equal to m.

Recall that a set D is (*weakly*) *homogeneous* for a formula $\varphi(\mathbf{x}, \mathbf{a})$ in a structure $\mathfrak{M}$, $\mathbf{a} \in \mathfrak{M}$, if every m-tuple $\mathbf{d}$ of (distinct) elements of D satisfies $\varphi(\mathbf{x}, \mathbf{a})$ in $\mathfrak{M}$. For an ordinal α, the $\aleph_\alpha$-interpretation Q_α^m of the m-placed Malitz quantifier Q^m is defined for a structure $\mathfrak{M}$ of power not less than $\aleph_\alpha$ by "$\mathfrak{M} \models Q_\alpha^m x_0 \ldots x_{m-1}\, \varphi(x_0, \ldots, x_{m-1})$ iff there is a set of power $\aleph_\alpha$ in $\mathfrak{M}$ which is weakly homogeneous for φ".

Warning. As to the elimination procedure given in the third subsection of this chapter, it is essential to interpret Q^m in the way in which it was there interpreted, with emphasis on "weakly". This attribute does not play any rôle in the investigation of eliminability, since the corresponding two quantifiers (one as given above, and the other having "weakly" omitted) are expressible one by the other. This the reader can easily verify. Accordingly, in the next subsection we will use this quantifier with "weakly" omitted, in the interpretation the omission being for the sake of simplicity.

The reader should consult Chang–Keisler [1973] or Shelah [1978a] for the fundamental concepts of stability theory.

1.2. Eliminability of Generalized Quantifiers

As we have already mentioned, our aim here is to find model-theoretic properties of first-order theories which are equivalent to eliminability of certain generalized quantifiers.

Convention. In this subsection "theory" means "first-order theory having infinite models only", and T will denote such a theory. Moreover, terms such as "definable" or "formula" are used for "first-order definable" or "first-order formula". Two other points are worth mentioning at this juncture.

First, remember the warning given in the first subsection; and, second, we note that although the general concept to be treated here is due to Tuschik, the material was unfortunately, not published in full detail until the work of Baudisch–Seese–Tuschik–Weese [1980]. In this connection, the reader should also see Tuschik [1975, 1977a].

We will begin our exposition with the unary quantifier Q_0 having the interpretation "there are infinitely many", examining first the following

Definition. A formula $\varphi(x, z_1, \ldots, z_n)$ is said to *have a degree* relative to T if there is a natural number k such that, for every model $\mathfrak{M}$ of T and elements $a_1, \ldots, a_n$ of $\mathfrak{M}$, the following holds:

> if $\varphi(x, a_1, \ldots, a_n)$ has finitely many solutions in $\mathfrak{M}$, then it has at most k-many.

T is said to be *graduated* if every formula has a degree relative to T.

The facts stated in the following examples can be derived from the corresponding elimination of (elementary) quantifiers.

Examples. The first-order theories of the following classes of structures are graduated:

(1) The class ACF of algebraically closed fields;
(2) The class RCF of real closed fields;
(3) The class DCF_0 of differentially closed fields of characteristic 0,
(4) The class DLO of dense linear orderings;
(5) The class ABA of atomless boolean algebras;
(6) The class AG_p of infinite elementary abelian p-groups, where p is a prime.

Ryll-Nardzewski's theorem (see Chang–Keisler [1973, Theorem 2.3.13]) yields a wealth of graduated theories:

Proposition. *Every countable $\aleph_0$-categorical theory is graduated.* □

Examples (4), (5), and (6) above are special cases of the assertion in the proposition. The next result shows what graduatedness is related to our general topic.

1.2.1 Theorem. *T is graduated* iff *Q_0 is eliminable in T.*

Clearly, in a graduated theory, Q_0 is eliminable by $\exists^{>n}$, where n ranges over the degrees of all formulas. If Q_0 is eliminable in such a simple manner, then we say that Q_0 is *definable in T*. The theorem just stated thus asserts that if Q_0 is eliminable, then it is definable. This is, *mutatis mutandis*, true for Malitz and other "Malitz-like" quantifiers; and, moreover, it is basic for eliminability investigations. For more on this, the reader should see Baldwin–Kueker [1980]; Baudisch–Seese–Tuschik–Weese [1980]; Rothmaler–Tuschik [1982]; Vinner [1975]. We will prove it here in the following general form, a form which is appropriate for our purposes. The reader may extend it to a more general concept of quantifiers, including that of Steinhorn [1980]. Before proceeding further in this direction, however, we need some additional notation.

For the m-placed Malitz quantifier Q^m, we also introduce finitary interpretations: for a given natural number n, the n-interpretation of Q^m, in terms $Q^m_{(n)}$, is given by "$\mathfrak{M} \models Q^m_{(n)} x_0 \dots x_{m-1}\, \varphi(x_0, \dots, x_{m-1})$ iff there is a set of power n in $\mathfrak{M}$ which is homogeneous for $\varphi(x_0, \dots, x_{m-1})$".

Definition. The quantifier Q^m_α is called *definable in T* if for every (first-order) formula $\varphi(\bar{x}, \bar{z})$, there is a number n_φ such that

$$T \cup \{Q^m_\alpha \bar{x}(\bar{x} = \bar{x})\} \vdash \forall \bar{z}(Q^m_{(n_\varphi)} \bar{x}\, \varphi(\bar{x}, \bar{z}) \to Q^m_\alpha \bar{x}\, \varphi(\bar{x}, \bar{z})).$$

Notice that if φ is first-order, then $Q^m_{(n)} \bar{x}\, \varphi(\bar{x})$ is first order also. Hence, Q^m_α is eliminable in T, if it is definable in T. The definability lemma given below asserts that the converse is true.

The Definability Lemma. *A Malitz quantifier is eliminable in T iff it is definable in T.*

Proof. One direction has been already mentioned. As for the other direction, suppose that $\varphi(\bar{x}, \bar{z})$ and $\psi(\bar{z})$ are (first-order) formulas such that, for $T' = T \cup \{Q^m_\alpha \bar{x}(\bar{x} = \bar{x})\}$, the following holds:

$$(*) \qquad T' \vdash \forall \bar{z}(Q^m_\alpha \bar{x}\, \varphi(\bar{x}, \bar{z}) \leftrightarrow \psi(\bar{z})).$$

We have to show that a number n_φ exists with

$$T' \vdash \forall \bar{z}(Q^m_{(n_\varphi)} \bar{x}\, \varphi(\bar{x}, \bar{z}) \to \psi(\bar{z})).$$

Thus, we assume the contrary. Then, for arbitrarily large numbers n, there are models $\mathfrak{M}_n$ of T' containing sequences $\bar{a}_n$ with $\mathfrak{M}_n \models \neg\psi(\bar{a}_n)$ and sets A_n homogeneous for $\varphi(\bar{x}, \bar{a}_n)$ which have power not less than n. Let $C = \{c_i: i < \aleph_\alpha\}$ be a set and $\bar{a}$ be a sequence of new and distinct constant symbols, and let S denote the union of the following sets of sentences in the corresponding inessential extension of the (first-order) language of T:

(1) $\{c_i \neq c_j: i < j < \aleph_\alpha\}$;

(2) $\{\varphi(\bar{c}, \bar{a}): \bar{c} \in C^m\}$;

(3) $T \cup \{\neg\psi(\bar{a})\}$.

By assumption, every finite subset of S can be realized in some $\mathfrak{M}_n$. Thus, the compactness theorem (for first-order logic) implies the existence of a model $\mathfrak{M}$ of T consisting of a sequence $\bar{a}$ with $\mathfrak{M} \models \neg\psi(\bar{a})$ and a set A which is homogeneous for $\varphi(\bar{x}, \bar{a})$ and has power $\aleph_\alpha$. But this contradicts the assertion in (*). □

Having proven the definability lemma in the most general form, we now return to unary Malitz quantifiers = usual cardinality quantifier. Theorem 1.2.1 is a special case of that lemma. Together with the proposition above, it implies that Q_0 is eliminable in every countable $\aleph_0$-categorical theory as well as in all the theories of Examples (1) through (6). Let us turn now to the $\aleph_1$-interpretation. We are going to prove a theorem which is due to Tuschik and which links the eliminability of Q_1 with the following well-known property of first-order model theory. First, recall that T has the *Vaught property* if it has a model $\mathfrak{M}$ containing an infinite definable set of power less than $|\mathfrak{M}|$.

We need also Vaught's two-cardinal theorem which asserts that a countable theory having the Vaught property possesses a model of power $\aleph_1$ containing an infinite countable definable set. For a proof of this, consult Chang–Keisler [1973, Theorem 3.2.12] or Sacks [1972, Section 22]. Interestingly enough, a good portion of it yields the next lemma; and, in fact, does so without any restriction on the cardinality of the theory.

Lemma. *A nongraduated theory has the Vaught property.* □

Now we are able to prove the promised theorem.

1.2.2 Theorem. *Let T be countable, then Q_1 is eliminable in T iff T does not have the Vaught property.*

Proof. If T has the Vaught property, then, by the two-cardinal theorem, Q_1 is not definable. Hence, it is not eliminable in T. For the other direction, suppose Q_1 is not eliminable in T. Then, by the definability lemma, there is a formula $\varphi(x, \bar{z})$ and models $\mathfrak{M}_n$ of T containing sequences $\bar{a}_n$ such that $\aleph_1 > |\varphi(\mathfrak{M}_n, \bar{a}_n)| \geq n$, for every number n. If one of these latter sets is infinite, then we are done.

If not, then T is not graduated. Thus, by the above lemma, again T has the Vaught property. ▯

Vaught's two-cardinal theorem implies no $\aleph_1$-categorical countable theory has the Vaught property. Thus, a corollary follows which was independently obtained by several investigators (see Tuschik [1975], Vinner [1975], or Wolter [1975b]).

Corollary. *Q_1 is eliminable in every countable $\aleph_1$-categorical theory.* ▯

Together with the above lemma, the preceding theorems on Q_0 and Q_1 yield the next result.

Corollary. *Let T be countable. If Q_1 is eliminable in T, then Q_0 is also.* ▯

Similarly, two-cardinal considerations show that the eliminability of Q_1 is equivalent to the eliminability of each of the following quantifiers in a given countable theory: Chang's quantifier Q_c (= the unary cardinality quantifier in the equicardinality interpretation) and Härtig's quantifier I. As a further consequence, we remark that Q_1 (and also Q_0) is eliminable in the theories of Examples (1), (2), and (6). For ACF and RCF, this fact was also shown by Vinner [1975].

In the remainder of this subsection, we will present some material that is due to Baldwin–Kueker [1980]. This material concerns the eliminability of Ramsey quantifiers (= Malitz quantifiers in the $\aleph_0$-interpretation) in complete theories. Moreover, we will eventually prove a theorem describing this eliminability within the class of stable theories in terms of the following notion of first-order model theory, a notion that was introduced by Keisler [1967b]. The reader should also see Shelah [1978a] in this connection.

$\varphi(\bar{x}, \bar{z})$ has the *finite cover property* (abbreviated f.c.p.) in T if, for arbitrarily large numbers n, there are models $\mathfrak{M}_n$ containing sequences $\bar{a}_0, \ldots, \bar{a}_{n-1}$ which satisfy

$$\mathfrak{M}_n \models \neg\exists\bar{x} \bigwedge_{j<n} \varphi(\bar{x}, \bar{a}_j) \wedge \bigwedge_{i<n} \exists\bar{x} \bigwedge_{i\neq j<n} \varphi(\bar{x}, \bar{a}_j).$$

T is said to have the *f.c.p.* if some formula has. Note that

$$\psi(x, \bar{v}^\frown u) \leftrightarrow (\varphi(x, \bar{v}) \wedge x \neq u)$$

has the f.c.p. if $\varphi(x, \bar{v})$ is not graduated.

By Keisler [1967b] a countable $\aleph_1$-categorical theory does not have the f.c.p. On the other hand, Shelah proved that every unstable one does have this property (See Shelah [1978a]). The first half of the theorem of Baldwin and Kueker is contained in the next lemma.

Lemma. *If T does not have the f.c.p., then all Ramsey quantifiers are eliminable in T.*

Proof. Assume Q_0^m is not eliminable in T. By the definability lemma, we then have a formula $\varphi(x_0, \ldots, x_{m-1}, \bar{z})$ as well as models $\mathfrak{M}_n$ of T containing sequences $\bar{a}_n$

and finite sets A_n of power not less than n such that A_n is homogeneous for $\varphi(\bar{x}, \bar{a}_n)$ in $\mathfrak{M}_n$ and maximal with respect to that property ($n < \omega$). Let $\psi(x, \bar{z}'{}^\frown\bar{z})$ be the formula $\varphi(x, z_1, \ldots, z_{m-1}, \bar{z}) \wedge x \neq z_1$, where $\bar{z}' = (z_1, \ldots, z_{m-1})$. We will show that $\psi(x, \bar{z}'{}^\frown\bar{z})$ has the f.c.p.

To this end, let B_n denote the set of all $(m-1)$-tuples from A_n. Choose a subset C_n of B_n minimal with respect to the property

$$(1)_n \qquad \mathfrak{M}_n \models \neg\exists x \bigwedge_{\bar{c} \in C_n} \psi(x, c^\frown\bar{a}_n).$$

This is possible, since B_n itself is a finite set having that property, for A_n is maximally homogeneous for $\varphi(\bar{x}, \bar{a}_n)$. Thus, the following holds:

$$(2)_n \qquad \mathfrak{M}_n \models \bigwedge_{\bar{c}' \in C_n} \exists x \bigwedge_{\bar{c}' \neq \bar{c} \in C_n} \psi(x, c^\frown\bar{a}_n).$$

For every subset of B_n consisting of less than n elements, we can choose an element of A_n different from all first components of elements of that subset. Hence, no such subset has the property given in $(1)_n$. Consequently, C_n has at least n elements. This conclusion, together with $(1)_n$ and $(2)_n$, for all n, shows that $\psi(x, \bar{z}'{}^\frown\bar{z})$ has the f.c.p. in T. □

In the other direction of the theorem below we shall utilize Shelah's f.c.p. theorem which asserts that a stable complete theory has the f.c.p. iff there is a formula $\varphi(x, y, \bar{z})$ satisfying the following: For every number n there is a sequence $\bar{c}_n$ of elements in some model $\mathfrak{M}_n$ of T such that $\varphi(x, y, \bar{c}_n)$ defines on the universe of $\mathfrak{M}_n$ an equivalence relation having not less than n, but only finitely many equivalence classes (see Shelah [1978a; Chapter II, Theorem 4.4]).

1.2.3 Theorem. *Let T be stable and complete. Then All Ramsey quantifiers are eliminable in T iff the Ramsey quantifier Q_0^2 is eliminable in T iff T does not have the f.c.p.*

Proof. For the remaining implication, assume T has the f.c.p., then we must show that Q_0^2 is not definable in T. To this end, we choose a formula $\varphi(x, y, \bar{z})$, as well as sequences $\bar{c}_n$ and models $\mathfrak{M}_n$ according to the f.c.p. theorem. Then, clearly we have that

$$\mathfrak{M}_n \models Q_{(n)}^2 x_0 x_1 \neg\varphi(x_0, x_1, \bar{c}_n) \wedge \neg Q_0^2 x_0 x_1 \neg\varphi(x_0, x_1, \bar{c}_n)$$

holds for all n. Whence, the assertion follows. □

Using the aforementioned observation of Keisler, we can easily derive the following

Corollary. *All Ramsey quantifiers are eliminable in every countable $\aleph_1$-categorical theory.* □

This corollary generalizes the corresponding result for ACF_0 as proven by Cowles [1979a].

Further Results. We will close this section with a few brief remarks sketching some further pertinent results.

(1) Tuschik has provided some further results with regard to the relative strength and effectiveness of eliminability of the unary cardinality quantifiers Q_α. The reader should consult Tuschik [1977a or 1982a]; or Baudisch–Seese–Tuschik–Weese [1980]; or Rothmaler–Tuschik [1982]. Vinner [1975] is also informative.

(2) In Rothmaler [1981 or 1984] it is shown that Q_0 is eliminable in every complete first-order theory of modules. Baudisch [1984] extended this to all Ramsey quantifiers. See the next subsection for more on this.

(3) Further algebraic results can be found in the papers of Cowles, Pinus, and Rothmaler that are cited in the bibliography.

(4) Baudisch [1977b or 1979], and Baldwin–Kueker [1980] prove independently that all Ramsey quantifiers are eliminable in a countable $\aleph_0$-categorical first-order theory, thus showing that the stability assumption made in Theorem 1.2.3 of this section is necessary.

(5) Schmerl–Simpson [1982] provided an effective elimination of all Ramsey quantifiers in Presburger arithmetic. In contrast, however, Kierstead–Remmel [1983] constructed decidable first-order theories admitting elimination of these quantifiers which cannot be made effective.

(6) Baldwin–Kueker [1980] proved the eliminability of the Malitz quantifiers Q_c^m (in the equi-cardinality interpretation) in countable $\aleph_1$- categorical first-order theories. Clearly, this is then true for all other interpretations. This result generalizes the corresponding result for ACF_0 which had been proven by Cowles [1979a].

(7) Rothmaler–Tuschik [1982] generalized the result that is here given as Theorem 1.2.2 to Malitz quantifiers Q_1^m $(m < \omega)$ so as to obtain an analog of Theorem 1.2.3 for these. Furthermore, as a corollary, they independently obtained the result mentioned in the preceding remark.

(8) Theorem 1.2.3 asserts, among other things, that in stable theories the eliminability of Q_α^2 implies that of all Q_α^m $(m < \omega)$ in the case $\alpha = 0$. Rapp [1982 or 1983] proved that this is also true in the case $\alpha = 1$. Moreover, he showed that in stable theories the eliminability of Q_1^2 implies that of all Malitz quantifiers Q_α^m $(m < \omega,\ \alpha \geq 0$; for $\alpha = 0$ this was already noticed by Rothmaler–Tuschik [1982]).

1.3. Elimination Procedures for Modules and Abelian Groups

The elementary theory of groups is undecidable (see Tarski in Tarski–Mostowski–Robinson [1953]). Furthermore, a good number algebraically interesting classes of groups have an undecidable elementary theory. From Ershov [1974] and Samjatin [1978], it is known that the elementary theory of every non-abelian variety of groups is undecidable. In contrast to this, however, Szmielew [1955] proved the decidability of the elementary theory T_Z of abelian groups. Extending the ordinary language $(+, -, 0)$ of group theory by predicates "$p^n | x$" and defining some core sentences, she gave an effective elimination procedure: every formula is equivalent modulo T_Z to a boolean combination of Szmielew core sentences and atomic formulas. One can extend this elimination procedure to the logics $\mathscr{L}_{\omega\omega}(Q_\alpha)$, $\mathscr{L}_{\omega\omega}(Q_\alpha^{<\omega})$, $\mathscr{L}_{\omega\omega}(\text{aa})$, and $\mathscr{L}_{\omega\omega}(I)$, provided the set of core sentences is extended in an appropriate way. For concrete results and references, the reader should see the list below. Moreover, one can find corresponding elimination procedures for arbitrary R-modules. Here we will present just such a procedure for Malitz quantifiers in regular interpretations (Baudisch [1984]).

Convention. Throughout this subsection R is an associative ring with unit 1, $\mathfrak{A}$ is a left R-module, and $\mathbf{a}$ is a sequence from $\mathfrak{A}$. As is usual in first-order model theory of modules, we will consider the first-order language having the following nonlogical symbols: 0, $+$, and, for every $r \in R$, a unary function symbol expressing the left multiplication by r.

For the sake of simplicity we will use L_R to denote the set of all first-order formulas in this language. Then the elementary theory of all (unital) left R-modules can be axiomatized by a set of L_R-sentences. Let $L_R(Q_\alpha^{<\omega})$ and $T_R(Q_\alpha^{<\omega})$ denote the extensions of L_R and T_R respectively to the logic $\mathscr{L}_{\omega\omega}(Q_\alpha^{<\omega})$.

A *positive primitive* (abbreviated *p.p.*) formula is a formula of the form $\exists \mathbf{y}\, \psi(\mathbf{x}, \mathbf{y})$, where ψ is a finite conjunction of equations (with coefficients from R). Notice that a p.p. formula $\chi(x_0, \ldots, x_{m-1})$ defines an additive subgroup $\chi(\mathfrak{A}^m)$ in the module $\mathfrak{A}^m$ (and if R is commutative, this is even a submodule), and a p.p. formula $\chi(\mathbf{x}; \mathbf{a})$ defines a coset of the subgroup $\chi(\mathfrak{A}^m; \mathbf{0})$ in $\mathfrak{A}^m$.

Notation. Throughout this discussion we will let $\chi(\mathbf{x}; \mathbf{z})$ be a p.p. formula. Moreover, we will use $\chi^j(x)$ to denote the formula $\chi(0, \ldots, 0, x, 0, \ldots, 0; \mathbf{0})$ obtained from $\chi(\mathbf{x}; \mathbf{0})$ by substituting x for x_j and 0 for the other components of $\mathbf{x}$, $\chi'(x)$ to denote $\bigwedge_{j<m} \chi^j(x)$, and $\chi^d(x; \mathbf{a})$ to denote the formula $\chi(x, \ldots, x; \mathbf{a})$.

Note that by the additivity of p.p. formulas, $\bigwedge_{j<m} \chi^j(x)$ implies $\chi^d(x; \mathbf{0})$. The following implication can be easily derived from additivity of the p.p. formula $\chi(\mathbf{x}; \mathbf{z})$:

$$(1) \qquad T_R \vdash \chi(x_0, \ldots, x_{j-1}, w, x_{j+1}, \ldots, x_{m-1}; \mathbf{z}) \rightarrow [\chi(x_0, \ldots, x_{j-1}, v, x_{j+1}, \ldots, x_{m-1}; \mathbf{z}) \leftrightarrow \chi^j(v - w)].$$

It can be easily seen from the first part of the next lemma that, for a p.p. formula, a sufficiently large set is homogeneous if it is weakly homogeneous.

Lemma. *Let $\chi(\mathbf{x}; \mathbf{z})$ be a p.p. formula, then we have*

(i) *A set C of power greater than m which is weakly homogeneous for $\chi(\mathbf{x}; \mathbf{a})$ in $\mathfrak{A}$ is contained in $\chi^d(\mathfrak{A}; \mathbf{a})$ and in $c + \chi'(\mathfrak{A})$, for every $c \in C$; and*

(ii) *Every subset C of $c + \chi'(\mathfrak{A})$, for some $c \in \chi^d(\mathfrak{A}; \mathbf{a})$, is weakly homogeneous for $\chi(x; \mathbf{a})$.*

Proof. To establish (i), we let $\{c_0, \ldots, c_m\}$ be an $(m+1)$-element-subset of C. Since it is weakly homogeneous for $\chi(\mathbf{x}; \mathbf{a})$ in $\mathfrak{A}$, it is not difficult to derive $\mathfrak{A} \models \bigwedge_{j<m} \chi^j(c_i - c_k)$, where $i, k \leq m$. Hence, all elements of C lie in the same coset of $\chi'(\mathfrak{A})$. Using (1), $C \subseteq \chi^d(\mathfrak{A}; \mathbf{a})$ follows. The proof of part (ii) is an immediate consequence of (1). □

Key Lemma. *Let C be an infinite subset of $\mathfrak{A}$ of regular cardinality such that for every $j < m$, there are less than $|C|$ elements of C in every coset of $\chi^j(\mathfrak{A})$. Then C contains a subset of the same cardinality which is weakly homogeneous for $\neg\chi(\mathbf{x}; \mathbf{a})$.*

Proof. We first note that by definition a set of power less than m is weakly homogeneous for arbitrary m-placed formulas; this fact provides the initial step of the following induction

It suffices to show that to every subset E of C of power less than $|C|$ which is weakly homogeneous for $\neg\chi(\mathbf{x}; \mathbf{a})$ one can add some $c \in C - E$ and still not disturb the weak homogeneity for $\neg\chi(\mathbf{x}; \mathbf{a})$. To do so, however, we must prove that the set of elements in C which one cannot add to E has power less than $|C|$. To this purpose, then, let $c \in C - E$ such that $E \cup \{c\}$ is not weakly homogeneous for $\neg\chi(\mathbf{x}; \mathbf{a})$. Then there are $j < m$ and distinct elements $e_0, \ldots, e_{j-1}, e_{j+1}, \ldots, e_{m-1}$ in E such that

$$(*) \qquad \mathfrak{A} \models \chi(e_0, \ldots, e_{j-1}, c, e_{j+1}, \ldots, e_{m-1}; \mathbf{a}).$$

By (1) above, all c's satisfying $(*)$ lie in the same coset of $\chi^j(\mathfrak{A})$. Hence, by hypothesis, there are less than $|C|$ such elements.

Since C is infinite and $|E| < |C|$, there are less than $|C|$ $(m-1)$-tuples in E. Consequently, the whole set of elements that one cannot add to E must have power less than $|C|$ also. □

This lemma enables us to prove a strong "Ramsey-like" property for p.p. formulas.

Lemma. *Every infinite subset of $\mathfrak{A}$ of regular cardinality contains a subset of the same cardinality which is weakly homogeneous either for $\chi(\mathbf{x}; \mathbf{a})$ or for $\neg\chi(\mathbf{x}; \mathbf{a})$.*

Proof. Using induction on m, we can clearly assume the assertion is true for $m - 1 \geq 1$. Let C be an infinite set in $\mathfrak{A}$ not containing a subset of cardinality $|C|$ which is weakly homogeneous for $\neg\chi(\mathbf{x}; \mathbf{a})$. By the Key Lemma there are some $j < m$ (for the sake of simplicity, say $j = 0$), some $c \in C$, and some subset E of power $|C|$ in $\chi^0(\mathfrak{A})$ with $c + E \subseteq C$. By the induction hypothesis, E contains a subset D of power $|E| = |C|$ which is weakly homogeneous for $\chi(c, c + x_1, \ldots, c + x_{m-1}; \mathbf{a})$ or for its negation. Since $D \subseteq \chi^0(\mathfrak{A})$, by (1) above, $c + D$ is then weakly homogeneous for $\chi(\mathbf{x}; \mathbf{a})$ or $\neg\chi(\mathbf{x}; \mathbf{a})$, respectively. □

Corollary. *Let $\varphi_i(\mathbf{x}; \mathbf{z})$ be a conjunction of p.p. and negated p.p. formulas $(i < n)$. Then every infinite set of regular cardinality weakly homogeneous for $\bigvee_{i<n} \varphi_i(\mathbf{x}; \mathbf{a})$ in $\mathfrak{A}$ contains a subset of the same cardinality which is weakly homogeneous for some $\varphi_{i_0}(\mathbf{x}; \mathbf{a})$, $i_0 < n$.*

Proof. Let C be infinite and weakly homogeneous for $\bigvee_{i<n} \varphi_i(\mathbf{x};\mathbf{a})$. Using induction on n, we assume that there is no subset of C of power $|C|$ which is weakly homogeneous for $\bigvee_{0<i<n} \varphi_i(\mathbf{x};\mathbf{a})$ in $\mathfrak{A}$. Let $\varphi_0(\mathbf{x};\mathbf{a})$ be $\bigwedge_{i<k} \chi_i(x;a)$, where the χ_i are p.p. or negated p.p. formulas. Step by step, we will construct a subset of C of power $|C|$ which is weakly homogeneous for $\varphi_0(\mathbf{x};\mathbf{a})$ in $\mathfrak{A}$. For this, assume that $C' \subseteq C$ is weakly homogeneous for $\bigwedge_{i<j} \chi_i(\mathbf{x};\mathbf{a})$ and $|C'| = |C|$, where $j \leq k$ (if $j = 0$, let $C = C'$). By the preceding lemma, it suffices to show that C' contains no subset of power $|C|$ which is weakly homogeneous for $\neg\chi_j(\mathbf{x};\mathbf{a})$. But this is clear, since every subset D of C that is weakly homogeneous for $\neg\chi_j(\mathbf{x};\mathbf{a})$ would be weakly homogeneous for $\bigvee_{0<i<n} \varphi_i(\mathbf{x};\mathbf{a})$, thus contradicting the assumption. ☐

Using an infinitary version of B. H. Neumann's lemma, we obtain the next lemma. (See Baudisch [1984]).

Lemma. *Let χ, η_i be p.p. formulas ($i < n$). Then*

$$T_R(Q_\alpha^2) \vdash Q_\alpha^2 x_0 x_1\Big(\chi(x_0) \wedge \bigwedge_{i<n} \neg\eta_i(x_0 - x_1)\Big)$$
$$\leftrightarrow \bigwedge_{i<n} Q_\alpha^2 x_0 x_1(\chi(x_0) \wedge \neg\eta_i(x_0 - x_1)).$$

Before we prove the main theorem of this subsection, we will introduce some more notation and state a theorem on the existence of an elementary elimination procedure which is due to Baur [1976] and Monk [1975] and which is basic for the the first-order model theory of modules.

If χ and η are p.p. formulas with $T_R \vdash \eta(x) \to \chi(x)$ then let $(\chi/\eta)(\mathfrak{A})$ denote the cardinality of the factor group $\chi(\mathfrak{A})/\eta(\mathfrak{A})$. Clearly there are elementary $\exists\forall$-sentences expressing $(\chi/\eta)(\mathfrak{A}) \geq k$ for every natural number k. Call these *elementary core sentences.* Now the theorem of Baur and Monk states that every formula of L_R is equivalent modulo T_R to a boolean combination of elementary core sentences and p.p. formulas.

Our goal is to prove an analogue to this theorem for the language $L_R(Q_\alpha^{<\omega})$. First note that in this language we can express $(\chi/\eta)(\mathfrak{A}) \geq \aleph_\alpha$ by $Q_\alpha^2 x_0 x_1(\chi(x_0) \wedge \neg\eta(x_0 - x_1))$. Those sentences, together with the elementary core sentences, will be called *Q_α^2-core sentences.*

Theorem. *Every formula of $L_R(Q_\alpha^{<\omega})$ is equivalent modulo $T_R(Q_\alpha^{<\omega})$ to a boolean combination of Q_α^2-core sentences and p.p. formulas. This* boolean combination *can be effectively found relative to the elementary procedure provided by the Theorem of Baur and Monk.*

Proof. We show the theorem for regular ω_α only. For a complete proof, see Baudisch [1984]. By the theorem of Baur and Monk and induction on the complexity of formulas, it suffices to consider the case $Q_\alpha^m \mathbf{x}\ \varphi(\mathbf{x};\mathbf{z})$, where $\varphi(\mathbf{x};\mathbf{z})$ is a boolean combination of p.p. formulas. The above corollary thus reduces this to the case in which φ is a conjunction of p.p. and negated p.p. formulas. Since a conjunction of p.p. formulas is equivalent to a p.p. formula, we can further suppose that $\varphi(\mathbf{x}; \mathbf{z})$ is of the form $\chi(\mathbf{x}; \mathbf{z}) \wedge \bigwedge_{i<k} \neg\eta_i(\mathbf{x}; \mathbf{z})$, where χ and η_i are p.p. ($i < k$). We will now construct the desired boolean combination in the following development.

Let H be the set of all partitions $\{I, J\}$ of $\{(i,j): i < k, j < m\}$. For each $\{I, J\} \in H$ define $F(I) = \{i < k: \text{for all } j < m\ (i,j) \in I\}$. Now let $\psi(\mathbf{z})$ be the disjunction of the following formulas, where $\{I, J\}$ runs over all the partitions in H:

$$\exists y\left[\chi^d(y; \mathbf{z}) \wedge \bigwedge_{i \in F(I)} \neg\eta_i^d(y; \mathbf{z})\right]$$

$$\wedge\ Q_\alpha^2 x_0 x_1\left[\chi'(x_0) \wedge \bigwedge_{(i,j)\in I} \eta_i^j(x_0) \wedge \bigwedge_{(i,j)\in J} \neg\eta_i^j(x_0 - x_1)\right].$$

Using the preceding lemma and the elementary elimination procedure, it is not difficult to show that ψ is indeed equivalent to a boolean combination of p.p. formulas and Q_α^2-core sentences. Thus, it suffices to verify that

$$T_R(Q_\alpha^{<\omega}) \vdash \forall\mathbf{z}(Q_\alpha^m \mathbf{x}\ \varphi(\mathbf{x}; \mathbf{z}) \leftrightarrow \psi(\mathbf{z})).$$

To prove this in the direction from left to right, we let C be a set of power $\aleph_\alpha$ which is weakly homogeneous for $\varphi(\mathbf{x}; \mathbf{a})$ in $\mathfrak{A}$. By part (i) of the first lemma, C is weakly homogeneous for $\chi'(x_0 - x_1)$. Thus it is trivially so for

$$\bigvee_{\{I,J\}\in H}\left(\chi'(x_0 - x_1) \wedge \bigwedge_{(i,j)\in I} \eta_i^j(x_0 - x_1) \wedge \bigwedge_{(i,j)\in J} \neg\eta_i^j(x_0 - x_1)\right).$$

The corollary above yields some $\{I, J\} \in H$ and some set $C' \subseteq C$ of power $\aleph_\alpha$ which is weakly homogeneous for

$$\chi'(x_0 - x_1) \wedge \bigwedge_{(i,j)\in I} \eta_i^j(x_0 - x_1) \wedge \bigwedge_{(i,j)\in J} \neg\eta_i^j(x_0 - x_1).$$

Let $c \in C'$ and E a set with $c + E = C'$. By (i) of the first lemma, $\mathfrak{A} \models \chi^d(c; \mathbf{a})$. Further,

(2) $\qquad E$ is weakly homogeneous for $\bigwedge_{(i,j)\in J} \neg\eta_i^j(x_0 - x_1)$

and

(3) $\qquad E \subseteq \chi'(\mathfrak{A}) \cap \bigcap_{(i,j)\in I} \eta_i^j(\mathfrak{A}).$

Notice that (3) implies

(4) $\qquad E \subseteq \eta_i'(\mathfrak{A}) \quad \text{for all} \quad i \in F(I),$

since $\vdash \bigwedge_{j<n} \eta_i^j(x) \leftrightarrow \eta_i'(x)$. Then $\mathfrak{A} \models \bigwedge_{i \in F(I)} \neg \eta_i^d(c; \mathbf{a})$; for, otherwise $\mathfrak{A} \models \eta_i^d(c; \mathbf{a})$ together with (4) and (ii) of the first lemma would imply that $c + E$ were weakly homogeneous for $\eta_i(\mathbf{x}; \mathbf{a})$. Recalling that $c + E \subseteq C$, we thus have a contradiction.

To establish the other direction of the above equivalence, we first choose a partition $\{I, J\} \in H$, a set E of power $\aleph_\alpha$ satisfying (2) and (3) (and hence, must satisfy (4) also), and an element $c \in \mathfrak{A}$ with

$$\mathfrak{A} \models \chi^d(c; \mathbf{a}) \wedge \bigwedge_{i \in F(I)} \neg \eta_i^d(c; \mathbf{a}).$$

We will eventually show that $c + E$ contains a subset of power $\aleph_\alpha$ which is weakly homogeneous for $\varphi(\mathbf{x}; \mathbf{a})$. By (ii) of the first lemma, $c + E$ is weakly homogeneous for $\chi(\mathbf{x}; \mathbf{a})$ as well as for $\bigwedge_{i \in F(I)} \neg \eta_i(\mathbf{x}; \mathbf{a})$; since, otherwise, by additivity, (4) would imply $\mathfrak{A} \models \eta_i^d(c; \mathbf{a})$, thus contradicting the choice of c.

It thus remains to prove the following

Claim. *For every subset E' of E of the same power and for every $i < n$ with $i \notin F(I)$, $c + E'$ contains a subset of the same power which is weakly homogeneous for $\neg \eta_i(x; \mathbf{a})$.*

To establish this claim, we fix some $i \notin F(I)$ and, without loss of generality, assume that $(i, m - 1) \in J$. That done, we first consider the case $\{(i, j): j < m\} \subseteq J$. Then, by (2), all the elements of $c + E'$ lie in different cosets of $\eta_i^j(\mathfrak{A})$ for all $j < m$. Thus, the hypothesis of the Key Lemma is trivially satisfied. Thereby establishing the claim for this case.

Turning now to the general case, we let the variables be ordered in such a way that "$(i, j) \in I$ iff $j < k$" for some $k < m$. We then apply the same argument to the formula $\eta_i(c, \ldots, c, x_k, \ldots, x_{m-1}; \mathbf{a})$ in order to obtain a subset $c + E'' \subseteq c + E'$ which has the same power and which is weakly homogeneous for its negation. Since $E'' \subseteq \bigcap_{j<k} \eta_i^j(\mathfrak{A})$ by (3), it is easy to see that $c + E''$ is weakly homogeneous even for $\neg \eta_i(\mathbf{x}; \mathbf{a})$; whence, the claim is proven. ▯

Corollary. *For modules, $\mathscr{L}_{\omega\omega}(Q_\alpha^2)$ has the same expressive power as $\mathscr{L}_{\omega\omega}(Q_\alpha^{<\omega})$.* ▯

Corollary. *All Ramsey quantifiers Q_0^m $(m < \omega)$ are eliminable in every complete (first-order) extension of T_R.* ▯

By Baur [1975], every complete first-order theory of modules is stable. Hence, the preceding corollary, together with Theorem 1.2.3, has as a consequence the following

Corollary. *No complete first-order theory of modules has the f.c.p.* ▯

Finally, we specify the theorem to the case of abelian groups. We begin by making a general remark.

Assume Σ_1 to be a set of p.p. formulas that is closed under substitution of free variables by 0 and Σ_2 to be a set of elementary core sentences such that the elementary elimination procedure only needs formulas from Σ_1 and Σ_2. Then, the theorem holds true for boolean combinations of formulas from $\Sigma_1 \cup \Sigma_2$ and Q_α^2-core sentences of the form $Q_\alpha^2 x_0 x_1(\chi(x_0) \wedge \neg\eta(x_0 - x_1))$, where χ is a conjunction of formulas from Σ_1 and η is in Σ_1.

Now let R be the ring Z of all integers, Σ_1 the set of all atomic L_Z-formulas and all formulas $\exists y(p^n y = \sum_{i<k} r_i x_i)$, where p is a prime, n a natural number, and r_i an integer. Furthermore, let Σ_2 be the set of all Szmielew core sentences; that is, Σ_2 is the set of all sentences $(\chi/\eta) \geq k$, where

(#)	either	$\chi(x)$ is $px = 0 \wedge p^{n-1}\mid x$	and	$\eta(x)$ is $x = 0$;
	or	$\chi(x)$ is $p^{n-1}\mid x$	and	$\eta(x)$ is $p^n\mid x$;
	or	$\chi(x)$ is $px = 0 \wedge p^{n-1}\mid x$	and	$\eta(x)$ is $px = 0 \wedge p^n\mid x$;
	or	$\chi(x)$ is $rx = 0$	and	$\eta(x)$ is $x = 0$,

for some prime p and natural numbers n and r, with $1 \leq n$.

Call all sentences of Σ_2 and all sentences $Q_\alpha^2 x_0 x_1(\chi(x_0) \wedge \neg\eta(x_0 - x_1))$, for (χ, η) from (#), *Q_α^2-Szmielew-core-sentences.* The new sentences express that the corresponding Szmielew invariants are of power at least $\aleph_\alpha$. By Szmielew [1955], the elementary elimination procedure for Z-modules (= Abelian groups) only needs formulas from Σ_1 and Σ_2. The above theorem can be sharpened in this context.

Theorem. *For every formula of $L_Z(Q_\alpha^{<\omega})$, we can effectively find a boolean combination of formulas from Σ_1 and Q_α^2-Szmielew-core-sentences to which it is equivalent modulo $T_Z(Q_\alpha^{<\omega})$.* ∎

Corollary. *$T_Z(Q_\alpha^{<\omega})$ is decidable.* ∎

We now collect corresponding results into the following table.

Table of Elimination Procedures and Decidability

	Abelian Groups	Modules
$\mathscr{L}_{\omega\omega}$	Szmielew [1955]	Baur [1976], Monk [1975]
$\mathscr{L}_{\omega\omega}(Q_\alpha)$	Baudisch [1976]	Rothmaler [1981 or 1984]
$\mathscr{L}_{\omega\omega}(Q_\alpha^{<\omega})$	Baudisch [1983]	Baudisch [1984]
$\mathscr{L}_{\omega\omega}$(aa)	Eklof–Mekler [1979] Baudisch [1981a]	Eklof–Mekler [1979]
$\mathscr{L}_{\omega\omega}(I)$	Baudisch [1977b or c]	Similar to Baudisch [1977c]
	Decidability Problem is open	

That $T_Z(I)$ is decidable iff the I-theory of all finite abelian groups is decidable follows from Baudisch [1977c]. Moreover, in Baudisch [1980] the I-theory of abelian p-groups is shown to be decidable. In the same vein, Schmitt [1982] has shown the decidability of the $\mathscr{L}(Q_\alpha)$-theory of ordered abelian groups for $\alpha = 0$ and $\alpha = 1$. Furthermore, in the language, considered, he allows quantification (with $\exists$ and Q_α) over convex subgroups generalizing Gurevich [1977a]. By adding suitable definable predicates, an elimination procedure for first-order quantifiers is given. In order to decide the remaining sentences, the order structure of the convex subgroups is considered in appropriate elementary languages.

In the logics that we have mentioned above, elimination procedures are also applied to other classes of structures. Thus, for example, Cowles has results for certain fields (see Cowles [1977, 1979a, b]) and Wolter for Pressburger arithmetic and for well-orderings (see Wolter [1975a, b]).

In the case of the Henkin-quantifier (the reader is referred to Section VI.2.13 of this volume), Krynicki–Lachlan [1979] used this method to prove the decidability of the corresponding theory of finitely many unary predictes with equality. For more on boolean algebras, the reader should also see the results of Molzahn [1981b] that are cited at the end of the third section of this chapter. Finally, some material on the elimination of quantifiers in stationary logic and its applications are given in the next subsection.

1.4. Elimination of Quantifiers for Stationary Logic

The reader should consult Chapter IV for the basic notions concerning $L(\mathrm{aa})$. Throughout the present subsection, L will be taken as a countable elementary language and T as an $L(\mathrm{aa})$-theory. Since generalization over second-order variables is not allowed in $L(\mathrm{aa})$, the appropriate notion of eliminability of quantifiers is the one that is defined below (see Eklof and Mekler [1979] where it is called *strong elimination of quantifiers*).

Definition. T is said to *admit elimination of quantifiers* if, for every formula $\varphi(\bar{s}, \bar{x})$, there is a quantifier-free formula $\psi(\bar{s}, \bar{x})$ such that $T \vdash \mathrm{aa}\, \bar{s}\, \forall \bar{x}(\varphi(\bar{s}, \bar{x}) \leftrightarrow \psi(\bar{s}, \bar{x}))$.

By generalizing ideas of Eklof–Mekler [1979], Mekler [1984] found the following criterion for eliminability of quantifiers in $L(\mathrm{aa})$-theories. This criterion is an analogue of that for the elementary case and the notation "$\equiv^0$" is used to denote equivalence with respect to quantifier-free formulas.

Theorem. *T admits elimination of quantifiers iff whenever $\mathfrak{A}, \mathfrak{B} \models T$ and $|\mathfrak{A}|$, $|\mathfrak{B}| \leq \aleph_1$, there are cubs C and D for $\mathfrak{A}$ and $\mathfrak{B}$ such that for all $\bar{A} \in C$, and $\bar{B} \in D$ and $\bar{a} \in \mathfrak{A}$ and $\bar{b} \in \mathfrak{B}$, if $\langle \mathfrak{A}, \bar{A}, \bar{a} \rangle \equiv^0 \langle \mathfrak{B}, \bar{B}, \bar{b} \rangle$ holds, then $\langle \mathfrak{A}, \bar{A}, \bar{a} \rangle \equiv_{\mathrm{aa}} \langle \mathfrak{B}, \bar{B}, \bar{b} \rangle$.*

Proof. We will present the proof for the nontrivial direction. Assume, then, that $\varphi(\bar{s}, \bar{x})$ is not equivalent to a quantifier-free formula. Let $\{\psi_n(\bar{s}, \bar{x}) : n < \omega\}$ be an enumeration of all the corresponding quantifier-free formulas. For $t \in {}^k2$, define $\psi^t = \bigwedge_{i<k} \psi_i(\bar{s}, \bar{x})^{t(i)}$, where $\psi_i^0 = \psi_i$ and $\psi_i^1 = \neg \psi_i$.

Let $\mathfrak{T}$ be the tree of all $t \in {}^{<\omega}2$ such that for all $t' < t$

$$(*)\qquad \begin{array}{ll} \text{neither} & \text{(i) } T \vdash \text{aa } \bar{s}\, \forall \bar{x}(\psi^{t'}(\bar{s}, \bar{x}) \to \varphi(\bar{s}, \bar{x})) \\ \text{nor} & \text{(ii) } T \vdash \text{aa } \bar{s}\, \forall \bar{x}(\psi^{t'}(\bar{s}, \bar{x}) \to \neg\varphi(\bar{s}, \bar{x})). \end{array}$$

Then $\mathfrak{T}$ must be infinite, because otherwise there would be some $k < \omega$ such that for all $t' \in {}^k2$ either (i) or (ii). This would imply $T \vdash \text{aa } \bar{s}\, \forall \bar{x}(\varphi(\bar{s}, \bar{x}) \leftrightarrow \bigvee_{t' \in I} \psi^{t'}(\bar{s}, \bar{x}))$, where I is the set of all $t' \in {}^k2$ with property (i). By König's lemma, there is an infinite branch $\eta \in {}^{\omega}2$ of $\mathfrak{T}$. So, by $(*)$ and the construction of $\mathfrak{T}$, we have

$$(**)\qquad \begin{array}{l} T \cup \{\text{stat } \bar{s}\, \exists \bar{x}(\psi^{\eta \restriction k}(\bar{s}, \bar{x}) \wedge \neg\varphi(\bar{s}, \bar{x}))\} \quad \text{and} \\ T \cup \{\text{stat } \bar{s}\, \exists \bar{x}(\psi^{\eta \restriction k}(\bar{s}, \bar{x}) \wedge \varphi(\bar{s}, \bar{x}))\} \end{array}$$

are consistent for every $k < \omega$. Assume now that $\bar{s} = (s_1, \ldots, s_n)$ and $\bar{x} = (x_1, \ldots, x_m)$. We will introduce new predicates $U_i(s_1, \ldots, s_i)$ and functions $f_j(\bar{s})$, where $0 < i \le n$, $0 < j \le m$. We also define T' to be the extension of T by the following axioms:

$$\text{stat } s_1 U_1(s_1),\ \text{aa } s_1 \text{ stat } s_2 U_2(s_1, s_2), \ldots;$$

$$\text{aa } s_1 \text{aa } s_2 \ldots aa\ s_{n-1} \text{ stat } s_n U_n(s_1, \ldots, s_n); \text{ and}$$

$$\text{aa } s_1 \cdots s_n(U_1(s_1) \wedge U_2(s_1, s_2) \wedge \cdots \wedge U_n(s_1, \ldots, s_n)$$
$$\to \psi^{\eta \restriction k}(\bar{s}, f_1(\bar{s}), \ldots, f_m(\bar{s})) \quad \text{for every} \quad k < \omega.$$

Using compactness, we see that $(*\,*)$ implies the consistency of

$$T_0 = T' \cup \{\text{aa } s_1 \ldots s_n(U_1(s_1) \wedge \cdots \wedge U_n(s_1, \ldots, s_n)$$
$$\to \neg\varphi(\bar{s}, f_1(\bar{s}), \ldots, f_m(\bar{s}))\};$$

and

$$T_1 = T' \cup \{\text{aa } s_1 \ldots s_n(U_1(s_1) \wedge \cdots \wedge U_n(s_1, \ldots, s_n)$$
$$\to \varphi(\bar{s}, f_1(\bar{s}), \ldots, f_m(\bar{s}))\}.$$

Now let $\mathfrak{A}$ and $\mathfrak{B}$ be the reducts to L of models of T_0 and T_1, and let C and D be the cubs in $\mathfrak{A}$ and $\mathfrak{B}$, given by the criterion. By the axioms of T', there are chains $A_1 \subseteq \cdots \subseteq A_n$ of elements of C and chains $B_1 \subseteq \cdots \subseteq B_n$ of elements of D all of which fulfill $U_1(s_1) \wedge \cdots \wedge U_n(s_1, \ldots, s_n)$. Furthermore, $\bar{A}$ and $\bar{B}$ can be chosen so that

$$\langle \mathfrak{A}, \bar{A}, f_1(\bar{A}), \ldots, f_m(\bar{A}) \rangle \equiv^0 \langle \mathfrak{B}, \bar{B}, f_1(\bar{B}), \ldots, f_m(\bar{B}) \rangle;$$

and

$$\mathfrak{A} \models \neg\varphi(\bar{A}, f_1(\bar{A}), \ldots, f_m(\bar{A})) \quad \text{and} \quad \mathfrak{B} \models \varphi(\bar{B}, f_1(\bar{B}), \ldots, f_m(\bar{B})).$$

However, this contradicts the condition of the criterion. $\square$

Applying this notion to modules, Eklof–Mekler [1979] found an elimination procedure for the L(aa)-theory of all R-modules. They used L(aa)-core-sentences of the form

$$\text{aa}\, s\, \forall x(\chi(x) \rightarrow \exists y(y \in s \wedge \chi(x - y) \wedge \eta(x - y))),$$

where χ and η are p.p. formulas. It is easy to see that such a core sentence is equivalent to $(\chi/\eta)(\mathfrak{A}) \leq \aleph_0$—which is the negation of a Q_1^2-core sentence—so that L(aa)-equivalence and Q_1^2-equivalence coincide. To verify that the criterion does indeed hold on modules, Eklof and Mekler used the work of Fisher [1977] on injective elements in abelian classes, which continues the work of Eklof–Fisher [1972] on the description of saturated abelian groups to give a model-theoretic proof of the results of Szmielew [1955].

Specifying this development to abelian groups, Eklof and Mekler proved decidability of the L(aa)-theory. Similar results on abelian groups were independently obtained by Baudisch [1981a]. Along these same lines, we note that further applications of this method to fields and orderings can be found in Eklof–Mekler [1979]. There decidability is shown for the L(aa)-theories of complex, real, and p-adic numbers.

2. *Interpretations*

The method of syntactic interpretation was used by Tarski–Mostowski–Robinson [1953] to deduce decidability or undecidability of theories from other theories (see also Rabin [1965]).

The actual method has many applications to decidability problems, and we will give a short description of it here. Let K and K' be model classes in languages L and L' respectively, where L and L' are not necessarily elementary. Then, we say that an *interpretation* I assigns to every relational symbol R of L a formula ψ_R of L', and the formula $x = x$ corresponds to a formula $\varphi(x)$ of L'. The interpretation of the basic symbols of the language L is inductively extended to all formulas χ of L. The interpreted formula is then denoted by χ^I and is built according to the following rules, where, for the sake of notational simplicity, we let L be an elementary language with only one binary relation symbol R.

(i) $(x = y)^I := (x = y)$;
(ii) $(R(x, y))^I := \psi_R(x, y)$;
(iii) $(\neg\chi)^I := \neg(\chi^I)$;
(iv) $(\chi_1 \vee \chi_2)^I := \chi_1^I \vee \chi_2^I$; and
(v) $(\exists x\chi)^I := \exists x(\varphi(x) \wedge \chi^I)$.

If $\mathfrak{B} = (B, \ldots)$ is a model of K', then we obtain—with the help of I—a model $\mathfrak{B}^I$ for L. The domain of $\mathfrak{B}^I$ is simply the set of all elements of the domain of $\mathfrak{B}$ satisfying the formula $\varphi(x)$. The symbol R of L is interpreted by the relation

$$\{(a, b) \in B^2/\mathfrak{B} \models \varphi(a) \wedge \varphi(b) \wedge \psi_R(a, b)\}.$$

The next lemma is easily proven by induction on the complexity of formulas.

Lemma (Rabin [1965]). *For each formula χ of L and for each structure $\mathfrak{B}$ of K':*

$$\mathfrak{B} \models \chi^I \text{ iff } \mathfrak{B}^I \models \chi. \quad \square$$

The theory $\mathrm{Th}_L(K)$ is said to be *interpretable in* $\mathrm{Th}_{L'}(K')$ if

(i) for every structure $\mathfrak{A} \in K$ there is a $\mathfrak{B} \in K'$ so that $\mathfrak{B}^I$ and $\mathfrak{A}$ are isomorphic;
(ii) for every structure $\mathfrak{B} \in K'$, the structure $\mathfrak{B}^I$ is isomorph to a structure of K.

The main property of interpretations with respect to decidability is expressed in the following result.

2.1 Theorem (Rabin [1965]). *Let K and K' be model classes and let L and L', respectively, be suitable languages, where L is assumed to be elementary. If* $\mathrm{Th}_L(K)$ *is interpretable in* $\mathrm{Th}_{L'}(K')$, *then the decidability of* $\mathrm{Th}_{L'}(K')$ *implies the decidability of* $\mathrm{Th}_L(K)$. $\square$

The proof is a straightforward application of the preceding lemma. There are obvious generalizations of the notion of interpretability, and a result similar to Theorem 2.1 can be proven for them. Thus, for example, we may admit

(a) any finite signature;
(b) the identity can be handled as a non-logical symbol; that is to say, it is interpreted by a congruence relation;
(c) n-tuples of elements from the domain of $\mathfrak{B}$ can be used as individuals of $\mathfrak{B}^I$; and
(d) both languages can be non-elementary.

In the next subsection, we will give some examples that show how to apply interpretability in investigations on decidability. In particular, we will embed these examples in an investigation of well-orderings.

Well-orderings. The elementary theory of the class WO of all well-orderings was proven to be decidable by Mostowski–Tarski [1949]. The proof uses the method of elimination of quantifiers and was published in Doner–Mostowski–Tarski [1978].

One of the simplest methods used to prove the undecidability of a theory is that of trying to show that a theory which is known to be undecidable is interpretable in it. This holds also for extended logics.

The following example shows that the expressive power of the logic with the equicardinality quantifier I is great enough to make the theory of well-orderings undecidable.

Let $K = \{\mathfrak{N}\}$, where $\mathfrak{N}$ is the structure of natural numbers with addition and multiplication, and let $K' = \{\mathfrak{M}\}$, where $\mathfrak{M} = (M, <)$ is a linear ordering of order type ω^2. Furthermore, L and L', respectively, will denote the corresponding elementary languages. $L'(I)$ arises from L' by adding the equicardinality quantifier

I. We shall show that $\mathrm{Th}_L(K)$ is interpretable in $\mathrm{Th}_{L'(I)}(K')$. In fact, the interpretation J is defined as follows:

(i) $$(x = y)^J := \exists y(y < x \wedge \neg\exists y(y < x \wedge \forall z(z \leq y \vee x \leq z))) = \varphi(x);$$

(ii) $$(x + y = z)^J := Iu(\varphi(u) \wedge u \leq x, \varphi(u) \wedge y \leq u \wedge u \leq z); \text{ and}$$

(iii) $$(x \cdot y = z)^J := Iu(\varphi(u) \wedge u \leq z, \exists v\, \exists w(\varphi(v) \wedge v < y \wedge v \leq u \wedge u < w \wedge Iw'(\varphi(w') \wedge w' < x, v \leq w' \wedge w' < w))).$$

Here φ defines the elements of $\mathfrak{N}$ as limit-elements of $\mathfrak{M}$ (see Fig. 1). The formula

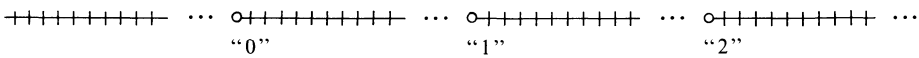

Fig. 1

on the right side of (ii) defines the addition by using the fact that between b and $a + b$ there must be just a elements. See Fig. 2, where the addition "2" + "4" = "6" is presented.

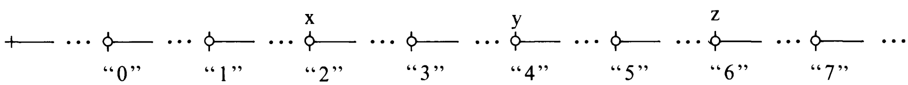

Fig. 2

To illustrate the meaning of the formula on the right side of (iii), we present the example "2" · "3" = "6" in Fig. 3. Here $x = 2$, $y = 3$, and $z = 6$.

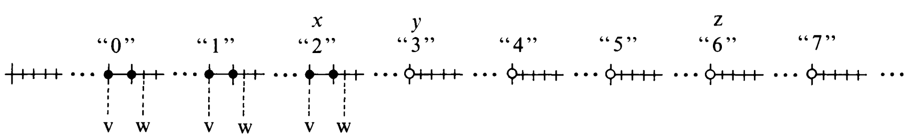

Fig. 3

The solid circles are precisely points satisfying the formula:

$$\exists v\, \exists w(\varphi(v) \wedge v < 3 \wedge Iw'(\varphi(w') \wedge w' < 2, v \leq w' \wedge w' < w) \wedge v \leq u \wedge u < w).$$

But there are just $x \cdot y = 2 \cdot 3$ points u, satisfying this formula. $\mathrm{Th}_L(K)$ is interpretable in $\mathrm{Th}_{L'(I)}(K')$ iff $(M, <)^J \cong \mathfrak{N}$. But this can be easily verified if we regard the meaning of the formulas $(x = x)^J$, $(x + y = z)^J$, and $(x \cdot y = z)^J$. Hence, we obtain the undecidability of $\mathrm{Th}_{L'(I)}$ by Theorem 2.1, since it is well-known that $\mathfrak{N}$ has an undecidable elementary theory.

From the above example, we infer the following result from Weese [1977c].

Lemma. $\mathrm{Th}_{L'(I)}(\mathrm{WO})$ *is undecidable.*

Proof. An easy way to prove the lemma is to use the fact that $\mathfrak{N}$ is strongly undecidable (see Shoenfield [1967, Theorem 2 and Theorem 3 on pages 134 and 135, respectively]). We go another way in demonstrating how to extend the notion of interpretability to languages containing the quantifier I.

We first show that $\mathrm{Th}_{L'}(\mathfrak{M})$ is interpretable in $\mathrm{Th}_{L'}(\mathrm{WO})$, where, as above, $\mathfrak{M}$ is a well-ordering of order-type ω^2. Let $\varphi_0(x)$ be a formula in L' expressing the notion

"x is the least limit-point which is a limit of limit-points".

Assume that $\mathfrak{M}'$ is a well-ordering of order-type greater than ω^2 and let a be an element of $\mathfrak{M}'$ with $\mathfrak{M}' \models \varphi_0(a)$. Then obviously

$$\mathfrak{M}' \restriction \{b/b \in |\mathfrak{M}'| \text{ and } b < a\}$$

has order-type ω^2. Hence, we get the desired interpretation, an interpretation defining $\varphi(x)$ to be $\exists y(\varphi_0(y) \wedge x < y)$ and defining $\psi_<(x, y)$ to be $x < y$.

Now we can extend this interpretation to an interpretation of $\mathrm{Th}_{L'(I)}(\mathfrak{M})$ in $\mathrm{Th}_{L'(I)}(\mathrm{WO})$. To this, we add rule (vi) as given below to rules (i) through (v) in the definition of χ^I:

(vi) $(Ix(\chi_1, \chi_2))^I := Ix(\varphi(x) \wedge \chi_1^I, \varphi(x) \wedge \chi_2^I)$.

It is easy to prove Theorem 2.1 as well as the lemma preceding it for this notion of interpretation. Hence, we obtain the undecidability of $\mathrm{Th}_{L'(I)}(\mathrm{WO})$ by the above example. $\square$

The strongest result for the decidability of classes of well-orderings in extended logics are the results for monadic second-order theories (see Chapter XIII, Section 4.2 of this volume). They imply many other results using the method of interpretability. The following result was proved first by Slomson (see Slomson [1976]) using the method of dense systems and a game-theoretical examination of the structure of well-orderings.

2.2 Theorem. $\mathrm{Th}_{Q_0^{<\omega}}(\mathrm{WO})$ *and* $\mathrm{Th}_{Q_1^{<\omega}}(\mathrm{WO})$ *are decidable.*

Proof. Shelah [1975e] proved the decidability of the monadic theory of the class of all well-orderings $<\omega_2$, which is briefly denoted by $\mathrm{Th}_{II}(\{(\alpha, <)/\alpha < \omega_2\})$. (The reader should also see Chapter XIV, Section 4.2 or this volume for more on this). We will use this result to prove Theorem 2.2 by interpretability. Obviously the following relations are expressible by formulas in the monadic language for ordinals:

"$Y \subseteq X$" is expressible by $\forall y(y \in Y \to y \in X)$;

"$X \neq \emptyset$" is expressible by $\exists x\, x \in X$;

"x has a successor in Y" is expressible by $\exists y(y \in Y \wedge x < y)$;

"x is not the first element of Y" is expressible by $\exists y(y < x \wedge y \in Y)$;

"x is a limit in Z" is expressible by

$$\forall z(z \in Z \wedge z < x \to \exists y(y \in Z \wedge z < y < x));$$

and

"Z is confinal in Y" is expressible by $\forall y(y \in Y \to \exists z(y \leq z \wedge z \in Z))$.

Then define $\chi_0(X)$ and $\chi_1(X)$ as follows:

$$\chi_0(X) := \exists Y(\text{“}Y \subseteq X\text{”} \wedge \text{“}Y \neq \emptyset\text{”} \wedge \forall x(x \in Y \to \text{“}x \text{ has a successor in } Y\text{”}));$$

$$\chi_1(X) := \exists Y(\text{“}Y \subseteq X\text{”} \wedge \chi_0(Y) \wedge \forall Z(\text{“}Z \subseteq Y\text{”} \wedge \text{“}Z \text{ is cofinal in } Y\text{”} \to \exists x(x \in Z \wedge \text{“}x \text{ is limit in } Z\text{”}))).$$

For each well-ordering $\mathfrak{M}$ and each subset B of the domain of $\mathfrak{M}$, the following holds:

(∗) $\quad \mathfrak{M} \models \chi_i(B)$ iff $|B| \geq \aleph_i$ for each $i = 0, 1$.

The downward Löwenheim–Skolem Theorem for $\mathscr{L}_{\omega\omega}(Q_i^{<\omega})$ gives

$$\mathrm{Th}_{Q_i^{<\omega}}(\mathrm{WO}) = \mathrm{Th}_{Q_i^{<\omega}}(\{(\alpha, <)/\alpha < \omega_2\}) \quad \text{for} \quad i = 0, 1.$$

We will show that $\mathrm{Th}_{Q_i^{<\omega}}(\{(\alpha, <)/\alpha < \omega_2\})$ is interpretable in

$$\mathrm{Th}_{II}(\{(\alpha, <)/\alpha < \omega_2\}).$$

To this end, let $\varphi(x)$ be the formula $x = x$ and define $\psi_<(x, y)$ to be $x < y$. We extend the rules (i) through (v) by one of the following sets of rules

(vi)$_0$ $(Q_0^n \mathbf{x}\chi)^I := \exists X(\chi_0(X) \wedge \forall \mathbf{x}\; \mathbf{x} \in X \to \chi)$ for each $0 < n \in \omega$;

(vi)$_1$ $(Q_1^n \mathbf{x}\chi)^I := \exists X(\chi_1(X) \wedge \forall \mathbf{x}\; \mathbf{x} \in X \to \chi)$ for each $0 < n < \omega$.

Condition (*) guarantees that $(Q_0^n x\chi)^I$ and $(Q_1^n x\chi)^I$ get the correct interpretation by $(\mathrm{vi})_0$ and $(\mathrm{vi})_1$. We will leave it to the reader to verify that Theorem 2.1 also holds for this notion of interpretability, which proves Theorem 2.2. ▯

At first sight stationary logic is a strengthening of $\mathscr{L}_{\omega\omega}(Q_1)$ which stands closer to monadic second-order logic than does $\mathscr{L}_{\omega\omega}(Q_1^{<\omega})$. Although the theory of well-orderings in stationary logic is decidable, there are models of set theory in which monadic second-order theory is undecidable (see, for example, Chapter XIII of this volume).

The former was proven by Mekler [1984] who used elimination of quantifiers and by Seese [1981b] who employed dense systems. Hence, it would be interesting to know whether or not this result might be inferred by interpretability from the decidability of $\mathrm{Th}_{II}(\{(\alpha, <)/\alpha < \omega_2\})$. For each natural number n, this is indeed the case for $\mathrm{Th}_{\mathrm{aa}}(\{(\alpha, <)/\alpha < \omega_1 \cdot n)$.

Exercise. Show that $\mathrm{Th}_{\mathrm{aa}}((\omega_1 \cdot n, <))$ is interpretable in

$$\mathrm{Th}_{II}(\{(\alpha, <)/\alpha < \omega_2) \quad \text{for each} \quad n \in \omega.$$

Hint. Extend the interpretability result Theorem 2.1 to $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ and then use the definability of $\omega_1 \cdot n$ in the monadic second-order logic and the fact that the initial-intervals of ω_1 build a canonical closed and unbounded set-system for $(\omega_1, <)$.

Aside from what has already been pointed out, this interpretability result gives the decidability of the theory of $(\omega_1 \cdot n, <)$, for all $n \in \omega$, in the language $\mathscr{L}_{II}(\mathrm{aa})$. It is not possible to extend this interpretability to $(\omega_1 \cdot \omega, <)$, as the following example will show. Moreover, the theorem given below shows that even an extension of $\mathrm{Th}_{\mathrm{aa}}((\omega_1 \cdot \omega, <))$ by unary predicates yields an undecidable theory.

2.3 Theorem. *Let* WOP *denote the following class of structures*

$$\{(\alpha, <, P)/(\alpha, <) \in \mathrm{WO} \text{ and } P \subseteq \alpha\}.$$

Then $\mathrm{Th}_{\mathrm{aa}}(\mathrm{WOP})$ *is undecidable.*

Proof. The proof falls into three steps. First, we prove that the elementary theory of countable, symmetric, and reflexive graphs, a theory that is known to be undecidable (see, for example, Rabin [1965]), is interpretable in

$$\mathrm{Th}_{\mathrm{aa}}\left(\left\{\dot{\bigcup_{i<\gamma}} (\omega_1, <, P_i) : \gamma \leq \omega, P_i \subseteq \omega_1 \ (i < \gamma)\right\}\right).$$

Moreover, we should here remark that $\dot{\bigcup}$ is the disjoint union and not the sum of orders. The basic idea used here is the following.

Let $\mathfrak{G} = (G, R)$ be a countable, symmetric, and reflexive graph. We assume that $G = \gamma$, for some $\gamma \leq \omega$, and show that $\mathfrak{G}$ can be defined in a uniform way in a structure of the above class. By a theorem of Ulam [1930], there is a set

$$\{S_{\alpha\beta}: \alpha \leq \beta < \omega_1\}$$

of pairwise disjoint stationary subsets of ω_1. For $i < \gamma$, let

$$A_i := \bigcup \{S_{ij}: (i, j) \in R \text{ and } i < j\}.$$

Then each A_i is stationary on ω_1 and

(#) $\qquad A_i \cap A_j$ is stationary iff $(i, j) \in R$.

This is used to define $\mathfrak{G}$ in $\dot{\bigcup}_{i<\gamma} (\omega_1, <, A_i)$ by the following formulas:

$$\varphi_0(x) := \neg \exists y(y < x),$$

and

$$\begin{aligned}\varphi_1(x, y) := \varphi_0(x) \wedge \varphi_0(y) \wedge (\text{stat } s)\, \exists z\, \exists u(\text{“}\sup(s \cap \{v: v < z\}) = z\text{”}\\ \wedge \text{“}\sup(s \cap \{v: v < u\}) = u\text{”}\\ \wedge x < z \wedge y < u \wedge P(z) \wedge P(u)).\end{aligned}$$

Here “$\sup(s \cap \{v: v < z\}) = z$” and “$\sup(s \cap \{v: v < u\}) = u$” are abbreviations of the corresponding formulas. $\varphi_1(x, y)$ expresses just the left side of (#), while $\varphi_0(x)$ defines the domain of $\mathfrak{G}$. Hence, by using as $\varphi(x)$, $\varphi_R(x, y)$ the formulas $\varphi_0(x)$, $\varphi_1(x, y)$, respectively, we get the desired interpretation. Moreover, we must have to add the rules (vi) and (vii) to the rules (i) through (v):

(vi) $(s(x))^I := s(x) \wedge \varphi(x)$; and

(vii) $(\text{aa } s\chi)^I := \text{aa } s\chi$.

Theorem 2.1 also holds for this notion of interpretability. Thus,

$$\mathrm{Th}_{\mathrm{aa}}\left(\left\{\dot{\bigcup_{i<\gamma}} (\omega_1, <, P_i): \gamma \leq \omega, P_i \subseteq \omega_1\ (i < \gamma)\right\}\right)$$

is undecidable.

The second step in our argument is to interpret this latter theory in

$$\mathrm{Th}_{\mathrm{aa}}(\{(\omega_1 \cdot \omega, <, P): P \subseteq \omega_1 \cdot \omega\}).$$

This can be easily done using the notion of interpretability given above for stationary logic and we leave the details of it to the reader. Finally, we notice that this theory is interpretable in $\mathrm{Th}_{\mathrm{aa}}(\mathrm{WOP})$. We give only a hint for this third interpretation: For each well-ordering $(\alpha, <)$ with $\alpha > \omega_1 \cdot \omega$ the point $\omega_1 \cdot \omega$ is uniformly definable that is, it is definable independently of α in $(\alpha, <)$ by a formula from $\mathscr{L}_{\omega\omega}(Q_1)$. This completes the proof. □

We will conclude this subsection with a few additional facts and some historical notes.

Further Results. In addition to Theorem 2.2, Slomson [1976] proved that $\mathrm{Th}_{Q_\alpha^{<\omega}}(\mathrm{WO})$ is decidable for all ordinals α. He used the method of dense systems and a game-theoretical examination of the structure of well-orderings. Moreover, he proved that for all α, $\beta > 0$ the theory $\mathrm{Th}_{Q_\alpha^{<\omega}}(\mathrm{WO})$ equals the theory $\mathrm{Th}_{Q_\beta^{<\omega}}(\mathrm{WO})$, while $\mathrm{Th}_{Q_0^{<\omega}}(\mathrm{WO})$ differs from these theories. These results had already been proven by Vinner [1972] for $\mathscr{L}_{\omega\omega}(Q_\alpha)$ rather than for $\mathscr{L}_{\omega\omega}(Q_\alpha^{<\omega})$. The reader should also consult Lipner [1970] and Slomson [1972] for more on this. A further generalization of Theorem 2.2 was proven in Tuschik [1982b]. In particular, let Δ be a set of ordinals, and let $L(\Delta)$, $L(\Delta)^{<\omega}$, respectively, be the language L with the additional quantifiers Q_α (for $\alpha \in \Delta$) and Q_α^n, for $\alpha \in \Delta$ and $n \geq 1$, respectively. Assuming GCH, Tuschik [1982b] proceed to prove that $\mathrm{Th}_{L(\Delta)}(\mathrm{WO})$ is decidable for each finite set Δ of non-limit ordinals. In this connection we note that Wolter [1975b] proved this for $\Delta = \{0, \alpha\}$. Moreover, assuming GCH, Tuschik [1982b] proved that, for any finite Δ, $L(\Delta)^{<\omega}$ is reducible to $L(\Delta)$ with respect to the class WO and that $\mathrm{Th}_{L(\Delta)^{<\omega}}(\mathrm{WO})$ is decidable. By performing ordered sums of finitely determinate linear orderings, the proof of the decidability of $\mathrm{Th}_{\mathrm{aa}}(\mathrm{WO})$ given in Seese [1981b] used the method of dense systems and an investigation of the preservation of $\equiv_n (L(\mathrm{aa}))$. Interestingly enough, the proof yields that all well-orderings are finitely determinate, a fact which was also proven by Mekler [1984]. Moreover, Mekler [1984] proved that a simple extension of $\mathrm{Th}_{\mathrm{aa}}(\mathrm{WO})$ by unary predicates and defining axioms for it admits elimination of second-order quantifiers.

Some further results on this can be found in Caicedo [1978], Kaufmann [1978a, b], and Mekler [1984], as well as in Seese–Weese [1982]. The reader should also see Chapter XIII of this volume for material on these notions. Finally, we note that the results cited at the end of the next section also provide some material on boolean algebras.

3. *Dense Systems*

The method of dense systems was used by Ershov [1964b] and by Läuchli–Leonard [1966] to obtain the decidability of the theories of boolean algebras and linear orderings, respectively. The method used in these studies can be formulated in a general form, a form which is applicable both to axiomatizable and non-axiomatizable logics. In order to develop this form, we first let K be a class of models and L any logic, and then make the following

Definition. A countable subset $M \subseteq K$ of models is:

(i) *dense for* K *(with respect to* L*)* if any sentence of L which is satisfiable in K already has a model in M; and
(ii) *is uniformly recursive with respect to* L if the relation "$A \models \varphi$" is recursive, where A varies over models of M and φ over sentences of L (we assume a fixed Gödel-numbering).

3.1 Theorem. *Suppose K and L are as above and M is dense for K and uniformly recursive with respect to L. Then the theory* $\mathrm{Th}_L(K)$ *is decidable if either:*

(i) *L and K are (recursively) axiomatizable; or*
(ii) *there is a recursive function f so that for each sentence φ of L having a model in K, there is a model $A \in M$, $A \models \varphi$ with $\ulcorner A \urcorner < f(\ulcorner \varphi \urcorner)$. Here, the notation $\ulcorner A \urcorner$ and $\ulcorner \varphi \urcorner$ denote the corresponding Gödel-numbers.*

Generally, to obtain a Gödel-numbering, the set M is generated from simple structures by some operations such as sums, products *etc.* A logic L which preserves L-elementary equivalence for these operations is especially convenient to obtaining decidability. If it preserves L-elementary equivalence for the direct product it has the *product property*. For instance, the elementary logic has the product property as well as the logic with the additional quantifier Q_α. But, on the other hand, the logic with Malitz quantifiers Q_α^n $(n > 1)$ and stationary logic do not possess this property. However, stationary logic does have the product property if only finitely determinate structures are considered.

In the following two subsections, the sets M are constructed for the classes of linear orderings and boolean algebras, respectively. This will clarify the abstract notions that are given above. Furthermore, we obtain some insight into the expressive power of cardinality quantifiers for these two classes.

3.1. Linear Orderings

Let us consider the class of linear orderings LO in the logic $L(Q_1)$ which has the cardinality quantifier Q_1. The decidability of the elementary theory of LO was established in Ehrenfeucht [1959b].

3.1.1 Theorem. *The theory* Th_{Q_1} (LO) *is decidable.*

This result was shown by Tuschik [1977b] and by Herre–Wolter [1977]. The proof closely follows the line given by Läuchli–Leonard [1966] for the elementary case, but with some important exception: the Ramsey theorem cannot be used, since it is no longer valid in the uncountable case. However, Shelah's theorem for additive colourings [1975e] is a useful substitute.

As to Theorem 3.1, it follows that it is enough to have a dense set M which is uniformly recursive. The models in M are called *term-models*. In order to define M, we need some special dense linear orderings $\sigma^{m,n}$ which we will briefly describe as follows: The $\sigma^{m,n}$ are uncountable dense linear orderings with finitely many predicates $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ which form a partition of the underlying set of $\sigma^{m,n}$, so that $X_1, \ldots, X_m$ are countable dense subsets and $Y_1, \ldots, Y_n$ are ω_1-dense subsets, where Y_i is said to be ω_1-dense if between any two elements there are uncountably many elements of Y_i. Suppose that $F = (A_1, \ldots, A_m)$ and $G = (B_1, \ldots, B_n)$ are two finite sequences of linear orderings, then $\sigma(F, G)$ results from $\sigma^{m,n}$ by replacing each point from X_i or Y_j by a copy of A_i or B_j, respectively. $\sigma(F, G)$ is thus called the *shuffle* of (F, G).

Now, the set M is the smallest set containing **1** (the unique one-element order), so that we have the following:

(i) If $A, B \in M$ then $A + B \in M$;
(ii) If $A \in M$, *so are* $A \cdot \omega$, $A \cdot \omega^*$, $A \cdot \omega_1$, and $A \cdot \omega_1^*$; and
(iii) If F and G are finite sequences of models from M, then $\sigma(F, G)$ belongs to M also.

The operations above are defined as usual for linear orderings. To show that M is dense, it is convenient to use n,1-isomorphisms, these having been introduced in Chapter II, Section 4.2. In the original papers, the game-theoretic equivalent of $\simeq_{n,1}$ was used (see Lipner [1970], Brown [1972]). To mark this difference we will denote $\simeq_{n,1}$ by $\overset{n}{\sim}$ in the following. The proof of the following lemma is omitted.

Lemma. *The operations which generate M preserve* $\overset{n}{\sim}$. $\square$

A linear ordering A is called *n-term-like* iff there is a term-model B such that $A \overset{n}{\sim} B$. The crucial point consists in proving the following fact:

Lemma. *Suppose every bounded convex subset of A is n-term-like, then A itself is n-term-like.*

Proof. We may suppose that A has a least element (otherwise, we can partition $A = B + C + D$, where B and D have a greatest or least element, respectively, and C is bounded and convex). By the Löwenheim–Skolem theorem for $L(Q_1)$, we can assume A has cardinality $\aleph_1$.

However, A then possesses an increasing cofinal κ-sequence, where κ is 1, ω, or ω_1. In the first case, the stated property follows immediately, since then A is bounded. To establish the other two cases, we remark that the equivalence relation $\overset{n}{\sim}$ has only finitely many equivalence classes. Hence, $\overset{n}{\sim}$ induces a colouring by assigning to the pair $\langle a, b\rangle$ the equivalence class of the interval $(a, b]$ (as an ordered set) with respect to $\overset{n}{\sim}$. This colouring is additive since $+$ preserves $\overset{n}{\sim}$ as was stated in the lemma above. Now, we can apply Shelah's theorem on additive colourings to choose homogeneous subsets. Hence, there is a subset $X \subseteq A$ of order-type ω_1 (or ω in the second case, respectively) so that for any element $a < b$ and $c < d$ of X we have

$$(a, b] \overset{n}{\sim} (c, d].$$

We refer to the original papers concerning the relation

$$A \overset{n}{\sim} A_0 + A_1 \cdot (\omega^* + \omega) \cdot \omega_1,$$

where A_0 and A_1 are bounded segments of A (namely, $A_0 = A^{\leq x_0}$ and $A_1 = (x_0, x_1]$ and x_0 and x_1 are the first two elements of X).

By the hypothesis, A_0 and A_1 are n-term-like, say $A_0 \overset{n}{\sim} B_0$ and $A_1 \overset{n}{\sim} B_1$ with $B_0, B_1 \in M$. Thus, we also have

$$A \overset{n}{\sim} B_0 + B_1 \cdot \omega + B_1 \cdot (\omega^* + \omega) \cdot \omega_1.$$

However, the right side itself is a term-model. Thus, A is n-term-like, and the lemma is proven. □

From the following lemma we can easily conclude that M is dense in LO.

Lemma. *Every linear ordering A is n-term-like.*

Proof. By the Löwenheim–Skolem theorem for $L(Q_1)$, we may again suppose that A is of cardinality $\leq \aleph_1$. We define an equivalence relation $\approx$ on A as follows:

$x \approx y$ iff every segment of the closed interval $[x, y]$ is n-term-like. Clearly, $\approx$ is convex. Furthermore, by the preceding lemma, every equivalence class itself is n-term-like.

Claim. *There is only one equivalence class.*

Assume there are two different equivalence classes $C < D$ in $A/\approx$. If D is a successor of C, then we can prove that the elements of C and D are equivalent, since M is closed under addition. However, this would contradict the assumption about $C < D$. But otherwise $A/\approx$ has to be dense. The elements of $A/\approx$ are themselves linear orderings, and, as we have already proven, they are n-term-like. Thus, there are term-models $A_1, \ldots, A_k$ so that every $C \in A/\approx$ is equivalent to some A_i, $1 \leq i \leq k$. For $C < D \in A/\approx$, we let $F(C, D) \subseteq \{A_1, \ldots, A_k\}$ be the subset of those term-models A_i which are $\overset{n}{\sim}$-equivalent to some E between C and D. Similarly, let $G(C, D) \subseteq \{A_1, \ldots, A_k\}$ be the subset of all term-models A_i so that there are uncountably many E between C and D, with $E \overset{n}{\sim} A_i$. Now, choose $C < D$ in $A/\approx$ with $F(C, D)$ and $G(C, D)$ minimal. Clearly, this implies that, for $C < E < F < D$, $F(C, D) = F(E, F)$ and $G(C, D) = G(E, F)$. Then, it is not difficult to prove that

$$\bigcup_{N \in (E,F)} N \overset{n}{\sim} \sigma(F(C, D), G(C, D)),$$

for any E and F, with $C < E < F < D$. Before continuing our argument, we should remark that (E, F) is the open interval in $A/\approx$ with endpoints E and F. Returning to our line of argument we note that by definition, $\sigma(F(C, D), G(C, D))$ is again a term-model. Thus, we may conclude that the elements of C and D are $\approx$-equivalent.

But this would be a contradiction to $C < D$. Thus, the claim holds and the lemma is proven. □

Corollary. *If some sentence φ of $L(Q_1)$ has an ordered set as a model, then it also has a term-model as a model.*

Proof. Let A be a model of φ. By the Löwenheim–Skolem theorem, we may assume that A has cardinality $\leq \aleph_1$. Suppose the quantifier rank of φ is n. Then, by the preceding lemma, there is a term-model B, so that $A \overset{n}{\sim} B$. However, using claim $(*)$ from the proof of Corollary 4.2.4 in Chapter II, this implies $B \models \varphi$. □

From the definition of the set M, we know that its members have a very determined structure. This idea is used to prove that M is uniformly recursive with respect to $L(Q_1)$.

Lemma. *M is uniformly recursive with respect to $L(Q_1)$.* □

The proof of the above lemma can be accomplished by induction on the complexity of the term-models and the sentences.

Now, using Theorem 3.1 we obtain the decidability of $\mathrm{Th}_{Q_1}(\mathrm{LO})$, and hence Theorem 3.1.1 is proven.

We have illustrated the main idea in order to prove the decidability of the theory of linear orderings in a language with the quantifier Q_1. Now, let us mention some further results about the class of linear orderings for logics with other generalized quantifiers.

(1) First of all, we refer the reader to Chapter XIII where second-order quantifiers are considered.

(2) (GCH) $\mathrm{Th}_{Q_\alpha}(\mathrm{LO})$ is decidable for every ordinal α. The case $\alpha = 0$ follows from Läucnli [1968]. Since, for regular $\aleph_\alpha$, this theory is the same as $\mathrm{Th}_{Q_1}(\mathrm{LO})$, it is clear that its decidability follows from Theorem 3.1.1, Herre–Wolter [1979b] provides a proof of it for singular $\aleph_\alpha$.

(3) Let Δ be a finite set of ordinals such that for all $\alpha \in \Delta$, $\aleph_\alpha$ is regular. L_Δ denotes the language of linear orderings with the additional generalized quantifiers Q_α, $\alpha \in \Delta$. Under some conditions that are weaker than GCH, Tuschik [1980] proved the decidability of $\mathrm{Th}_{L_\Delta}(\mathrm{LO})$.

(4) Let $L_\Delta^{<\omega}$ be the language L with the additional Malitz quantifiers Q_α^m, for all $\alpha \in \Delta$. If we only add the binary Malitz quantifier, the extended language will then be denoted by L_Δ^2. Suppose that, for all $\alpha \in \Delta$, $\aleph_\alpha$ is regular, then Tuschik [1982b] has shown that $L_\Delta^{<\omega}$ is reducible to L_Δ^2 for the class of linear orderings. Furthermore, $\mathrm{Th}_{L_\Delta^{<\omega}}(\mathrm{LO})$ is decidable. For the limit cardinal number $\aleph_\omega$, it is also shown that $L_{\{\omega\}}^{<\omega}$ is reducible to $L_{\{\omega\}}^2$ for linear orders.

(5) In contrast to the results mentioned above, the theories $\mathrm{Th}_I(\mathrm{LO})$ and $\mathrm{Th}_{\mathrm{aa}}(\mathrm{LO})$ are undecidable. The undecidability of $\mathrm{Th}_I(\mathrm{LO})$ follows immediately from that of $\mathrm{Th}_I(\mathrm{WO})$ (see Section 2), while that of $\mathrm{Th}_{\mathrm{aa}}(\mathrm{LO})$ is proven in Seese–Tuschik–Weese [1982].

3.2. Boolean Algebras

The decidability of the elementary theory of boolean algebras Th(BA) was proved by Tarski [1949]. Some years later, Ershov [1964b] showed that the theory of boolean algebras with a distinguished prime ideal is also decidable. Here we will consider the class of boolean algebras in the logic with the additional cardinality quantifier Q_α, for arbitrary ordinals α.

First, we will compare the various cardinality quantifiers with each other. Therefore, throughout this subsection we will work in a fixed model of set theory, where δ is that ordinal which satisfies $\aleph_\delta = \beth_\omega$. Weese [1976b] showed the following

Theorem. *For every ordinal* $\alpha > 0$, *we have*

(i) $\mathrm{Th}_{Q_\alpha}(\mathrm{BA}) = \mathrm{Th}_{Q_1}(\mathrm{BA})$ *iff there is some* $\beta < \alpha$, *with* $2^{\aleph_\beta} \geq \aleph_\alpha$;
(ii) $\mathrm{Th}_{Q_\alpha}(\mathrm{BA}) = \mathrm{Th}_{Q_\delta}(\mathrm{BA})$ *iff* $2^{\aleph_\beta} < \aleph_\alpha$, *for every* $\beta < \alpha$. ☐

Remark. In fact, $L(Q_\alpha)$ and $L(Q_\beta)$ represent one and the same language $L(Q)$. The ordinal subscript only serves to mark the different interpretation. If we make comparisons such as the above, we can consider the theories Th_{Q_α} as subsets of $L(Q)$.

From the theorem, we see that there are at most three different theories of boolean algebras in logics with cardinality quantifiers, namely $\mathrm{Th}_{Q_0}(\mathrm{BA})$, $\mathrm{Th}_{Q_1}(\mathrm{BA})$, and $\mathrm{Th}_{Q_\delta}(\mathrm{BA})$. The connection between these theories is illustrated in the next proposition.

Proposition. $\mathrm{Th}_{Q_1}(\mathrm{BA}) \subsetneqq \mathrm{Th}_{Q_\delta}(\mathrm{BA}) \subsetneqq \mathrm{Th}_{Q_0}(\mathrm{BA})$.

Proof. We will only prove that the inclusions are proper. Let At(x), at(x), and atl(x) be formulas of the elementary language of boolean algebras which express the properties "x is an atom", "x is atomic", and "x is atomless", respectively. Set

$$\varphi := \forall x(\mathrm{atl}(x) \to Qy(y \leq x)),$$

and

$$\psi := \forall x(\mathrm{at}(x) \wedge Qy(y \leq x) \to Qy(\mathrm{At}(y) \wedge y \leq x)).$$

Then it is immediately seen that

$$\varphi \in \mathrm{Th}_{Q_0}(\mathrm{BA}) \setminus \mathrm{Th}_{Q_\delta}(\mathrm{BA}) \quad \text{and} \quad \psi \in \mathrm{Th}_{Q_\delta}(\mathrm{BA}) \setminus \mathrm{Th}_{Q_1}(\mathrm{BA}). \quad ☐$$

Now, to prove the decidability of these theories, we want to establish dense sets M_0, M_1, and M_σ. For the sake of simplicity, we will restrict ourselves to the construction of M_0 in the following discussion. The constructions of M_1 and M_δ would require some further operations, so that we will omit them entirely and refer the reader to the literature. Before we can define the set M_0, we must introduce two operations for boolean algebras. Let η be the set of rational numbers. Then $\bigoplus_\eta B$ and $\prod^\eta B$ are subalgebras of the Cartesian product $\prod_{i\in\eta} B_i$, where $B_i = B$ for all $i \in \eta$. $\bigoplus_\eta B$ is the subalgebra generated by the elements $\{a_i: i \in \eta\}$, where $\{i \in \eta: a_i \neq 0\}$ is finite. This kind of product is also called a *direct sum.*

Let $I(\eta)$ be the boolean subalgebra of the power set of η, which is generated by the intervals. Then $\prod^\eta B$ is generated by the elements $\{a_i: i \in \eta\}$ with the properties that $\{i \in \eta: a_i \neq 0\}$ belongs to $I(\eta)$, and $\{i \in \eta: a_i \neq 0 \text{ and } a_i \neq 1\}$ is finite.

We are now ready to define M_0. Let **2** be the unique boolean algebra with only two elements and let P be any fixed countable atomless boolean algebra. Then, M_0 is the smallest set containing **2** and P such that the following hold:

(i) if A and B belong to M_0, then so does their direct product $A \times B$;
(ii) if $B \in M_0$, then $\bigoplus_\eta B$ and $\prod^\eta B$ also belong to M_0.

The algebras in M_0 are called *term-models*. To show that M_0 is dense it is convenient to use n, 0-isomorphisms, these latter having been introduced in Chapter II, Section 4.2. In the original paper the game-theoretic equivalent of $\cong_{n,0}$ was used (see Lipner [1970] and Brown [1972]). We observe that it has an especially simple form for boolean algebras. To mark this difference, we denote $\cong_{n,0}$ by $\overset{n}{\sim}$ in the following discussion. The proof of the lemma given below is omitted.

Lemma. *The operations which generate M_0 preserve $\overset{n}{\sim}$ also.* □

A boolean algebra A is *n-term-like* iff there is a term-model B so that $A \overset{n}{\sim} B$. If a is an element of the boolean algebra B, then the ideal generated by a is denoted by $(a)_B = \{b \in B: b \leq a\}$. If no confusion can arise, we omit the subscript B altogether. By interpreting the constant 1 by the element a, we see that the structure $(a)_B$ becomes a boolean algebra. For each boolean algebra B, we can thus define the subset $D_n(B)$ of n-term-like elements as

$$D_n(B) = \{a \in B: \text{for every non-zero } b \in (a), (b) \text{ is } n\text{-term-like}\}.$$

Lemma. *$D_n(B)$ is an ideal.*

Proof. Clearly, if $a \in D_n(B)$ and $b \leq a$, then $b \in D_n(B)$. Let be $a, b \in D_n(B)$. If $a \leq b$ or if $b \leq a$, then obviously $a \cup b \in D_n(B)$. Otherwise, $a \cup b = a \cup (b \backslash a)$ and $a \neq 0$ and $b \backslash a \neq 0$. Since a and $(b\backslash a)$ are n-term-like, there are term-models A_1 and A_2 such that $(a) \overset{n}{\sim} A_1$ and $(b\backslash a) \overset{n}{\sim} A_2$. However, since a and $(b\backslash a)$ are disjoint, we get that $a \cup (b\backslash a) \overset{n}{\sim} A_1 \times A_2$. By definition, $A_1 \times A_2$ is again a term-model. Hence, $a \cup b$ is n-term-like. If $c \leq a \cup b$, then we can repeat the proof for $a \cap c$ and $(b\backslash a) \cap c$. Hence, the element $a \cup b$ belongs also to $D_n(B)$. □

From the next lemma we can easily conclude that M_0 is dense in BA.

Lemma. *Every boolean algebra is n-term-like.*

Proof. By the preceding lemma, we know that $D_n(B)$ is an ideal for every boolean algebra B. We will show that $D_n(B)$ is not proper. Then $B = (1)$ is n-term-like and the lemma is proved. Assume that $1 \notin D_n(B)$. Since $\overset{n}{\sim}$ has only finitely many equivalence classes, there are $A_1, \ldots, A_k \in M_0$ such that any n-term-like boolean algebra is $\overset{n}{\sim}$-equivalent to some A_i, $1 \leq i \leq k$. For each $b \in B \setminus D_n(B)$, let

$$T_n(b) = \{i\colon \text{there is some } c \in D_n(B) \text{ with } c \leq b \text{ such that } (c) \overset{n}{\sim} A_i\}.$$

Let $a \in B \setminus D_n(B)$ be minimal. That is, for every $b \in B \setminus D_n(B) \cap (a)$ $T_n(b) \supseteq T_n(a)$. Clearly, we may assume that either $a/D_n(B)$ is an atom or atomless. We will show that in either cases a is n-term-like.

Case 1. $a/D_n(B)$ is an atom.
If $D_n(B)$ restricted to (a) is the zero-ideal, then a is an atom in B also; thus $(a) \overset{n}{\sim} \mathbf{2}$ and a is n-term-like. Otherwise, $D_n(B)$ is not the zero-ideal and we can prove that

$$(a) \overset{n}{\sim} \bigoplus_{\eta} A, \quad \text{where} \quad A = \prod_{i \in T_n(a)} A_i.$$

Since M_0 is closed under direct product, the algebra A belongs to M_0. Furthermore, M_0 is also closed under the direct sum of an algebra. Hence, $\bigoplus_\eta A$ is a term-model and (a) is n-term-like.

Case 2. $a/D_n(B)$ is atomless.
If $D_n(B)$ restricted to (a) is the zero-ideal, then a is atomless in B also. Thus, $(a) \overset{n}{\sim} P$ and a is n-term-like. Otherwise, $D_n(B)$ is not the zero-ideal, and we can prove that

$$(a) \overset{n}{\sim} \prod\nolimits^{\eta} A, \quad \text{where} \quad A = \prod_{i \in T_n(a)} A_i.$$

As in the first case, $\prod^\eta A$ is a term-model, and hence (a) is n-term-like.

If $b \leq a$, then either $b \in B \setminus D_n(B)$ or $b \in D_n(B)$. In both cases b is n-term-like (in the first case, the proof is the same as for the element a above). However, a must then be an element of $D_n(B)$, which is a contradiction. Hence $D_n(B) = B$. ∎

Corollary. *M_0 is dense for* BA *with respect to $L(Q_0)$.* ∎

The proof is similar to the corresponding proof of the corollary of Theorem 3.1.1.

An easy construction of the term-models is used to prove the following

Lemma. *M_0 is uniformly recursive with respect to $L(Q_0)$.*

Proof. The proof is by induction on the complexity of the term-models and the sentences. ∎

As a conclusion we obtain the following theorem, a result that was proved by Pinus [1976] and by Weese [1977a].

Theorem. *The theory* $\mathrm{Th}_{Q_0}(\mathrm{BA})$ *is decidable.* □

In a similar way (by using rather complicated term-models), we can prove the decidability of the theories $\mathrm{Th}_{Q_1}(\mathrm{BA})$ and $\mathrm{Th}_{Q_\delta}(\mathrm{BA})$. In connection with the first theorem of this subsection, we may conclude the following result due to Weese [1976b].

Theorem. *For every ordinal number* α, *the theory* $\mathrm{Th}_{Q_\alpha}(\mathrm{BA})$ *is decidable.* □

Now, we want to compare the expressive power of $L(Q_0)$ with those of the elementary language L and weak second-order logic L_{ws}. Let F be the boolean subalgebra of the power set of ω generated by the finite sets. Then $F \equiv F \times F\ (L)$; however, in $L(Q_0)$, they can be distinguished by the sentence φ, where

$$\varphi := \exists x\ \exists y (x \cap y = 0 \wedge Q_0 z(z \leq x) \wedge Q_0 z(z \leq y)).$$

Hence, $L(Q_0)$ is really more expressive. On the other hand, we have, for any boolean algebras A and B,

$$A \equiv B\ (L_{\mathrm{ws}}) \quad \text{iff} \quad A \equiv B\ (L(Q_0)).$$

Thus, L_{ws} and $L(Q_0)$ are of the same expressive power. However, while $\mathrm{Th}_{Q_0}(\mathrm{BA})$ is decidable, $\mathrm{Th}_{\mathrm{ws}}(\mathrm{BA})$ is not, as was proved by Paljutin [1971].

In the following discussion, we will mention further decidability results for the class of boolean algebras.

(1) First of all, we refer to the results of Rabin [1969, 1977], who proved the decidability of the theory $\mathrm{Th}_{LI}(P)$, where P is a countable atomless boolean algebra and LI is a second-order language appropriate for boolean algebras whose set variables range over ideals. Rabin interpreted this theory in $S2S$, the monadic theory of two successor functions. Using the fact that for each countable boolean algebra A there is an ideal I on P so that $A \cong P/I$, he concluded that the theory of all countable boolean algebras in the logic LI is also decidable. As a corollary, he obtained the decidability of the elementary theory of boolean algebras with a sequence of distinguished ideals, an accomplishment generalizing the result of Ershov that was mentioned at the beginning of the subsection.

(2) In this discussion, CH is assumed. Using a result of Sierpinski on the existence of special families of linear orderings, Rubin [1982] established the undecidability of $\mathrm{Th}_{Q_1^2}(\mathrm{BA})$, the theory of boolean algebras in the logic with the binary Malitz quantifier in the $\aleph_1$-interpretation.

(3) In contrast to the preceding fact, Molzan [1981b] proved the decidability of $\mathrm{Th}_{Q_0^2}(\mathrm{BA})$ by a quantifier elimination procedure.

(4) The undecidability of the theory $\mathrm{Th}_I(\mathrm{BA})$ in the logic with the Härtig quantifier I was proved by Weese [1976c] by means of interpretation.

(5) Interpretability also yields the undecidability of the theory $\mathrm{Th}_{\mathrm{aa}}(\mathrm{BA})$ of boolean algebras in the stationary logic. This fact was proven by Seese–Tuschik–Weese [1982].

Open Problems

(1) Find appropriate "first-order" conditions equivalent to the eliminability of all Ramsey quantifiers Q_0^m or to the eliminability of all Malitz quantifiers Q_1^m ($m < \omega$) in unstable (countable) complete first-order theories. For stable theories this is known (see Theorem 1.2.3 and Remark 7 at the end of Section 1.2).

(2) Investigate the relative strength of eliminability of Q_α^m for various ordinals α and fixed $m < \omega$. For stable theories, this is known in the case $m = 1$ (see Remark 1 at the end of Section 1.2). In the case $m > 1$, only some partial information is presently available (see Remark 8 at the end of Section 1.2).

(3) Investigate the relative strength of eliminability of Q_α^m for various numbers m (and fixed ordinals α). For stable theories, this is known in case $\alpha = 0$ and $\alpha = 1$ (see Theorem 1.2.3 and Remark 8 at the end of Section 1.2, respectively).

(4) Is $T_Z(I)$, the theory of abelian groups in the logic with the Härtig quantifier, decidable?

(5) Is the theory of well-founded trees in the logic with Q_1 decidable?

(6) Is it consistent with ZFC that $\mathrm{Th}_{Q_1^2}(\mathrm{BA})$ is decidable? Under CH it is not (see Remark 2 at the end of Section 3.2).