Chapter III
Characterizing Logics

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The model theory of first-order logic is well developed. It provides general results and methods which enable us to study and classify the models of systems of first-order axioms. Among these general results of wide applicability are the completeness theorem, the compactness theorem, and the Löwenheim–Skolem theorem. Thus, for example, the completeness theorem leads to decidability results; in many cases we obtain for a given system of axioms models with special properties using a compactness argument; finally the Löwenheim–Skolem theorem tells us that we can restrict to countable structures when classifying—with respect to its first-order properties—models of a system of axioms.

Much effort was spent in finding languages which strengthen the first-order language and which are

(i) sufficiently strong to allow the formulation of interesting systems of axioms and properties of structures which are not expressible in first-order logic, and

(ii) still simple enough to yield general principles and results which are useful in investigating and classifying models.

Taking into account the situation for first-order logic, it is not surprising that many logicians attempted to find logics satisfying the analogues of the completeness, the compactness, and the Löwenheim–Skolem theorems. That this search could not be successful was shown by the following two results, both of which are due to Lindström [1969]:

(1) First-order logic is a maximal logic with respect to expressive power satisfying the compactness theorem and the Löwenheim–Skolem theorem.

(2) First-order logic is a maximal logic satisfying the completeness theorem and the Löwenheim–Skolem theorem.

Let us point out some consequences of these results.

(a) They tell us that first-order logic is a natural logic, if one accepts the completeness (or the compactness) and the Löwenheim–Skolem property as natural properties. I suspect that most mathematicians do not accept the Löwenheim–Skolem property as natural. Quite the contrary, as Wang [1974, p. 154] remarked: "When we are interested in set theory or classical analysis, the Löwenheim theorem is usually taken as a sort of defect (often thought to be inevitable) of the first-order logic. Therefore, what is established (by Lindström's theorems) is not that first-order logic is the only possible logic but rather that it is the only possible logic when we in a sense deny reality to the concept of uncountability . . . ".

(b) Lindström's results show that it makes no sense to classify logics as either good or bad, depending on whether they are complete (compact) and have the Löwenheim–Skolem property or not. On the contrary, Lindström's result gave special emphasis to the proposal—already expressed by Kreisel in 1963—that there must be a balance between the syntax and the semantics of a logic and that the semantic properties we consider must be adapted to the expressive power and the special features of the given logic.

(c) Lindström's results were the starting point

(i) for investigations which were trying to find some order in the diversity of extensions of first-order logic, and
(ii) for a systematic study of the relationship between model-theoretic properties of logics.

In particular, these investigations have led to characterizations of other logics by means of suitable model-theoretic properties.

(d) Robinson [1973] specified the following task "... to develop topological model theory. What I have in mind is a theory which is related to algebraic-topological structures, such as topological groups and fields, as ordinary model theory is related to algebraic structures." There were some approaches to this problem which led to different logics for topological structures. However, when Ziegler [1976] proved that a certain logic $\mathscr{L}_t$ is a maximal logic—in the sense of Lindström's results—for topological structures, there was strong confidence in the fact that $\mathscr{L}_t$ is the logic for topological structures corresponding to first-order logic; and, in particular, that $\mathscr{L}_t$ should prove helpful for the investigation and classification of topological structures. It turned out that this is actually the case.

Section 1 of the present chapter is mainly devoted to a proof of Lindström's theorems. Section 2 contains some further characterizations of first-order logic by means of model-theoretic properties. In Section 3 we show that $\mathscr{L}_{\omega\omega}$ is a maximal logic satisfying properties which can be viewed as model-theoretic generalizations or substitutes for compactness and the Löwenheim–Skolem property. In Section 4 we prove that among the logics of the form $\mathscr{L}_{\omega\omega}(Q)$ with a unary monotone quantifier Q the logics $\mathscr{L}_{\omega\omega}(Q_\alpha)$, where Q_α is the quantifier "there are at least $\aleph_\alpha$-many" are the only ones with the relativization property. Finally in Section 5 an "abstract maximality theorem" is established. This result not only covers Lindström's result but it also tells us how to obtain maximal logics for other kinds of structures, such as topological structures, for example.

1. Lindström's Characterizations of First-Order Logic

We first present a proof of Lindström's first theorem ("compactness + Löwenheim–Skolem property characterize $\mathscr{L}_{\omega\omega}$"), which does not presuppose knowledge of any special model-theoretic results. We then try to minimize the assumptions and prove a lemma which, on the one hand, isolates the main step in the derivation of Lindström's first and second theorems ("recursive enumerability for validity + Löwenheim–Skolem property characterize $\mathscr{L}_{\omega\omega}$") and, on the other, makes visible the relationship between maximality and a separation property. Later, when we are characterizing $\mathscr{L}_{\infty\omega}$ and some other logics as maximal logics, we will see that a proof of the maximality along the same lines leads to a separation theorem. In this way we will obtain in a unified form some results which now appear to be scattered throughout the literature. In the second part of this section, we list some examples which show that it is not possible to strengthen Lindström's theorems in some more or less plausible ways. We will close this section by giving a characterization of the monadic part of first-order logic (and of some monadic extensions of first-order logic).

Throughout this chapter, given any vocabulary τ we denote by τ' a disjoint copy of τ. For f, R, c in τ let f', R', c' be the corresponding symbols in τ'. If $\mathscr{L}$ is a logic and ψ an $\mathscr{L}[\tau]$-sentence, then ψ' will be the $\mathscr{L}[\tau']$-sentence associated with ψ by the renaming property. Finally, for a τ-structure $\mathfrak{A}$ let $\mathfrak{A}'$ be the corresponding τ'-structure. If $\mathfrak{B} = \mathfrak{A}'$, we set $\mathfrak{B}^{-\prime} = \mathfrak{A}$.

For definiteness let us assume that all logics are one-sorted. In this section, if not explicitly stated otherwise, all logics are assumed to be closed under (finitary) boolean operations (that is, they are assumed to have the Boole property).

1.1. Lindström's First and Second Theorems

We begin by proving a version of Lindström's first theorem.

1.1.1 Theorem. *Let $\mathscr{L}$ be a logic, $\mathscr{L}_{\omega\omega} \leq \mathscr{L}$, with the compactness property and the Löwenheim–Skolem property for countable sets of sentences. Then $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$.*

Proof. The proof proceeds in three steps. First, we will show that each $\mathscr{L}$-sentence depends on finitely many symbols. Then, in case $\psi \in \mathscr{L}[\tau]$ is not equivalent to a first-order sentence, we will get elementarily equivalent structures $\mathfrak{A}$ and $\mathfrak{B}$ such that

$$(+) \qquad \mathfrak{A} \models \psi \quad \text{and} \quad \mathfrak{B} \models \neg\psi.$$

Finally, we will see that it is even possible to obtain isomorphic $\mathfrak{A}$ and $\mathfrak{B}$ with (+)—a contradiction.

Let us start with the first step (see Proposition II.5.1.2).

(1) Given $\psi \in \mathscr{L}[\tau]$, there is a finite $\tau_0 \subset \tau$ such that for any τ-structures $\mathfrak{A}$ and $\mathfrak{B}$

$$\mathfrak{A} \restriction \tau_0 \cong \mathfrak{B} \restriction \tau_0 \quad \text{implies} \quad (\mathfrak{A} \models \psi \text{ iff } \mathfrak{B} \models \psi).$$

To prove (1) let $\Phi \subset \mathscr{L}[\tau \cup \tau']$ be the set

$$\begin{aligned}\Phi = &\{\forall x_1 \ldots \forall x_n(Rx_1 \ldots x_n \leftrightarrow R'x_1 \ldots x_n) \mid n \geq 1, R \in \tau \text{ } n\text{-ary}\}\\ &\cup \{\forall x_1 \ldots \forall x_n f(x_1, \ldots, x_n) = f'(x_1, \ldots, x_n) \mid n \geq 1, f \in \tau \text{ } n\text{-ary}\}\\ &\cup \{c = c' \mid c \in \tau\}.\end{aligned}$$

Clearly $\Phi \models \psi \leftrightarrow \psi'$. Hence, by $\mathscr{L}$-compactness, there is a finite Φ_0 such that $\Phi_0 \models \psi \leftrightarrow \psi'$. But then any finite τ_1 such that $\Phi_0 \subset \mathscr{L}[\tau_1]$ leads to a finite τ_0 satisfying (1).

We now assume that the conclusion of the theorem fails; and, hence we suppose that some $\psi \in \mathscr{L}[\tau]$ is not equivalent to a first-order sentence. Choose a finite $\tau_0 \subset \tau$ according to (1). We now prove:

(2) There are τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ with

$$A = B, \qquad \mathfrak{A} \models \psi, \qquad \mathfrak{B} \models \neg\psi \quad \text{and} \quad \mathfrak{A} \upharpoonright \tau_0 \equiv \mathfrak{B} \upharpoonright \tau_0.$$

To establish (2), let $\varphi_1, \varphi_2, \ldots$ be a complete list of the $\mathscr{L}_{\omega\omega}[\tau_0]$-sentences. By induction, we obtain a sequence $\psi_1, \psi_2, \ldots$ such that for each n, $\psi_n = \varphi_n$ or $\psi_n = \neg\varphi_n$, and $\psi \wedge \psi_1 \wedge \cdots \wedge \psi_n$ is not equivalent to a first-order sentence. Then also $\neg\psi \wedge \psi_1 \wedge \cdots \wedge \psi_n$ is not equivalent to a first-order sentence. Hence, both $\psi \wedge \psi_1 \wedge \cdots \wedge \psi_n$ and $\neg\psi \wedge \psi_1 \wedge \cdots \wedge \psi_n$ are satisfiable. Let $\Psi = \{\psi_n \mid n \geq 1\}$. By $\mathscr{L}$-compactness and by the assumed Löwenheim–Skolem property for $\mathscr{L}$ there are countable structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \models \Psi \cup \{\psi\}$ and $\mathfrak{B} \models \Psi \cup \{\neg\psi\}$. But then $\mathfrak{A} \upharpoonright \tau_0 \equiv \mathfrak{B} \upharpoonright \tau_0$ and by (1), $\mathfrak{A} \upharpoonright \tau_0 \not\cong \mathfrak{B} \upharpoonright \tau_0$. Therefore, A and B are countable and infinite. Hence, without loss of generality, $A = B$.

In the last step we obtain the desired contradiction passing in (2) to structures having isomorphic—instead of elementarily equivalent—τ_0-reducts. For this purpose, choose a disjoint copy τ' of τ and new $(2n + 1)$-ary function symbols f_n and g_n. Set $\tau^* = \tau \cup \tau' \cup \{f_n, g_n \mid n \in \omega\}$.

For each n, fix an enumeration $\langle \chi_i(x_1, \ldots, x_n, x) \mid i \in \omega \rangle$ of all $\mathscr{L}_{\omega\omega}[\tau_0]$-formulas with free variables among $x_1, \ldots, x_n$, x. Let Γ consist of the $\mathscr{L}[\tau^*]$-sentences

$\psi, \neg\psi'$

("the τ-reduct is a model of ψ, the τ'-reduct a model of $\neg\psi'$"),

$\varphi \leftrightarrow \varphi'$ for each $\mathscr{L}_{\omega\omega}[\tau_0]$-sentence φ

("the τ_0-reduct and the τ_0'-reduct are elementarily equivalent"),

and of the following sentences which enable us to construct in a countable model, step by step, an isomorphism of the τ_0-reduct onto the τ_0'-reduct (let $\bar{x} = x_1 \ldots x_n$ and $\bar{y} = y_1 \ldots y_n$)

$$
(*)\qquad
\begin{aligned}
&\forall\bar{x}\,\forall\bar{y}\,\forall x\left(\exists y\left(\bigwedge_{i=0}^{r}(\chi_i(\bar{x},x)\leftrightarrow\chi_i'(\bar{y},y))\right)\right.\\
&\qquad\left.\to\bigwedge_{i=0}^{r}(\chi_i(\bar{x},x)\leftrightarrow\chi_i'(\bar{y},f_n(\bar{x},\bar{y},x)))\right),\\
&\forall\bar{x}\,\forall\bar{y}\,\forall y\left(\exists x\left(\bigwedge_{i=0}^{r}(\chi_i(\bar{x},x)\leftrightarrow\chi_i'(\bar{y},y))\right)\right.\\
&\qquad\left.\to\bigwedge_{i=0}^{r}(\chi_i(\bar{x},g_n(\bar{x},\bar{y},y))\leftrightarrow\chi_i'(\bar{y},y))\right),\qquad n,r\in\omega.
\end{aligned}
$$

Note that given a finite set Ψ_0 of sentences in $(*)$ we can expand an arbitrary $\tau_0\cup\tau_0'$-structure to a model of Ψ_0. Hence, by (2) each finite subset of Γ is satisfiable. Using the compactness and the Löwenheim–Skolem property of $\mathscr{L}$, we obtain a countable model $\mathfrak{D}$ of Γ. Let $\mathfrak{A}=\mathfrak{D}\upharpoonright\tau$ and $\mathfrak{B}=(\mathfrak{D}\upharpoonright\tau')^{-\prime}$ (where $'\colon\tau\to\tau'$ is the given renaming). Clearly, $A=B=D$, $\mathfrak{A}\models\psi$, $\mathfrak{B}\models\neg\psi$ and $\mathfrak{A}\upharpoonright\tau_0\equiv\mathfrak{B}\upharpoonright\tau_0$. We will show that $\mathfrak{A}\upharpoonright\tau_0\cong\mathfrak{B}\upharpoonright\tau_0$, which contradicts (1).

Let $d_1, d_2, \ldots$ be an enumeration of D. Since $\mathfrak{A}\upharpoonright\tau_0\equiv\mathfrak{B}\upharpoonright\tau_0$, then we have by $(*)$

$$
\begin{aligned}
(\mathfrak{A}\upharpoonright\tau_0,d_1)&\equiv(\mathfrak{B}\upharpoonright\tau_0,f_0(d_1)),\\
(\mathfrak{A}\upharpoonright\tau_0,d_1,g_1(d_1,f_0(d_1),d_1))&\equiv(\mathfrak{B}\upharpoonright\tau_0,f_0(d_1),d_1),\\
(\mathfrak{A}\upharpoonright\tau_0,d_1,g_1(d_1,f_0(d_1),d_1),d_2)&\equiv(\mathfrak{B}\upharpoonright\tau_0,f_0(d_1),d_1,f_1(\ldots))\ldots
\end{aligned}
$$

Continuing in this way (see the proof of Theorem II.4.3.1), one obtains sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that $A=\{a_n\mid n\in\omega\}$, $B=\{b_n\mid n\in\omega\}$ and

$$
(\mathfrak{A}\upharpoonright\tau_0,a_1,a_2,\ldots)\equiv(\mathfrak{B}\upharpoonright\tau_0,b_1,b_2,\ldots).
$$

But then $\pi\colon\mathfrak{A}\upharpoonright\tau_0\cong\mathfrak{B}\upharpoonright\tau_0$ for π defined by $\pi(a_n)=b_n$ for $n\in\omega$. $\square$

The following lemma contains the main step in Lindström's derivation of his theorems. We state it in the form of a "separation theorem". In this way, we will be able to obtain some further applications.

Recall that a logic $\mathscr{L}$ is said to have the finite occurrence property, if for arbitrary τ we have $\mathscr{L}[\tau]=\bigcup\{\mathscr{L}[\tau_0]\mid\tau_0\subset\tau,\ \tau_0\text{ finite}\}$.

1.1.2 Lemma. *Let $\mathscr{L}$ be a logic with the finite occurrence property. Assume $\mathscr{L}_{\omega\omega}\leq\mathscr{L}$ and that $\mathscr{L}$ is closed under conjunctions and disjunctions but not necessarily under negations. Let $\mathscr{L}$ have the Löwenheim–Skolem property and suppose that there are disjoint $\mathscr{L}$-classes which cannot be separated by an elementary class, i.e. for* *some* τ_0 *there are* $\mathscr{L}[\tau_0]$*-sentences* φ *and* ψ *with* $\mathrm{Mod}(\varphi) \cap \mathrm{Mod}(\psi) = \varnothing$ *such that there is no* $\chi \in \mathscr{L}_{\omega\omega}[\tau_0]$ *with*

$$\mathrm{Mod}(\varphi) \subset \mathrm{Mod}(\chi) \quad \textit{and} \quad \mathrm{Mod}(\chi) \cap \mathrm{Mod}(\psi) = \varnothing.$$

Then there is for some vocabulary $\boldsymbol{\sigma}$ *containing (at least) a unary relation symbol* U *an* $\mathscr{L}[\boldsymbol{\sigma}]$*-sentence* ϑ *such that* (i) *and* (ii) *hold*:

(i) *if* $\mathfrak{A} \models \vartheta$ *then* U^A *is finite and non-empty,*
(ii) *for each* $n \geq 1$ *there is an* $\mathfrak{A} \models \vartheta$ *with* $|U^A| = n$.

Before proving this lemma, let us state some of its consequences.

1.1.3 Theorem. *Let* $\mathscr{L}$ *be a logic with the finite occurrence property,* $\mathscr{L}_{\omega\omega} \leq \mathscr{L}$, *and assume that* $\mathscr{L}$ *is closed under conjunctions and disjunctions but not neccessarily under negations. If* $\mathscr{L}$ *has the Löwenheim–Skolem property and is countably compact, then any disjoint* $\mathscr{L}$*-classes can be separated by an elementary class.*

Proof. Otherwise, there exists an $\mathscr{L}$-sentence ϑ satisfying (i) and (ii) of Lemma 1.1.2. But then

(+) $\quad \{\vartheta\} \cup \{$"there are more than n elements x with Ux"$|n \geq 1\}$

is a finitely satisfiable set which has no model. ∎

Recall that in this section we assume that all logics have the Boole property, if not explicitly stated otherwise.

1.1.4 Lindström's First Theorem. *Let* $\mathscr{L}$, $\mathscr{L}_{\omega\omega} \leq \mathscr{L}$, *be a logic with the finite occurrence property. If* $\mathscr{L}$ *has the Löwenheim–Skolem property and is countably compact, then* $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$.

Proof. Given any $\mathscr{L}$-sentence φ the model classes of φ and $\neg\varphi$ are disjoint, hence, by the preceding theorem, there is a first-order sentence χ separating $\mathrm{Mod}(\varphi)$ and $\mathrm{Mod}(\neg\varphi)$. But then χ is equivalent to φ. ∎

1.1.5 Lindström's Second Theorem. *Assume that* $\mathscr{L}$ *is an effectively regular logic (see Chapter II for definitions). If* $\mathscr{L}$ *has the Löwenheim–Skolem property and is recursively enumerable for validity then* $\mathscr{L}_{\omega\omega}$ *effectively contains* $\mathscr{L}$.

Proof. For the sake of contradiction, suppose that some $\mathscr{L}$-sentence φ is not equivalent to a first-order sentence. Since the model classes of φ and $\neg\varphi$ cannot be separated by an elementary class there is an $\mathscr{L}$-sentence ϑ, $\vartheta \in \mathscr{L}[\boldsymbol{\sigma}]$, with properties (i) and (ii) of Lemma 1.1.2. By a theorem of Trahtenbrot [1950], for some finite $\boldsymbol{\tau}$—we can assume $\boldsymbol{\tau} \cap \boldsymbol{\sigma} = \varnothing$–the set Φ of $\mathscr{L}_{\omega\omega}[\boldsymbol{\tau}]$-sentences true in all finite models is not recursively enumerable. On the other hand, we have for $\varphi \in \mathscr{L}_{\omega\omega}[\tau]$

$$(*) \qquad \varphi \in \Phi \quad \text{iff} \quad \models \vartheta \rightarrow \varphi^U$$

where φ^U denotes the relativization of φ to U (see Definition II.1.2.2). $\mathscr{L}$ is effectively regular and recursively enumerable for validity, hence by (*), the set Φ is recursively enumerable—a contradiction. To show that $\mathscr{L}_{\omega\omega}$ *effectively* contains $\mathscr{L}$, given an $\mathscr{L}$-sentence φ, we enumerate the validities of $\mathscr{L}$ until we arrive at a formula which expresses the equivalence of φ to a first-order sentence. □

1.1.6 Remarks. (a) If $\mathscr{L}$ effectively contains $\mathscr{L}_{\omega\omega}$ then the set (+) of $\mathscr{L}$-sentences in the proof of Theorem 1.1.3 is recursive. Therefore, in this case, the assumption "$\mathscr{L}$ is countably compact" which was made in Theorems 1.1.3 and 1.1.4 can be replaced by "$\mathscr{L}$ is compact for recursive sets".

(b) In Theorems 1.1.3 and 1.1.4 we can drop the assumption "$\mathscr{L}$ has the finite occurrence property", if we assume that $\mathscr{L}$ is compact and not merely countably compact. In fact, suppose that φ and ψ are $\mathscr{L}[\tau]$ classes with disjoint model classes, then we can obtain a finite $\tau_0 \subset \tau$ such that

$$\mathfrak{A} \models \varphi \quad \text{and} \quad \mathfrak{A} \upharpoonright \tau_0 \cong \mathfrak{B} \upharpoonright \tau_0 \quad \text{imply non } \mathfrak{B} \models \psi.$$

(Here we apply $\mathscr{L}$-compactness to the unsatisfiable set $\Phi \cup \{\varphi, \psi'\}$ where Φ is the set introduced in the first step of the proof of Theorem 1.1.1). Using this finite τ_0, one obtains—as in the following proof of Lemma 1.1.2—a formula ϑ with (i) and (ii).

Proof of Lemma 1.1.2. Let $\mathscr{L}$, τ_0, φ and ψ be given as in Lemma 1.1.2. Suppose by contradiction that there is no $\mathscr{L}$-sentence ϑ with the properties (i) and (ii). Then we can show:

(1) If χ is an $\mathscr{L}$-sentence not equivalent to a first-order sentence, then χ has a model of power $\aleph_0$.

In fact, given such a χ choose a finite τ such that χ is an $\mathscr{L}[\tau]$-sentence. If χ has an infinite model, then for a new unary function symbol f the $\mathscr{L}$-sentence

$$\chi \wedge \text{"}f \text{ is one-to-one but not onto"}$$

is satisfiable and by the Löwenheim–Skolem property has a countable model, which must be of power $\aleph_0$. Now suppose χ has only finite models. Since τ is finite, for each $n \in \omega$, there are only finitely many (non-isomorphic) τ-structures of size $\leq n$, and each one can be characterized by a first-order sentence. Therefore, for each n, χ must have a model with at least n elements (otherwise it would be equivalent to a first-order sentence). But in this case, $\vartheta := \chi \wedge \exists x\, Ux$ for a new unary relation symbol U is a sentence satisfying (i) and (ii). This completes the proof of (1).

By assumption, the $\mathscr{L}[\tau_0]$-sentences φ and ψ have disjoint model classes which cannot be separated by an elementary class. We can assume that τ_0 is finite. Let $\varphi_1, \varphi_2, \ldots$ be an enumeration of the set of $\mathscr{L}_{\omega\omega}[\tau_0]$-sentences. Then:

(2) For each $n \geq 1$ there are τ_0-structures $\mathfrak{A}$ and $\mathfrak{B}$ of power $\aleph_0$ such that
$$A = B, \qquad \mathfrak{A} \models \varphi, \qquad \mathfrak{B} \models \psi \quad \text{and for } i \leq n \quad (\mathfrak{A} \models \varphi_i \text{ iff } \mathfrak{B} \models \varphi_i).$$

To establish (2), by induction choose $\psi_1, \psi_2, \ldots$ such that for each n, $\psi_n = \varphi_n$ or $\psi_n = \neg\varphi_n$ and such that the model classes of $\varphi^n := \varphi \wedge \psi_1 \wedge \ldots \wedge \psi_n$ and $\psi^n := \psi \wedge \psi_1 \wedge \ldots \wedge \psi_n$ cannot be separated by an elementary class. In particular, neither φ^n nor ψ^n is equivalent to a first-order sentence. Thus, we obtain the desired models $\mathfrak{A}$ and $\mathfrak{B}$ applying (1).

Using the notions of partial isomorphism, k-partial isomorphic, ... and the corresponding results (see Section II.4.2) we may rewrite (2) in the following form:

(2′) For each $k \in \omega$, there are τ_0-structures $\mathfrak{A}$ and $\mathfrak{B}$ of power $\aleph_0$ such that
$$A = B, \qquad \mathfrak{A} \models \varphi, \qquad \mathfrak{B} \models \psi \quad \text{and} \quad \mathfrak{A} \cong_k \mathfrak{B}.$$

In the last step, we pass in (2′) to isomorphic structures $\mathfrak{A}$ and $\mathfrak{B}$. To achieve the corresponding result in the proof of Theorem 1.1.1, we applied the Löwenheim–Skolem property to a set Γ consisting of two $\mathscr{L}$-sentences and a recursive set of $\mathscr{L}_{\omega\omega}[\tau^*]$-sentences in a vocabulary τ^* including $\tau_0 \cup \tau_0'$. By a theorem of Craig and Vaught, there is a finite set of $\mathscr{L}_{\omega\omega}$-sentences having the same $\tau_0 \cup \tau_0'$-reducts. Therefore one really needs the Löwenheim–Skolem property only for single sentences. Nevertheless, we show here explicitly how to obtain isomorphic structures in (2′), since, in this way, we can become acquainted with a proof technique which is frequently used in soft model theory in general and in this chapter in particular.

For $k \in \omega$, take $\mathfrak{A}$ and $\mathfrak{B}$ as given by (2′) and choose $(I_m)_{m \leq k}$ such that $(I_m)_{m \leq k}$: $\mathfrak{A} \cong_k \mathfrak{B}$. By the results of Section II.4.2 we can assume that $\bigcup_{m \leq k} I_m$ is countable. Moreover, suppose without loss of generality that $\{0, \ldots, k\} \subset A$. Choose a one-to-one mapping from $\bigcup_{m \leq k} I_m$ into A. In the sequel, we shall identify $p \in \bigcup_{m \leq k} I_m$ with its value under this mapping. Take new relation symbols U, P (unary), $<$, I (binary) and G (ternary) and let $\boldsymbol{\sigma} = \tau_0 \cup \tau_0' \cup \{P, <, I, G\}$, where τ_0' is a disjoint copy of τ_0. Let $\mathfrak{C}$ $(= \mathfrak{C}_k)$ be the $\boldsymbol{\sigma}$-structure with domain A given by

$\mathfrak{C} \upharpoonright \tau_0 = \mathfrak{A}, \qquad \mathfrak{C} \upharpoonright \tau_0' = \mathfrak{B}' \qquad$ ($\mathfrak{B}'$ denotes the τ_0'-structure corresponding to $\mathfrak{B}$),

$U^C = \{0, \ldots, k\}$,

$<^C$ is the natural ordering on $\{0, \ldots, k\}$,

$P^C p \quad$ iff $\quad p \in \bigcup_{m \leq k} I_m$,

$I^C mp \quad$ iff $\quad m \leq k \quad$ and $\quad p \in I_m$,

$G^C pab \quad$ iff $\quad p \in \bigcup_{m \leq k} I_m \quad$ and $\quad p(a) = b$.

Then $\mathfrak{C}$ is a model of the conjunction ϑ of the following $\mathscr{L}[\boldsymbol{\sigma}]$-sentences

φ, ψ',

"$<$ is a discrete ordering with first and last element",

"U is the field of $<$",

"Each p on P is a (partial) injective mapping"

that is, $\forall p(Pp \to \forall x\, \forall y\, \forall u\, \forall v(Gpxu \wedge Gpyv \to (x = y \leftrightarrow u = v)))$,

"Each p in P preserves all symbols in $\boldsymbol{\tau}_0$"

for example, for a binary R in $\boldsymbol{\tau}_0$

$\forall p(Pp \to \forall x\, \forall y\, \forall u\, \forall v(Gpxu \wedge Gpyv \to (Rxy \leftrightarrow R'uv)))$,

"For each u in U the set I_u is non-empty"

that is, $\forall u(Uu \to \exists p(Pp \wedge Iup))$,

"The sequence of I_u's has the forth property"

that is, $\forall u\, \forall v(v < u \to \forall p(Iup \to \forall x\, \exists q\, \exists y(Ivq \wedge Gqxy$
$\wedge \forall z\, \forall w(Gpzw \to Gqzw))))$,

"The sequence of I_u's has the back property".

Clearly, by (2′) the sentence ϑ has property (ii). In fact, $\mathfrak{C}_k$ is a model with $|U^{\mathfrak{C}_k}| = k + 1$. We show that ϑ also satisfies (i). Otherwise, ϑ has a model with infinite U-part. Let f be a new unary function symbol. Then,

$\vartheta \wedge$ "f maps the U-part one-to-one onto a proper subset"

has a model, and hence a countable model $\mathfrak{D}$ by the Löwenheim–Skolem property. $<^{\mathfrak{D}}$ being a discrete ordering with last element of the infinite set $U^{\mathfrak{D}}$ contains an infinite descending sequence

$$\cdots <^{\mathfrak{D}} d_2 <^{\mathfrak{D}} d_1 <^{\mathfrak{D}} d_0.$$

Let $\mathfrak{A}_0 = \mathfrak{D} \restriction \boldsymbol{\tau}_0$, $\mathfrak{B}_0 = (\mathfrak{D} \restriction \boldsymbol{\tau}'_0)^{-\prime}$ and $J = \{p \mid I^{\mathfrak{D}} d_n p$ for some $n \in \omega\}$. Since $\mathfrak{D} \models \vartheta$, we can identify $p \in P^{\mathfrak{D}}$ with the partial isomorphism $\{(a, b) \mid G^{\mathfrak{D}}pab\}$ from $\mathfrak{A}_0$ to $\mathfrak{B}_0$. Moreover, by $\mathfrak{D} \models \vartheta$, we have $\mathfrak{A}_0 \models \varphi$, $\mathfrak{B}_0 \models \psi$ and $J\colon \mathfrak{A}_0 \cong_p \mathfrak{B}_0$; that is, $\mathfrak{A}_0$ and $\mathfrak{B}_0$ are partially isomorphic via J (that J has the back and forth property can be easily seen by using the fact that the d_n's form an infinite descending sequence). But D is countable and countable partially isomorphic structures are isomorphic (see Theorem II.4.3.1). Hence, $\mathfrak{A}_0 \cong \mathfrak{B}_0$. In particular, $\mathfrak{A}_0 \models \{\varphi, \psi\}$ and therefore $\mathfrak{A} \in \mathrm{Mod}(\varphi) \cap \mathrm{Mod}(\psi)$—a contradiction. $\square$

1.1.7 Examples. (a) Take as $\mathscr{L}$ in Theorem 1.1.3 the set of Σ_1^1-sentences over $\mathscr{L}_{\omega\omega}$ (that is, sentences of the form $\exists R_1 \ldots \exists R_n \varphi$, where φ is first-order). Then

Theorem 1.1.3 yields the $\mathscr{L}_{\omega\omega}$-interpolation theorem: Any two Σ_1^1-sentences with disjoint model classes can be separated by an elementary class.

(b) For $n \geq 1$ let Q_n be a quantifier binding n-ary relation variables. Fix the interpretation of Q_n by the clause

$$\mathfrak{A} \models Q_n R\varphi \quad \text{iff} \quad |\{R^A \mid (\mathfrak{A}, R^A) \models \varphi\}| \geq 2^{|A|}.$$

Q_n is a kind of "second-order Lindström quantifier". Call an $\mathscr{L}_{\omega\omega}(Q_n \mid n \geq 1)$-formula *positive*, if it is a member of the smallest set containing the first-order formulas and closed under $\wedge$, $\vee$, $\exists x$, $\forall x$ and $Q_n R$. Let $\mathscr{L}$ consist of the positive $\mathscr{L}_{\omega\omega}(Q_n \mid n \geq 1)$-sentences. Using the local Chang–Makkai theorem for recursively saturated structures (see Schlipf [1978]), one can show that $\mathscr{L}$ has the Löwenheim–Skolem property and is countably compact (it is even compact!). Hence, by Theorem 1.1.3, it follows that any two disjoint $\mathscr{L}$-classes can be separated by an elementary class.

The following proposition is related to a result obtained in the course of the proof of Theorem 1.1.1.

1.1.8 Proposition. *Let $\mathscr{L} \leq \mathscr{L}'$ where $\mathscr{L}$ and $\mathscr{L}'$ are logics and where $\mathscr{L}[\tau]$ is a set for any vocabulary τ. If $\mathscr{L}'$ is compact and*

$$(*) \qquad \mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B} \quad \textit{implies} \quad \mathfrak{A} \equiv_{\mathscr{L}'} \mathfrak{B}$$

then $\mathscr{L} \equiv \mathscr{L}'$.

Proof. For an arbitrary satisfiable $\mathscr{L}'[\tau]$-sentence φ we have by $(*)$

$$\models \varphi \leftrightarrow \bigvee_{\mathfrak{A} \models \varphi} \bigwedge_{\substack{\psi \in \mathscr{L}[\tau] \\ \mathfrak{A} \models \psi}} \psi.$$

Now standard compactness arguments will show that the disjunction and the conjunctions on the right-hand side can be replaced by finite ones. ☐

For further reference we state:

1.1.9 Remark. The preceding proof shows that if $\mathscr{L}'$ has the finite occurrence property and $\mathscr{L}[\tau]$ is countable for finite τ, then it suffices to assume that $\mathscr{L}'$ is countably compact. Moreover, if $\mathscr{L}'$ has the Löwenheim–Skolem property down to κ for countable sets of sentences, then $(*)$ must only be required for structures of cardinality $\leq \kappa$.

1.2. Some Counterexamples

In this part we list some more or less strange examples which will show that certain strengthenings of the theorems of Lindström are not possible. A further example is contained in Section 2.3. Already in Chapter II it has been shown that $\mathscr{L}_{\omega\omega}(Q^{\omega})$, the logic with the cofinality ω quantifier, is compact, recursively enumerable for validity, and has the Löwenheim–Skolem property down to $\aleph_1$. The reader should consult Shelah [1975d] where further examples of compact extensions of first-order logics are given. In particular, there a logic $\mathscr{L}_{\omega\omega}(Q)$ is introduced, which is regular, compact, and more expressive than first-order logic even for countable structures: Let λ be the first weakly compact cardinal. The binary quantifier Q is then defined as follows:

$$\mathfrak{A} \models Qxy\varphi(x, y) \quad \text{iff} \quad \begin{array}{l}\varphi^{\mathfrak{A}} := \{(a, b) \mid \mathfrak{A} \models \varphi[a, b]\} \text{ is an ordering and} \\ \text{there is a Dedekind cut } (A_1, A_2) \text{ of } \varphi^{\mathfrak{A}} \text{ with} \\ \text{cofinalities in } \{\aleph_0, \lambda\}.\end{array}$$

A Dedekind cut of an ordering $(B, <)$ is a pair (B_1, B_2) such that $B_1 \cap B_2 = \emptyset$, $B_1 \cup B_2 = B$ and $b_1 < b_2$ for $b_1 \in B_1$, $b_2 \in B_2$. The cofinalities of a Dedekind cut are the cofinalities of $(B_1, <)$ and of $(B_2, >)$.

Note that $(\mathbb{Z}, <)$ and $(\mathbb{Z}, <) + (\mathbb{Z}, <)$ are not $\mathscr{L}_{\omega\omega}(Q)$-equivalent. $\mathscr{L}_{\omega\omega}(Q)$ does not have the Löwenheim–Skolem property.

1.2.1 Example. Let $\mathscr{L}$ be the logic obtained from $\mathscr{L}_{\omega\omega}$ by adding a new "atomic" sentence χ_s. Let χ_s be in $\mathscr{L}[\tau]$ for each τ and set

$$\mathfrak{A} \models \chi_s \quad \text{iff} \quad |A| \text{ is a successor cardinal.}$$

Then $\mathscr{L}$ is compact, has the interpolation property and the Löwenheim–Skolem property down to $\aleph_1$.

1.2.2 Example. Given a τ-structure $\mathfrak{A}$, say $\mathfrak{A} = (A, R_1, R_2, \ldots, f_1, f_2, \ldots, c_1, \ldots)$ denote by $\mathfrak{A}^c$ the τ-structure $(A, R_1^c, R_2^c, \ldots, f_1, f_2, \ldots, c_1, \ldots)$, where for n-ary R_i, $R_i^c = A^n \backslash R_i$. Let $\mathscr{L}^c$ be the logic with the same syntax as $\mathscr{L}_{\omega\omega}$ and with the semantics given by

$$\mathfrak{A} \models_c \varphi \quad \text{iff} \quad \begin{cases} \mathfrak{A} \models \varphi, & \text{if } A \text{ is infinite,} \\ \mathfrak{A}^c \models \varphi, & \text{if } A \text{ is finite.} \end{cases}$$

Neither $\mathscr{L}_{\omega\omega} \leq \mathscr{L}^c$ nor $\mathscr{L}^c \leq \mathscr{L}_{\omega\omega}$ holds. $\mathscr{L}^c$ is a compact logic with the Löwenheim–Skolem property and is also a maximal logic with these two properties. We leave it to the reader to adapt the preceding proofs to show that $\mathscr{L}^c$ is a maximal logic.

1.3. The Monadic Case

Throughout this part of the discussion we will restrict ourselves to monadic vocabularies; that is, we will assume that all vocabularies only contain unary relation symbols. We will give Lindström-type characterizations of monadic first-order logic and of some extensions.

When analyzing the arguments of Section 1.1 for the monadic case one should note the following differences:

(a) For a finite vocabulary, any two elementarily equivalent structures are isomorphic (thus the last step in the proof of Theorem 1.1.1 is actually redundant).
(b) Using a single monadic $\mathscr{L}_{\omega\omega}$-sentence, one cannot force a structure to be infinite (as by "f is one-to-one but not onto" in the general case).
(c) For any monadic recursive vocabulary τ, the set of $\mathscr{L}_{\omega\omega}[\tau]$-sentences valid in all finite models is recursively enumerable (it is even recursive).

In fact, while the characterization of $\mathscr{L}_{\omega\omega}$ given in Theorem 1.1.1 carries over to the monadic case (see Theorem 1.3.2 below), the following examples show that in this characterization the Löwenheim–Skolem property is really needed for countable sets (and not just for single sentences), and that Lindström's second theorem no longer holds.

1.3.1 Examples. (a) The monadic part of the logic in Example 1.2.1, with the new atomic sentence χ_s true in structures, the cardinality of which is a successor cardinal, is an example of a logic more expressive than first-order logic which is compact and has the Löwenheim–Skolem property. The Löwenheim–Skolem property follows from the fact that every satisfiable monadic $\mathscr{L}_{\omega\omega}$-sentence has a finite model.

(b) Let $\mathscr{L}$ be the logic obtained from $\mathscr{L}_{\omega\omega}$ by adding a new "atomic" sentence which is true just in the models of even finite cardinality. Then the monadic part of $\mathscr{L}$ properly extends $\mathscr{L}_{\omega\omega}$, is decidable, and has the Löwenheim–Skolem property for countable sets of sentences.

(c) Let $\mathscr{L}$ be obtained from $\mathscr{L}_{\omega\omega}$ by adding a new "atomic" sentence which is true in models of even finite or uncountable cardinality. Then the monadic part of $\mathscr{L}$ is countably compact, decidable, and each satisfiable sentence has a finite model.

It is easy to extend the above-mentioned maximality result for the monadic part of $\mathscr{L}_{\omega\omega}$ to a more general situation. For the rest of this section, however, *we restrict ourselves to monadic logics $\mathscr{L}$ with the finite occurrence property.* For an ordinal β denote by Q_β the unary quantifier "there are $\aleph_\beta$-many". Fix an ordinal α and let $\mathscr{L} = \mathscr{L}(Q_\beta \mid \beta \leq \alpha, \beta$ successor ordinal). Since in a structure $\mathfrak{A}$ of finite vocabulary $\tau_0 = \{R_1, \ldots, R_n\}$, the cardinalities of the boolean atoms $P_1 \cap \cdots \cap P_n$, where $P_i = R_i^A$ or $P_i = A \backslash R_i^A$, determine the isomorphism type of $\mathfrak{A}$, we have for τ_0-structures $\mathfrak{A}$ and $\mathfrak{B}$

$$(*) \qquad \mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B} \quad \text{and} \quad |A|, |B| \leq \aleph_\alpha \quad \text{imply} \quad \mathfrak{A} \cong \mathfrak{B}.$$

Hence, for any logic $\mathscr{L}'$ with the finite occurrence property and an arbitrary monadic vocabulary τ, we obtain from $(*)$

$$(\overset{*}{*}) \qquad \mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B} \quad \text{and} \quad |A|, |B| \leq \aleph_\alpha \quad \text{imply} \quad \mathfrak{A} \equiv_{\mathscr{L}'} \mathfrak{B}.$$

We will make use of (∗) and ($\overset{*}{*}$) in proving the following maximality theorem.

1.3.2 Theorem. *For a countable ordinal α the monadic part of $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_\beta \mid \beta \leq \alpha$, β successor ordinal) is (among the monadic logics) a maximal countably compact logic with the Löwenheim–Skolem property down to $\aleph_\alpha$ for countable sets of sentences. Furthermore, $\mathscr{L}$ has the interpolation property.*

Before undertaking the proof of Theorem 1.3.2, let us state some consequences.

1.3.3 Corollary. (a) (Tharp [1973]) *The monadic part of $\mathscr{L}_{\omega\omega}$ is a maximal logic with countable compactness and the Löwenheim–Skolem property for countable sets of sentences.*

(b) (Caicedo [1981b]) *The monadic part of $\mathscr{L}_{\omega\omega}(Q_1)$ is a maximal logic with countable compactness and the Löwenheim–Skolem property down to $\aleph_1$ for countable sets of sentences. Moreover, this logic has the interpolation property.*

(c) (Caicedo [1981b]) *The monadic part of $\mathscr{L}_{\omega\omega}(Q_1)$ and of $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ are equivalent. (The reader is referred to Example* 4 *of Section* 2.2, *Chapter II for the definition of $\mathscr{L}_{\omega\omega}(\mathrm{aa})$).*

Proof of Theorem 1.3.2. We prove the "separation property" corresponding to the maximality assertion : Let $\mathscr{L}'$, $\mathscr{L} \leq \mathscr{L}'$, be a countably compact logic with the Löwenheim–Skolem property down to $\aleph_\alpha$ for countable sets. Suppose that $\mathrm{Mod}(\varphi)$ and $\mathrm{Mod}(\psi)$ are disjoint $\mathscr{L}'$-classes not separable by an $\mathscr{L}$-class. Since for a finite vocabulary τ there are only countably many $\mathscr{L}$-sentences—this is the point where we need the restriction to a countable ordinal α—we obtain as in the proof of (2) in Lemma 1.1.2, using the countable compactness of $\mathscr{L}'$, structures $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\leq \aleph_\alpha$ with

$$\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}, \qquad \mathfrak{A} \models \varphi \quad \text{and} \quad \mathfrak{B} \models \psi.$$

This is a contradiction in view of ($\overset{*}{*}$) above. It still remains to show that $\mathscr{L}$ is countably compact, and this will be accomplished by the next theorem. $\square$

1.3.4 Theorem. *Let α be an arbitrary ordinal and $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_\beta \mid 0 < \beta \leq \alpha$, cofinality of $\beta \neq \omega$). Then the monadic part of $\mathscr{L}$ is countably compact and has the interpolation property.*

Proof. Suppose each finite subset of $\Phi = \{\varphi_1, \varphi_2, \ldots\}$ is satisfiable. Choose $\tau = \{R_1, R_2, \ldots\}$ such that $\Phi \subset \mathscr{L}[\tau]$. We want to obtain a model $\mathfrak{A}$ of Φ, fixing step by step, the cardinalities of the boolean atoms determined by $\{R_1, \ldots, R_n\}$ in such a way that for any finite subset Φ_0 of Φ these cardinalities are realized in some model of Φ_0. For this let

$$F = \{f \mid \text{there is } k \geq 0 \text{ such that } f\colon \{1, \ldots, k\} \to \{0, 1\}\}.$$

For a τ-structure $\mathfrak{A}$ and $f \in F$ define A^f by

$$A^f := \begin{cases} A, & \text{if } f = \emptyset, \\ P_1 \cap \cdots \cap P_k, & \text{if } f\colon \{1, \ldots, k\} \to \{0, 1\}, \end{cases}$$

where

$$P_i = \begin{cases} R_i^A, & \text{if } f(i) = 1, \\ A \backslash R_i^A, & \text{if } f(i) = 0. \end{cases}$$

Suppose given pairwise distinct $f, g_1, \ldots, g_l \in F$ with $\mathrm{dom}(g_i) \subset \mathrm{dom}(f)$ for all i, and cardinals $\lambda_1, \ldots, \lambda_l$. For $n \in \omega$ *let*

$$\begin{aligned} C_n &= C(n, f, g_1, \ldots, g_l, \lambda_1, \ldots, \lambda_l) \\ &:= \{ |A^f| \mid \mathfrak{A} \models \{\varphi_1, \ldots, \varphi_n\}, |A^{g_1}| = \lambda_1, \ldots, |A^{g_l}| = \lambda_l \}. \end{aligned}$$

We show that

(a) $\sup\{\kappa_m \mid m \in \omega\} \in C_n$, for any sequence $\kappa_1 < \kappa_2 < \cdots$ of cardinals in C_n.
(b) $C_0 \supset C_1 \supset \cdots$
(c) If $C_n \neq \emptyset$ for all $n \in \omega$, then $\bigcap \{C_n \mid n \in \omega\} \neq \emptyset$.

To prove (a), choose m_0 large enough such that in $\varphi_1 \wedge \cdots \wedge \varphi_n$ no quantifier Q_β with $\kappa_{m_0} \leq \kappa_\beta < \sup \kappa_m$ appears (and hence by definition of $\mathscr{L}$ no quantifier Q_β with $\kappa_{m_0} \leq \kappa_\beta \leq \sup \kappa_m$). We assume that κ_{m_0} is infinite and leave the case "κ_{m_0} finite" to the reader. Take $\mathfrak{A} \models \{\varphi_1, \ldots, \varphi_n\}$ such that $|A^{g_1}| = \lambda_1, \ldots, |A^{g_l}| = \lambda_l$ and $|A^f| = \kappa_{m_0}$. Suppose $f\colon \{1, \ldots, k\} \to \{0, 1\}$ and choose $k' \geq k$ such that $\varphi_1 \wedge \cdots \wedge \varphi_n \in \mathscr{L}[\{R_1, \ldots, R_{k'}\}]$. Since $|A^f| = \kappa_{m_0}$, there is a boolean atom determined by $R_1, \ldots, R_{k'}$, of power κ_{m_0}, which is a part of A^f. Obtain $\mathfrak{A}'$ from $\mathfrak{A}$ blowing up this boolean atom to a set of cardinality $\sup \kappa_m$. Then $\mathfrak{A}'$ shows that $\sup \kappa_m \in C_n$ (since $\kappa_m \in C_n$ for all m, we have $\sup \kappa_m \leq \lambda_i$ for any i with $g_i \subset f$). (b) is clear by definition of the C_n's, and (c) follows immediately from (a) and (b).

We now construct the desired model $\mathfrak{A} = (A, R_1^A, \ldots)$ of Φ. Let $f_0 = \emptyset$. Choose $\kappa_0 \in \bigcap \{C(n, f_0) \mid n \in \omega\}$ and let A be a set of cardinality κ_0. Denote by f_1 and f_2 the functions given by $f_1, f_2\colon \{1\} \to \{0, 1\}$ with $f_1(1) = 1$ and $f_2(1) = 0$. Choose $\kappa_1 \in \bigcap \{C(n, f_1, f_0, \kappa_0) \mid n \in \omega\}$ and $\kappa_2 \in \bigcap \{C(n, f_2, f_0, f_1, \kappa_0, \kappa_1) \mid n \in \omega\}$. Let R_1^A be a subset of A of cardinality κ_1 with complement of cardinality κ_2. Now one defines R_2^A choosing cardinalities for the subsets A^f, where $\mathrm{dom}(f) = \{1, 2\}$ with the help of the appropriate sets $C(\ldots)$. In this way, one can fix inductively all the R_n^A's.

We leave it to the reader to verify the interpolation property. Observe that since in any $\mathscr{L}$-sentence only finitely many Q_β occur, one can restrict to a countable sublanguage and argue as in the proof of Theorem 1.3.2). $\square$

1.3.5 Notes. Theorems 1.1.4 and 1.1.5 were proven by Lindström [1966a], [1969]; Theorem 1.1.4 was later rediscovered by Friedman, to whom assertion (1) in the proof of Theorem 1.1.1 is due. The examples listed in 1.3.1 are due to Tharp [1973]. Example 1.2.2 and the results given in Theorems 1.3.2 and 1.3.4 on monadic logics are new here, (although the countable compactness of the monadic part of $\mathscr{L}_{\omega\omega}(Q_{\alpha_1}, \ldots, Q_{\alpha_n})$ for $\alpha_1, \ldots, \alpha_n > 0$ was proved by Fajardo [1980]. *Added in proof*: Theorem 1.3.2 can be generalized to uncountable α, as will be shown elsewhere.

2. *Further Characterizations of* $\mathscr{L}_{\omega\omega}$

In this section we present some further characterizations of first-order logic, first examining those logics having the Löwenheim–Skolem property or a related property, the Karp property. In the second part we drop these assumptions. We close this section with the study of compact sublanguages of $\mathscr{L}_{\infty\omega}$.

Throughout parts 1 and 2 we will assume that all logics $\mathscr{L}$ under consideration are *regular and have the finite occurrence property* (even though many results would continue to hold under weaker assumptions). Recall that a regular logic $\mathscr{L}$ has the substitution property (and hence possesses the relativization property) and also satisfies $\mathscr{L}_{\omega\omega} \leq \mathscr{L}$.

2.1. *The Löwenheim–Skolem Property and the Karp Property*

By definition, a logic $\mathscr{L}$ has the *Karp property* if partially isomorphic structures are $\mathscr{L}$-equivalent, that is, if

$$\mathfrak{A} \cong_p \mathfrak{B} \quad \text{implies} \quad \mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}.$$

In the presence of the substitution property, we can replace in Lindström's first characterization of first-order logic the Löwenheim–Skolem property by the weaker Karp property (The reader is referred to Proposition 2.1.7 below for the relationship between the Löwenheim–Skolem and the Karp properties). Indeed, we have:

2.1.1 Theorem. *If $\mathscr{L}$ has the Karp property and is countably compact, then $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$.*

Since the ordering $(\omega, <)$ cannot be defined in a countably compact logic, this theorem is a consequence of the following lemma.

2.1.2 Lemma. *If $\mathscr{L}_{\omega\omega} < \mathscr{L}$ and $\mathscr{L}$ has the Karp property, then $(\omega, <)$ is* RPC *in $\mathscr{L}$ (that is, there is a satisfiable $\mathscr{L}$-sentence $\varphi_0(U, <, \ldots)$ such that in each model $\mathfrak{A}$ of φ_0 the relativized reduct $(U^A, <^A)$ is isomorphic to $(\omega, <)$).*

Proof. Let φ be an $\mathscr{L}$-sentence not equivalent to a first-order sentence. Choose a finite τ_0 such that $\varphi \in \mathscr{L}[\tau_0]$. Then, for finitely many $\varphi_1, \ldots, \varphi_n \in \mathscr{L}_{\omega\omega}[\tau_0]$, there are $\mathfrak{A}$ and $\mathfrak{B}$ such that

$$\mathfrak{A} \models \varphi, \qquad \mathfrak{B} \models \neg\varphi \quad \text{and} \quad (\mathfrak{A} \models \varphi_i \text{ iff } \mathfrak{B} \models \varphi_i) \quad \text{for } i \leq n.$$

Hence,

$$(*) \qquad \begin{array}{l} \text{for each } k \in \omega, \text{ there are } \mathfrak{A}_k \text{ and } \mathfrak{B}_k \text{ such that} \\ \mathfrak{A}_k \cong_k \mathfrak{B}_k, \qquad \mathfrak{A}_k \models \varphi \quad \text{and} \quad \mathfrak{B}_k \models \neg\varphi. \end{array}$$

Let U, $<$, V, W be new relation symbols, U unary, $<$, V and W binary. Coding partial isomorphisms as in the proof of Theorem 1.1.2, we obtain in a suitable vocabulary τ, an $\mathscr{L}$-sentence φ_0 expressing

> "$<$ is a discrete linear ordering of its field U with first but no last element; for each x in U the set $Vx\cdot$ (i.e. $\{y \mid Vxy\}$) is a model of φ, the set $Wx\cdot$ is a model of $\neg\varphi$, and $Vx\cdot$ and $Wx\cdot$ are x-partially isomorphic, i.e. there is a sequence, indexed by the $<$-predecessors of x of non-empty sets of partial isomorphisms with the back and forth property."

(Compare Chapter II Proposition 5.2.4 to see how we can formulate this statement by an $\mathscr{L}$-sentence). By $(*)$, φ_0 has a model $\mathfrak{A}$, where $(U^A, <^A)$ is isomorphic to $(\omega, <)$ and where attached to the k-th element a of the ordering $<^A$ are the models $\mathfrak{A}_k$ and $\mathfrak{B}_k$; that is,

$$(\{b \mid V^A ab\}, \ldots) \cong \mathfrak{A}_k \quad \text{and} \quad (\{b \mid W^A ab\}, \ldots) \cong \mathfrak{B}_k.$$

Now let $\mathfrak{B}$ be any model of φ_0; we must show that $(U^B, <^B)$ is isomorphic to $(\omega, <)$. If $(U^B, <^B) \not\cong (\omega, <)$, a "non-standard" element x in $(U^B, <^B)$, gives rise—as in the proof of Lemma 1.1.2—to partially isomorphic models $Vx\cdot$ of φ and $Wx\cdot$ of $\neg\varphi$. This is, a contradiction, however, since we assumed that $\mathscr{L}$ has the Karp property. □

In case $\mathscr{L}$ has the Löwenheim–Skolem property, the structures $\mathfrak{A}_k$ and $\mathfrak{B}_k$ in $(*)$ of the preceding proof can be chosen of power $\aleph_0$ and hence $(*)$ can be coded in a countable model of φ_0. Thus we can require that $<$ is an ordering of the universe of the model. Accordingly, we obtain:

2.1.3 Corollary. *If $\mathscr{L}_{\omega\omega} < \mathscr{L}$ and $\mathscr{L}$ has the Löwenheim–Skolem property, then $(\omega, <)$ is* PC *in $\mathscr{L}$.*

This corollary can also be derived from the results of the preceding section: Since $\mathscr{L}_{\omega\omega} < \mathscr{L}$ there is a sentence $\vartheta = \vartheta(U, \ldots)$ having properties (i) and (ii) of Lemma 1.1.2. Now it is not difficult (using the substitution property of $\mathscr{L}$) to write down a sentence PC-characterizing $(\omega, <)$: This sentence will express that attached to each element x of the ordering $<$ is a model of ϑ whose U-part has as many elements as the set of $<$-predecessors of x.

In the following theorem, we collect some model-theoretic properties that characterize $\mathscr{L}_{\omega\omega}$ among the logics with the Löwenheim–Skolem property. However, we state the theorem in such a way that it provides information on the expressive power of proper extensions of $\mathscr{L}_{\omega\omega}$.

2.1.4 Theorem. *For a logic $\mathscr{L}$ satisfying the Löwenheim–Skolem property the following conditions are equivalent.*

(i) $\mathscr{L}_{\omega\omega} < \mathscr{L}$.
(ii) *$\mathscr{L}$ is not countably compact.*
(iii) *(The class of structures isomorphic to) $(\omega, <)$ is* PC *in $\mathscr{L}$.*
(iv) *Each countable structure in a countable vocabulary is* PC_δ *in $\mathscr{L}$; that is, it is characterizable using additional symbols by a countable set of sentences.*
(v) *$\mathscr{L}^{\text{HF}} \leq_{\text{RPC}} \mathscr{L}$, where $\mathscr{L}^{\text{HF}}$ is the second-order logic with quantification on hereditarily finite sets over the universe, and $\mathscr{L}_1 \leq_{\text{RPC}} \mathscr{L}_2$ means that each class of relativized reducts in $\mathscr{L}_1$ is such a class in $\mathscr{L}_2$.*
(vi) *There is an $\mathscr{L}$-sentence with an infinite but no uncountable model.*
(vii) *$\mathscr{L}_{\omega\omega} <_{\equiv} \mathscr{L}$; that is, there are $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A} \equiv \mathfrak{B}$ but $\mathfrak{A} \not\equiv_{\mathscr{L}} \mathfrak{B}$.*

Proof. Clearly each of the conditions in (ii)–(vii) implies (i). Hence, it suffices to show that (ii)–(vii) follow from (i).

(i) $\Rightarrow$ (ii). This was shown in Section 1.

(i) $\Rightarrow$ (iii). See the preceding corollary.

For the proofs of the following implications let φ_0 always denote an $\mathscr{L}[\tau_0]$-sentence PC-characterizing $(\omega, <)$.

(iii) $\Rightarrow$ (iv). Given a countable structure $\mathfrak{A}$ choose a one-to-one enumeration $\langle a_n \mid n \in \omega \rangle$ of A, write down the algebraic diagram Φ of $\mathfrak{A}$, where a_n is represented by the n-th element of an ordering $<$ of type ω. Then $\{\varphi_0\} \cup \Phi$ is a PC_δ-characterization of $\mathfrak{A}$.

(iii) $\Rightarrow$ (v). Use φ_0 (and hence $(\omega, <)$) to code the hereditarily finite sets over the universe.

(iii) $\Rightarrow$ (vi). φ_0 has no uncountable model.

(iii) $\Rightarrow$ (vii). Let $\mathfrak{A}$ be a countable model of φ_0. Then any uncountable model of $\text{Th}(\mathfrak{A})$, the first-order theory of $\mathfrak{A}$, is elementarily equivalent but not $\mathscr{L}$-equivalent to $\mathfrak{A}$. $\square$

2.1.5 Remarks. (a) In case $\mathscr{L}$ has the form $\mathscr{L}_{\omega\omega}(Q_1, \ldots, Q_n)$ where $Q_1, \ldots, Q_n$ are Lindström quantifiers we can add in Theorem 2.1.4 the condition

(viii) $\mathscr{L}$ does not have the definability property (Beth property).

The reader is referred to Chapter XVII for a proof of this result.

(b) We want to draw the reader's attention to the notion of an ω_1-securable quantifier (see Makowsky [1975b]) which captures the properties of the existential quantifier needed to prove that each structure has a countable elementary substructure. In fact, Makowsky proved the following: If $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_i | i \in I)$ is obtained from first-order logic adding ω_1-securable quantifiers, then $\mathscr{L}$ has the Löwenheim–Skolem property. Hence, for such an $\mathscr{L}$, the equivalences in Theorem 2.1.4 hold.

We use Theorem 2.1.4 to derive a further characterization of $\mathscr{L}_{\omega\omega}$. A logic $\mathscr{L}$ is said to have the *Robinson property* if the following holds: Let τ, τ_1 and τ_2 be vocabularies with $\tau = \tau_1 \cap \tau_2$. Let Φ be a set of $\mathscr{L}[\tau]$-sentences and Φ_i a set of $\mathscr{L}[\tau_i]$-sentences for $i = 1, 2$. If Φ is complete and $\Phi \cup \Phi_1$ and $\Phi \cup \Phi_2$ are satisfiable then so is $\Phi \cup \Phi_1 \cup \Phi_2$. In Chapter XIX it is shown that this is a very strong property of a logic. In fact, it is proved there that in case there are no measurable cardinals the Robinson property implies the compactness property.

2.1.6 Theorem. *If $\mathscr{L}$ has the Löwenheim–Skolem property and the Robinson property then $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$.*

Proof. Since we have the general assumption that $\mathscr{L}$ has the finite occurrence property there are countable structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in a countable vocabulary τ such that

$$\mathfrak{A}_1 \equiv_{\mathscr{L}} \mathfrak{A}_2 \quad \text{and} \quad \mathfrak{A}_1 \not\cong \mathfrak{A}_2$$

(e.g., take non-isomorphic $\mathfrak{A}_1$ and $\mathfrak{A}_2$ such that for any finite $\tau_0 \subset \tau$ the τ_0-reducts $\mathfrak{A}_1 \upharpoonright \tau_0$ and $\mathfrak{A}_2 \upharpoonright \tau_0$ are isomorphic). Suppose by contradiction, that $\mathscr{L}_{\omega\omega} < \mathscr{L}$. Then, by the equivalence (i) $\Leftrightarrow$ (iv) of Theorem 2.1.4, there are PC_δ-characterizations Φ_1 and Φ_2 in $\mathscr{L}$ of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. We use distinct additional symbols for $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Since $\mathfrak{A}_1 \equiv_{\mathscr{L}} \mathfrak{A}_2$, $\Phi \cup \Phi_1$ and $\Phi \cup \Phi_2$ are satisfiable; but $\Phi \cup \Phi_1 \cup \Phi_2$ has no model, as $\mathfrak{A}_1 \not\cong \mathfrak{A}_2$.

Note that in case we restrict attention to finite vocabularies, the preceding proof shows:

> If $\mathscr{L}_{\omega\omega} < \mathscr{L}$ and $\mathscr{L}$ has the Löwenheim–Skolem property and the Robinson property for countable sets of sentences (that is, countable Φ, Φ_1 and Φ_2), then $\equiv_{\mathscr{L}}$ coincides with the isomorphism relation on countable structures.

In particular, we see that weak second-order logic does not have the Robinson property; $\mathscr{L}_{\omega_1\omega}$ is a logic satisfying the hypothesis of this result.

We close this discussion with a result that clarifies the relationship between the Karp property and the Löwenheim–Skolem property.

2.1.7 Proposition. (a) *If $\mathscr{L}$ has the Löwenheim–Skolem property, then $\mathscr{L}$ has the Karp property.*

(b) *Assume $\mathscr{L}$ has both the Karp property and the interpolation property, then $\mathscr{L}$ has the Löwenheim–Skolem property.*

Proof. The proof of (a) is by contradiction. Suppose that $\mathscr{L}$ has the Löwenheim–Skolem property but that for some $\mathscr{L}$-sentence φ, we have

$$(*)\qquad \mathfrak{A} \cong_p \mathfrak{B}, \qquad \mathfrak{A} \models \varphi \quad \text{and} \quad \mathfrak{B} \models \neg\varphi.$$

Coding partial isomorphisms as in the precedings proofs, we obtain an $\mathscr{L}$-sentence ψ expressing that

> "the V-part is a model of φ, the W-part is a model of $\neg\varphi$, and the V-part and the W-part are partially isomorphic".

By $(*)$, the sentence ψ has a model, and hence one of power $\leq\aleph_0$. But then we obtain countable structures $\mathfrak{A}'$ (the V-part) and $\mathfrak{B}'$ (the W-part) such that $\mathfrak{A}' \models \varphi$, $\mathfrak{B}' \models \neg\varphi$ and $\mathfrak{A}' \cong_p \mathfrak{B}'$; hence $\mathfrak{A}' \cong \mathfrak{B}'$, a contradiction.

Turning now to the proof of (b), we let $\mathscr{L}$ be given as in (b) and suppose $\mathscr{L}_{\omega\omega} < \mathscr{L}$ (if $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$, the conclusion holds). Since $\mathscr{L}$ has the Karp property, by Lemma 2.1.2 $(\omega, <)$ is RPC in $\mathscr{L}$, say $\varphi_0(U, <, \ldots)$ is an $\mathscr{L}$-sentence RPC-characterizing $(\omega, <)$. If $\mathscr{L}$ does not have the Löwenheim–Skolem property, then there is an $\mathscr{L}$-sentence φ_1 having only uncountable models. Consider the classes

$$\mathfrak{K}_0 := \{(A, U^A) \mid \text{there is } <^A, --- \text{ such that } (A, U^A, <^A, ---) \models \varphi_0\},$$

$$\mathfrak{K}_1 := \{(A, U^A) \mid \text{there is} \ldots \text{such that } (U^A, \ldots) \models \varphi_1\}.$$

Since $(A, U^A) \in \mathfrak{K}_0$ (resp. $(A, U^A) \in \mathfrak{K}_1$) implies that U^A is countable (resp. uncountable), $\mathfrak{K}_0$ and $\mathfrak{K}_1$ are disjoint PC-classes of $\mathscr{L}$. Take an arbitrary (A, U^A) in $\mathfrak{K}_0$ and choose (B, U^B) in $\mathfrak{K}_1$ such that $|B\backslash U^B| = |A\backslash U^A|$. Then (A, U^A) and (B, U^B) are partially isomorphic. Hence, there is no $\mathscr{L}$-class separating $\mathfrak{K}_0$ and $\mathfrak{K}_1$, since $\mathscr{L}$ has the Karp property. But this contradicts the assumption that $\mathscr{L}$ has the interpolation property. $\square$

2.2. *The Tarski Union Property and the Omitting Types Property*

The following characterization of $\mathscr{L}_{\omega\omega}$ shows that an important model-theoretic tool of first-order logic, the Tarski union lemma, is not available in any proper compact extension.

First we introduce some terminology. Suppose given a logic $\mathscr{L}$. A structure $\mathfrak{B}$ is said to be an *$\mathscr{L}$-extension* of $\mathfrak{A}$, $\mathfrak{A} \prec_{\mathscr{L}} \mathfrak{B}$, if $\mathfrak{B}$ is an extension of $\mathfrak{A}$ and if for any finite $a_0, \ldots, a_{n-1} \in A$, we have $(\mathfrak{A}, a_0, \ldots, a_{n-1}) \equiv_{\mathscr{L}} (\mathfrak{B}, a_0, \ldots, a_{n-1})$. (For $\mathscr{L} = \mathscr{L}_{\omega\omega}$ we say that $\mathfrak{B}$ is an elementary extension and write $\mathfrak{A} \prec \mathfrak{B}$.)

Denote by $\mathrm{Th}_{\mathscr{L}}(\mathfrak{A})$ and $\mathrm{D}_{\mathscr{L}}(\mathfrak{A})$ the *$\mathscr{L}$-theory of* $\mathfrak{A}$ and the *$\mathscr{L}$-diagram of* $\mathfrak{A}$, respectively; that is,

$$\mathrm{Th}_{\mathscr{L}}(\mathfrak{A}) := \{\varphi \mid \varphi\ \mathscr{L}\text{-sentence}, \mathfrak{A} \models \varphi\}, \qquad \mathrm{D}_{\mathscr{L}}(\mathfrak{A}) := \mathrm{Th}_{\mathscr{L}}((\mathfrak{A}, (a)_{a \in A})),$$

where in the latter case we consider the $\mathscr{L}$-theory in an expanded vocabulary containing a new constant for each $a \in A$. In case $\mathscr{L} = \mathscr{L}_{\omega\omega}$, write $\mathrm{Th}(\mathfrak{A})$ and $\mathrm{D}(\mathfrak{A})$.

As for first-order logic, one can easily prove both $(+)$ and $(++)$ below (recall that all our logics are assumed to be regular and to have the finite occurrence property).

$(+)$ The (reducts of) models of $\mathrm{D}_{\mathscr{L}}(\mathfrak{A})$ are—up to isomorphism—the $\mathscr{L}$-extensions of $\mathfrak{A}$.

$(++)$ Assume $\mathscr{L}$ is compact. Suppose given $\mathfrak{A}$ and a set of Φ of $\mathscr{L}$-sentences If $\mathrm{Th}(\mathfrak{A}) \cup \Phi$ is satisfiable, then there is $\mathfrak{B}$ such that $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{B} \models \Phi$.

Now we say that $\mathscr{L}$ has the *Tarski union property*, if whenever

$$\mathfrak{A}_0 \prec_{\mathscr{L}} \mathfrak{A}_1 \prec_{\mathscr{L}} \mathfrak{A}_2 \prec \cdots$$

then $\mathfrak{A}_n \prec_{\mathscr{L}} \bigcup_m \mathfrak{A}_m$ for each n.

$\mathscr{L}_{\omega\omega}$ and $\mathscr{L}_{\omega\omega}(Q_1)$ have the Tarski union property. Moreover, we have (see Makowsky [1975b] for further examples and results):

2.2.1 Theorem. *If $\mathscr{L}$ is compact and has the Tarski union property, then $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$.*

Proof. If not $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$, then there is an $\mathscr{L}$-sentence φ and structures $\mathfrak{A}$, $\mathfrak{B}$ such that

$$(1) \qquad \mathfrak{A} \equiv \mathfrak{B}, \qquad \mathfrak{A} \models \varphi \quad \text{and} \quad \mathfrak{B} \models \neg\varphi$$

(see Proposition 1.1.8). We construct by induction a sequence $\mathfrak{A}_0, \mathfrak{A}_1, \ldots$ such that

$$\begin{tikzcd}[column sep=small]
\mathfrak{A} = \mathfrak{A}_0 \arrow[rr, bend right, dashed, no head, "\prec_{\mathscr{L}}"'] \arrow[r, phantom, "\prec"] & \mathfrak{A}_1 \arrow[rr, bend left, dashed, no head, "\prec_{\mathscr{L}}"] \arrow[r, phantom, "\prec"] & \mathfrak{A}_2 \arrow[rr, bend right, dashed, no head, "\prec_{\mathscr{L}}"'] \arrow[r, phantom, "\prec"] & \mathfrak{A}_3 \arrow[r, phantom, "\prec"] & \cdots
\end{tikzcd}$$

and $\mathfrak{A}_1 \models \neg\varphi$ as follows:

By (1), $\mathrm{Th}(\mathfrak{A}) \cup \{\neg\varphi\} = \mathrm{Th}(\mathfrak{B}) \cup \{\neg\varphi\}$ is satisfiable. Hence, by $(++)$ there is $\mathfrak{A}_1$ such that $\mathfrak{A} \prec \mathfrak{A}_1$ and $\mathfrak{A}_1 \models \neg\varphi$. Now suppose $\mathfrak{A}_n$ has already been defined. Since

$$\mathrm{Th}((\mathfrak{A}_n, (\mathrm{a})_{a \in A_{n-1}})) = \mathrm{Th}((\mathfrak{A}_{n-1}, (a)_{a \in A_{n-1}})) \subset \mathrm{D}_{\mathscr{L}}(\mathfrak{A}_{n-1}),$$

we have that

$$\mathrm{Th}((\mathfrak{A}_n, (a)_{a \in A_{n-1}})) \cup \mathrm{D}_{\mathscr{L}}(\mathfrak{A}_{n-1})$$

is a satisfiable set of $\mathscr{L}$-sentences. Using $(++)$ once more, we therefore obtain an elementary extension $(\mathfrak{A}_{n+1}, (a)_{a\in A_{n-1}})$ of $(\mathfrak{A}_n, (a)_{a\in A_{n-1}})$ which is a model of $D_{\mathscr{L}}(\mathfrak{A}_{n-1})$. Then,

$$\begin{tikzcd} \mathfrak{A}_{n-1} \arrow[r, phantom, "\prec"] \arrow[rr, dashed, bend right, no head, "\prec_{\mathscr{L}}"'] & \mathfrak{A}_n \arrow[r, phantom, "\prec"] & \mathfrak{A}_{n+1}. \end{tikzcd}$$

Let $\mathfrak{D} = \bigcup_n \mathfrak{A}_{2n} = \bigcup_n \mathfrak{A}_{2n+1}$. By the Tarski union property, we have $\mathfrak{A}_0 \prec_{\mathscr{L}} \mathfrak{D}$ and $\mathfrak{A}_1 \prec_{\mathscr{L}} \mathfrak{D}$. But since $\mathfrak{A}_0 \models \varphi$ and $\mathfrak{A}_1 \models \neg\varphi$, we obtain the contradiction: $\mathfrak{D} \models \varphi$ and $\mathfrak{D} \models \neg\varphi$. $\square$

Lindström [1983] introduced a kind of union property for direct limits and showed by refining the previous proof, that a logic is equivalent to first-order logic if it has this generalized union property and is countably compact.

We now turn to a characterization of $\mathscr{L}_{\omega\omega}$ by means of a single property, the omitting types property for an uncountable regular cardinal.

Let κ be an infinite cardinal and $\mathscr{L}$ be a logic. Given a set Φ of $\mathscr{L}$-sentences and a set $\Gamma(x)$ of $\mathscr{L}$-formulas having at most the free variable x (see II.1.1.2), we say that $\Gamma(x)$ is a *κ-free type of Φ*, if the following hold:

> $|\Phi \cup \Gamma(x)| \leq \kappa$, Φ is satisfiable and for every set $\Psi(x)$ of $\mathscr{L}$-formulas such that $|\Psi(x)| < \kappa$, if $\Phi \cup \Psi(x)$ has a model, then for some $\chi(x) \in \Gamma(x)$ the set $\Phi \cup \Psi(x) \cup \{\neg\chi(x)\}$ has a model.

We say that $\mathscr{L}$ has the *κ-omitting types property*, if whenever $\Gamma(x)$ is a κ-free type of Φ, there is a model of Φ omitting $\Gamma(x)$.

Thus the "classical" omitting types theorem is the result that $\mathscr{L}_{\omega\omega}$ has the ω-omitting types property. In Keisler [1971] it is shown that also $\mathscr{L}_{\omega_1\omega}$ has the ω-omitting types property. A logic with the ω-omitting types property has the Löwenheim–Skolem property for countable sets of sentences: Given a countable and satisfiable set Φ, apply the ω-omitting types property to the ω-free type $\Gamma(x)$ of Φ, where $\Gamma(x) := \{\neg x = c_n \mid n \in \omega\}$ for new constants c_n.

The κ-omitting types property is strongly related to the construction method of models from constants (the reader should consult Barwise [1980] where a different notion of omitting types property—more precisely, of ω-omitting types property—is introduced, which is more sensitive to the specific features of a given logic). Using the method of construction of a model from constants, one can show that $\mathscr{L}_{\omega\omega}$ has the κ-omitting types property for all κ (see Chang–Keisler [1977]). Moreover, we have

2.2.2 Theorem. *If κ is an uncountable regular cardinal and $\mathscr{L}$ has the κ-omitting types property then $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$.*

Proof. First, we show

(∗) Suppose the set Φ of $\mathscr{L}$-sentences, $|\Phi| \leq \kappa$, has a model $\mathfrak{A}$ such that $(U^{\mathfrak{A}}, \leq^{\mathfrak{A}})$ is an ordering without last element. Then Φ has a model $\mathfrak{B}$ such that $(U^{\mathfrak{B}}, \leq^{\mathfrak{B}})$ is an ordering of cofinality κ.

To prove (∗), take new constants c_α, $\alpha < \kappa$; then

$$\Gamma(x) = \{Ux\} \cup \{c_\alpha \leq x \mid \alpha < \kappa\}$$

is a κ-free type of

$$\Phi_1 = \Phi \cup \{\text{“} < \text{is an ordering of } U \text{ without last element”}\}$$
$$\cup \{c_\beta \leq c_\alpha \mid \beta < \alpha < \kappa\}.$$

In fact, if $|\Psi(x)| < \kappa$ and $\Phi_1 \cup \Psi(x) \cup \{Ux\}$ is satisfiable, then choose α sufficiently large such that c_β does not occur in $\Psi(x)$ for $\beta > \alpha$. In a model of $\Phi_1 \cup \Psi(x) \cup \{Ux\}$, all these c_β may be interpreted by a fixed element bigger than x. Thus $\Phi_1 \cup \Psi(x) \cup \{\neg c_{\alpha+1} \leq x\}$ is satisfiable. Now, since $\mathscr{L}$ has the κ-omitting types property there is a model $\mathfrak{B}$ of Φ_1 omitting $\Gamma(x)$. But then $(U^{\mathfrak{B}}, \leq^{\mathfrak{B}})$ has cofinality κ.

In particular, (∗) shows that the ordering $(\omega, <)$ is not RPC in $\mathscr{L}$. Using Lemma 2.1.2, we see that in case $\mathscr{L}_{\omega\omega} < \mathscr{L}$, the logic $\mathscr{L}$ does not have the Karp property; that is, there are $\mathfrak{A}$ and $\mathfrak{B}$ such that

$(^*_*)$ $\quad \mathfrak{A} \cong_p \mathfrak{B} \quad \text{and} \quad \mathfrak{A} \not\equiv_{\mathscr{L}} \mathfrak{B}.$

We will code $(^*_*)$ in a model in such a way that use of the κ-omitting types property leads to isomorphic but not $\mathscr{L}$-equivalent structures—a contradiction. Choose an $\mathscr{L}$-sentence ψ such that $\mathfrak{A} \models \psi$ and $\mathfrak{B} \models \neg\psi$. Let c_α, d_α, p_α, for $\alpha < \kappa$, be new constants and V, W, I be new unary relation symbols. Let Φ be a set of $\mathscr{L}$-sentences, $|\Phi| = \kappa$, expressing the following:

“$V \cap W = \varnothing$”,

“the V-part is a model of ψ”,

“the W-part is a model of $\neg\psi$”,

“the V-part and the W-part are partially isomorphic via I”,

“Ip_α, c_α is in the domain of p_α and d_α in the range of p_α” for $\alpha < \kappa$,

“p_β is an extension of p_α” for $\alpha < \beta < \kappa$.

By $(^*_*)$, Φ is satisfiable (choose a partial isomorphism p in I where $I\colon \mathfrak{A} \cong_p \mathfrak{B}$, p with non-empty domain, say $a \in \mathrm{dom}(p)$, and set for all α, $p_\alpha = p$, $c_\alpha = a$, and $d_\alpha = p(a)$).

Let $\Gamma(x)$ be the type

$$\Gamma(x) = \{Vx \vee Wx\} \cup \{\neg x = c_\alpha \wedge \neg x = d_\beta \mid \alpha, \beta < \kappa\}.$$

Clearly, in a model of Φ omitting $\Gamma(x)$, the function $\bigcup_{\alpha<\kappa} p_\alpha$ is an isomorphism of the V-part onto the W-part. Therefore, it suffices to prove that $\Gamma(x)$ is a κ-free type of Φ.

Let $\Psi(x)$ be a set of $\mathscr{L}$-formulas, $|\Psi(x)| < \kappa$ and suppose $\Phi \cup \Psi(x)$ is satisfiable, say $\mathfrak{C} \models \Phi$ and $\mathfrak{C} \models \Psi[a]$. We must show that $\Phi \cup \Psi(x) \cup \{\neg\chi(x)\}$ has a model for some $\chi(x) \in \Gamma(x)$. If $a \notin V^C \cup W^C$, then $\mathfrak{C} \models \neg\chi[a]$ for $\chi = Vx \vee Wx \in \Gamma(x)$. Let $a \in V^C \cup W^C$, say $a \in V^C$. Choose $\alpha < \kappa$ large enough so that for $\beta > \alpha$, the constants p_β, c_β and d_β do not occur in $\Psi(x)$. Using the forth property, we see that there is a partial isomorphism q in the model extending p_α and with a in its domain. For $\beta > \alpha$, change the interpretation of p_β to q, of c_β to a, and of d_β to $q(a)$. This shows that $\Phi \cup \Psi(x) \cup \{x = c_{\alpha+1}\}$ is satisfiable. $\square$

2.3. Compact Sublanguages of $\mathscr{L}_{\infty\omega}$

Let φ_0 be an $\mathscr{L}_{\omega_1\omega}$-sentence and denote by $\mathscr{L}_{\omega\omega}(\varphi_0)$ the smallest set of sentences containing φ_0 and closed under first-order operations. Clearly, $\mathscr{L}_{\omega\omega}(\varphi_0)$ has the Löwenheim–Skolem property. But, in general, $\mathscr{L}_{\omega\omega}(\varphi_0)$ does not have the renaming property. Therefore, in case $\mathscr{L}_{\omega\omega}(\varphi_0)$ is countably compact, we cannot apply the theorems already proven to conclude that $\mathscr{L}_{\omega\omega}(\varphi_0) \equiv \mathscr{L}_{\omega\omega}$, and hence that φ_0 is equivalent to a first-order sentence. Indeed we will show that there is a φ_0 such that $\mathscr{L}_{\omega\omega}(\varphi_0)$ is countably compact but stronger than first-order logic. On the other hand, if $\mathscr{L}_{\omega\omega}(\varphi_0)$ is assumed to be compact (that is, is fully compact and not merely countably compact) then φ_0 already expresses a first-order property. Finally, we will see that this result does not generalize to $\mathscr{L}_{\infty\omega}$: There is $\varphi_0 \in \mathscr{L}_{\infty\omega}$ such that $\mathscr{L}_{\omega\omega}(\varphi_0)$ properly extends $\mathscr{L}_{\omega\omega}$ and is compact.

To be precise, for an $\mathscr{L}_{\infty\omega}[\boldsymbol{\sigma}]$-sentence, define $\mathscr{L} = \mathscr{L}_{\omega\omega}(\varphi_0)$ by

$$\mathscr{L}[\boldsymbol{\tau}] = \begin{cases} \varnothing, & \text{if } \boldsymbol{\sigma} \not\subset \boldsymbol{\tau}, \\ \text{smallest subset of } \mathscr{L}_{\infty\omega}[\boldsymbol{\tau}] \text{ containing } \varphi_0 \text{ and the} \\ \text{atomic } \mathscr{L}_{\omega\omega}[\boldsymbol{\tau}]\text{-formulas and closed under first-} \\ \text{order operations (say } \neg, \vee, \exists x), & \text{if } \boldsymbol{\sigma} \subset \boldsymbol{\tau}. \end{cases}$$

Given any $\varphi_0 \in \mathscr{L}_{\infty\omega}[\boldsymbol{\sigma}]$ set $\mathfrak{K}_1 = \mathrm{Mod}^{\boldsymbol{\sigma}}(\varphi_0)$ and $\mathfrak{K}_2 = \mathrm{Mod}^{\boldsymbol{\sigma}}(\neg\varphi_0)$. Then $\mathscr{L}_{\omega\omega}(\varphi_0)$ is compact (countably compact) if and only if $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are compact (countably compact). Here a class $\mathfrak{K}$ of $\boldsymbol{\sigma}$-structures is called *compact* (*countably compact*) if the following holds: Given any set of $\mathscr{L}_{\omega\omega}[\boldsymbol{\tau}]$-sentences Φ, with $|\Phi| = \aleph_0$, where $\boldsymbol{\sigma} \subset \boldsymbol{\tau}$, if every finite subset of Φ has a model with $\boldsymbol{\sigma}$-reduct in $\mathfrak{K}$, then so has Φ.

2.3.1 Example. We will give an example of an $\mathscr{L}_{\omega_1\omega}$-sentence φ_0 such that $\mathscr{L}_{\omega\omega}(\varphi_0)$ is a proper countably compact extension of $\mathscr{L}_{\omega\omega}$. Let each natural number code in an one-to-one and effective way a finite sequence of natural numbers. Define the binary relation $\prec$ on ω by

$n \prec m$ iff the sequence corresponding to n is an initial segment of the one corresponding to m.

There is a recursive functional T which assigns to each $X \subset \omega$ a tree $(T(X), \prec^{T(X)}) \subset (\omega, \prec)$ recursive in X with an infinite branch but with no branch hyperarithmetic in X (see Rogers [1967]). In particular, for $n \in \omega$, there is $p(n) \subset P_\omega(\omega) \times P_\omega(\omega)$, where $P_\omega(\omega)$ denotes the set of finite subsets of ω, such that for any $X \subset \omega$

$$\text{(1)} \qquad n \in T(X) \quad \text{iff} \quad \text{there is } (X_1, X_2) \in p(n) \text{ with } X_1 \subset X \text{ and } X_2 \cap X = \emptyset.$$

Moreover, the binary relation $\mathfrak{R}$ on $P(\omega)$, the power set of ω, given by

$$\mathfrak{R}XY \quad \text{iff} \quad Y \text{ is an infinite branch of } T(X)$$

has the property

$$\text{(2)} \qquad \begin{aligned} &\forall X \in P(\omega)\, \exists Y \in P(\omega) \mathfrak{R}XY, \\ &\forall X \in P(\omega) \neg \exists Y \in P(\omega)\, (\mathfrak{R}XY \text{ and } Y \text{ hyperarithmetic in } X). \end{aligned}$$

Let $\boldsymbol{\sigma} = \{R_n \mid n \in \omega\}$, where R_n are unary relation symbols and let $\mathfrak{A}_0$ be the $\boldsymbol{\sigma}$-structure $(P(\omega), (R_n^{A_0})_{n \in \omega})$, where

$$R_n^{A_0}X \quad \text{iff} \quad n \in X.$$

By (1) we have for $n \in \omega$ and $X \in P(\omega)$,

$$\mathfrak{A}_0 \models \psi_n[X] \quad \text{iff} \quad n \in T(X),$$

where $\psi_n(x)$ is the $\mathscr{L}_{\omega_1\omega}(\boldsymbol{\sigma})$-formula

$$\psi_n(x) = \bigvee_{(X_1, X_2) \in p(n)} \left(\bigwedge_{m \in X_1} R_m x \wedge \bigwedge_{m \in X_2} \neg R_m x \right).$$

Now

$$\mathfrak{A}_0 \models \varphi[X, Y] \quad \text{iff} \quad \mathfrak{R}XY$$

holds for

$$\varphi(x, y) = \left(\bigwedge_{n \in \omega} \bigvee_{\substack{m \in \omega \\ n < m}} R_m y \right) \wedge \bigwedge_{n \in \omega} (R_n y \to \psi_n(x))$$

$$\wedge \bigwedge_{n \in \omega} \left(R_n y \to \bigwedge_{\substack{m \in \omega \\ m \prec n}} (R_m y \leftrightarrow \psi_m(x)) \right).$$

Moreover, one can easily verify that for any $\boldsymbol{\sigma}$-structure $\mathfrak{A}$ and $a, b \in A$,

$$\mathfrak{A} \models \psi_n[a] \quad \text{iff} \quad n \in T(\{m \mid R_m^A a\})$$

and hence, we have

$$(3) \qquad \mathfrak{A} \models \varphi[a, b] \quad \text{iff} \quad \mathfrak{R}\{n \mid R_n^A a\}\{n \mid R_n^A b\}.$$

Finally, take as φ_0 the $\mathscr{L}_{\omega_1\omega}[\boldsymbol{\sigma}]$-sentence

$$\varphi_0 = \bigwedge \mathrm{Th}(\mathfrak{A}_0) \wedge \forall x\, \exists y\, \varphi(x, y),$$

where $\mathrm{Th}(\mathfrak{A}_0)$, the theory of $\mathfrak{A}_0$, denotes the set of first-order sentences holding in $\mathfrak{A}_0$.

Clearly, φ_0 is not equivalent to a first-order sentence. But $\mathscr{L}_{\omega\omega}(\varphi_0)$ is countably compact: Set $\mathfrak{K}_1 = \mathrm{Mod}^{\boldsymbol{\sigma}}(\varphi_0)$ and $\mathfrak{K}_2 = \mathrm{Mod}^{\boldsymbol{\sigma}}(\neg \varphi_0)$. To prove that $\mathfrak{K}_1$ is countably compact (even compact) it suffices to show that every ω-saturated model $\mathfrak{A}$ of $\mathrm{Th}(\mathfrak{A}_0)$ is a model of $\forall x\, \exists y\, \varphi(x, y)$. But for each $Y \subset \omega$, $\mathfrak{A}$ being ω-saturated contains an element a such that $Y = \{n \mid R_n^A a\}$. Then by (2) and (3),

$$\mathfrak{A} \models \forall x\, \exists y\, \varphi(x, y).$$

Toprove that $\mathfrak{K}_2 = \mathrm{Mod}^{\boldsymbol{\sigma}}(\neg \varphi_0)$ is countably compact, it suffices to show that if $\Phi \cup \mathrm{Th}(\mathfrak{A}_0)$ is satisfiable, where Φ is a countable set of first-order sentences, then there is a model of $\Phi \cup \mathrm{Th}(\mathfrak{A}_0) \cup \{\neg \forall x\, \exists y\, \varphi(x, y)\}$: Take a subset $X \subset \omega$ such that $\Phi \cup \mathrm{Th}(\mathfrak{A}_0)$ is recursive in X. Inside $\mathrm{Hyp}(X)$, the smallest admissible set containing X, construct a model $\mathfrak{B}$ of

$$\Phi \cup \mathrm{Th}(\mathfrak{A}_0) \cup \{R_n c \mid n \in X\} \cup \{\neg R_n c \mid n \notin X\},$$

where c is a new constant. $\mathrm{Hyp}(X)$ only contains subsets of ω hyperarithmetic in X. Therefore by (2), $\mathfrak{B} \models \neg \exists y\, \varphi(c, y)$. ☐

On the other hand we have:

2.3.2 Theorem. *Suppose φ_0 is an $\mathscr{L}_{\infty\omega}[\boldsymbol{\sigma}]$-sentence for some countable $\boldsymbol{\sigma}$. If $\mathscr{L}_{\omega\omega}(\varphi_0)$ is compact, then φ_0 is equivalent to a first-order sentence.*

Observe that for each $\mathscr{L}_{\omega_1\omega}$-sentence φ_0, there is some countable $\boldsymbol{\sigma}$ such that $\varphi_0 \in \mathscr{L}_{\omega_1\omega}[\boldsymbol{\sigma}]$.

Proof. First, we prove:

(*) Suppose that $\mathrm{Mod}(\varphi)$ is compact, where $\varphi \in \mathscr{L}_{\infty\omega}[\boldsymbol{\sigma}]$ and $|\boldsymbol{\sigma}| \leq \aleph_0$. If $\mathfrak{A} \models \varphi$ then there is an ω-saturated $\mathfrak{A}'$ such that $\mathfrak{A}' \equiv \mathfrak{A}$ and $\mathfrak{A}' \models \varphi$.

To establish this, for each $(n+1)$-type $p \subset \mathscr{L}_{\omega\omega}[\boldsymbol{\sigma}]$,

$$p = \{\psi_m(x_1, \ldots, x_n, y) \mid m \in \omega\},$$

take a new n-ary function symbol f_p. Now, set

$$\Phi = \left\{\forall x_1 \ldots \forall x_n \left(\exists y \bigwedge_{i \le m} \psi_i(x_1, \ldots, x_n, y) \right.\right.$$
$$\left. \to \bigwedge_{i \le m} \psi_i(x_1, \ldots, x_n, f_p(x_1, \ldots, x_n))\right)$$
$$\left. \mid m, n \in \omega,\ p = \{\psi_m(x_1, \ldots, x_n, y) \mid m < \omega\} (n+1)\text{-type} \right\}.$$

Clearly, $\mathrm{Th}(\mathfrak{A}) \cup \Phi \cup \{\varphi\}$ is finitely satisfiable and hence satisfiable, say $\mathfrak{B} \models \mathrm{Th}(\mathfrak{A}) \cup \Phi \cup \{\varphi\}$. Let $\mathfrak{A}' = \mathfrak{B} \restriction \boldsymbol{\sigma}$. Then $\mathfrak{A}' \equiv \mathfrak{A}$, $\mathfrak{A}' \models \varphi$, and $\mathfrak{A}'$ is ω-saturated since $\mathfrak{B} \models \Phi$.

Now let φ_0 and $\boldsymbol{\sigma}$ be given as in the theorem and suppose that $\mathscr{L}_{\omega\omega}(\varphi_0)$ is compact. By (the proof of) Proposition 1.1.8, it suffices to show that

$$\mathfrak{A} \equiv \mathfrak{B} \quad \text{implies} \quad \mathfrak{A} \equiv_{\mathscr{L}_{\omega\omega}(\varphi_0)} \mathfrak{B},$$

or, equivalently, that

$$\mathfrak{A} \equiv \mathfrak{B} \quad \text{implies} \quad (\mathfrak{A} \models \varphi_0 \text{ iff } \mathfrak{B} \models \varphi_0).$$

For the sake of argument, suppose that $\mathfrak{A} \models \varphi_0$ and $\mathfrak{B} \models \neg\varphi_0$. Applying (*) twice, we obtain ω-saturated $\mathfrak{A}'$ and $\mathfrak{B}'$ such that

$$\mathfrak{A}' \equiv \mathfrak{B}', \qquad \mathfrak{A}' \models \varphi_0 \quad \text{and} \quad \mathfrak{B}' \models \neg\varphi_0.$$

But this is a contradiction, since any two ω-saturated elementarily equivalent models are $\mathscr{L}_{\infty\omega}$-equivalent. □

2.3.3 Example. We will now show that Theorem 2.3.2 does not remain valid, when we drop the assumption that $\boldsymbol{\sigma}$ is countable. In fact, we can give an example of a sentence $\varphi_0 \in \mathscr{L}_{\omega_2\omega}$ such that $\mathscr{L}_{\omega\omega}(\varphi_0)$ properly extends $\mathscr{L}_{\omega\omega}$ and is compact. For $\alpha < \omega_1$, let R_α be a binary relation symbol and set $\boldsymbol{\sigma} = \{R_\alpha \mid \alpha < \omega_1\}$. Call a pair $\mathscr{F} = (\mathscr{F}_0, \mathscr{F}_1)$ of finite sets $\mathscr{F}_0$ and $\mathscr{F}_1$ of non-empty finite subsets of ω_1 *good*, if $F \not\subset E$ holds for all $E \in \mathscr{F}_0$ and $F \in \mathscr{F}_1$. (E, F, $E^\alpha, \ldots$ will always denote finite non-empty subsets of ω_1). Denote by $\varphi_{\mathscr{F}}(x)$ the formula

$$\varphi_{\mathscr{F}}(x) = \bigwedge_{E \in \mathscr{F}_0} \exists y \bigwedge_{\alpha \in E} R_\alpha xy \wedge \bigwedge_{F \in \mathscr{F}_1} \neg \exists y \bigwedge_{\alpha \in F} R_\alpha xy$$

and set $\Phi_0 = \{\exists x\, \varphi_{\mathscr{F}}(x) \mid \mathscr{F} \text{ good}\}$. Let φ_0 be the sentence

$$\varphi_0 = \bigwedge \Phi_0 \to \psi_0,$$

where

$$\psi_0 = \forall x\left(\left(\bigwedge_F \exists y \bigwedge_{\alpha \in F} R_\alpha xy\right) \to \exists y \bigwedge_{\alpha < \omega_1} R_\alpha xy\right).$$

Clearly, $\varphi_0 \in \mathscr{L}_{\omega_2\omega}$. Furthermore, (1) and (2) below show that $\mathscr{L}_{\omega\omega}(\varphi_0)$ properly extends $\mathscr{L}_{\omega\omega}$ and is compact.

(1) φ_0 is not equivalent to a first-order sentence.

To show (1), we prove that there are elementarily equivalent structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \models \neg\varphi_0$ and $\mathfrak{B} \models \varphi_0$. Choose an enumeration $\langle \mathscr{F}^\beta \mid \beta < \omega_1 \rangle$ of all good pairs, say, $\mathscr{F}^\beta = (\mathscr{F}_0^\beta, \mathscr{F}_1^\beta)$ with $\mathscr{F}_0^\beta = \{E_1^\beta, \ldots, E_{m_\beta}^\beta\}$. Also let the $\boldsymbol{\sigma}$-structure $\mathfrak{A}$ be given by:

$$A = \omega_1 \cup \{\omega_1\}, \quad \text{and}$$

$$R_\alpha^A \omega_1 \gamma \quad \text{iff} \quad \alpha < \gamma < \omega_1, \qquad \text{and for } \beta < \omega_1\text{:}$$

$$R_\alpha^A \beta\gamma \quad \text{iff} \quad \gamma < m_\beta \quad \text{and} \quad \alpha \in E_{\gamma+1}^\beta.$$

Then $\mathfrak{A} \models \varphi_{\mathscr{F}^\beta}[\beta]$. Hence $\mathfrak{A} \models \Phi_0$, and $x := \omega_1$ shows that $\mathfrak{A} \not\models \psi_0$. Therefore, $\mathfrak{A} \not\models \varphi_0$. On the other hand, any ω-saturated structure $\mathfrak{B}$ elementarily equivalent to $\mathfrak{A}$ is a model of ψ_0 and hence of φ_0 also.

Set $\mathfrak{K}_1 = \mathrm{Mod}^{\boldsymbol{\sigma}}(\varphi_0)$ and $\mathfrak{K}_2 = \mathrm{Mod}^{\boldsymbol{\sigma}}(\neg\varphi_0)$.

(2) $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are compact.

Since any ω-saturated structure is a model of ψ_0, the class $\mathfrak{K}_1$ is compact. Now, assume that $\Phi \cup \{\neg\varphi_0\}$ is finitely satisfiable, where $\Phi \subset \mathscr{L}_{\omega\omega}[\boldsymbol{\tau}]$ with $\boldsymbol{\sigma} \subset \boldsymbol{\tau}$. We must show that $\Phi \cup \{\neg\varphi_0\}$ has a model. We may assume that the consistent set $\Phi \cup \Phi_0$ has built-in Skolem functions. Let $\Gamma_0(x)$ be maximal among the types $\Gamma(x)$, $\Gamma(x) \subset \mathscr{L}_{\omega\omega}[\boldsymbol{\tau}]$, with the property:

For any good $\mathscr{F}$: $\Phi \cup \Phi_0 \cup \Gamma(x) \cup \{\varphi_{\mathscr{F}}(x)\}$ is consistent.

(Note that $\Gamma(x) := \varnothing$ has this property.) By first-order compactness, there is a model $\mathfrak{A}$ and $a \in A$ such that

$$\mathfrak{A} \models \Phi \cup \Phi_0 \quad \text{and} \quad \mathfrak{A} \models \Gamma_0(x) \cup \left\{\bigwedge_F \exists y \bigwedge_{\alpha \in F} R_\alpha xy\right\}[a].$$

Let $\mathfrak{B}$ be the submodel generated by a. We will complete the proof by showing that $\mathfrak{B} \models \Phi \cup \{\neg\varphi_0\}$. Since $\mathfrak{B} \models \Phi \cup \Phi_0$, it suffices to prove that $\mathfrak{B} \not\models \neg\psi_0$.

In fact, we show $\mathfrak{B} \not\models \exists y \bigwedge_{\alpha<\omega_1} R_\alpha xy[a]$, where this assertion is obtained proving that for any unary Skolem function f

there is some α such that $\neg R_\alpha xf(x) \in \Gamma_0(x)$.

Otherwise, for each α there is a good $\mathscr{F}^{\beta(\alpha)}$ such that

$$(*) \qquad \Phi \cup \Phi_0 \cup \Gamma_0(x) \cup \{\varphi_{\mathscr{F}^{\beta(\alpha)}}(x)\} \models R_\alpha xf(x).$$

But then by a combinatorial argument which uses a result of Erdös and Hajnal, one obtains $\alpha, \alpha' \in \omega_1, \alpha \neq \alpha'$ such that for $\beta := \beta(\alpha)$ and $\beta' := \beta(\alpha')$ the following hold:

$$\alpha \notin E \quad \text{for } E \in \mathscr{F}_0^{\beta'}, \qquad \alpha' \notin E \quad \text{for } E \in \mathscr{F}_0^{\beta}$$
$$E \not\subset F \quad \text{for } (E, F) \in (\mathscr{F}_0^{\beta} \times \mathscr{F}_1^{\beta'}) \cup (\mathscr{F}_0^{\beta'} \times \mathscr{F}_0^{\beta}).$$

Hence, $\mathscr{F} = (\mathscr{F}_0^{\beta} \cup \mathscr{F}_0^{\beta'}, \mathscr{F}_1^{\beta} \cup \mathscr{F}_1^{\beta'} \cup \{\alpha, \alpha'\})$ is good, and $\models \varphi_{\mathscr{F}}(x) \to (\varphi_{\mathscr{F}^\beta}(x) \wedge \varphi_{\mathscr{F}^{\beta'}}(x))$,

But then, using $(*)$, we obtain

$$\Phi \cup \Phi_0 \cup \Gamma_0(x) \cup \{\varphi_{\mathscr{F}}(x)\} \models R_\alpha xf(x) \wedge R_{\alpha'} xf(x).$$

But this is a contradiction, since $\models \varphi_{\mathscr{F}}(x) \to \neg \exists y(R_\alpha xy \wedge R_{\alpha'} xy)$.

2.3.4 Notes. Nearly all results of Section 2.1 are contained in Barwise [1974a] or in Lindström's papers [1966a, 1969]. The characterizations of $\mathscr{L}_{\omega\omega}$ in Section 2.2 are due to Lindström [1973a, 1974]. The reader will find a further interesting characterization of $\mathscr{L}_{\omega\omega}$ in Barwise–Moschovakis [1978]: $\mathscr{L}_{\omega\omega}$ is the unique logic with "uniformly inductive" satisfaction relation. Observe also that criteria for first-order axiomatizability of classes of structures such as

$\mathfrak{K}$ is an elementary class iff $\mathfrak{K}$ and its complement are closed under ultraproducts and isomorphisms,

may be rewritten as characterizations of first-order logic. We owe Example 2.3.1 and Theorem 2.3.2 to Gold [1978]. Example 2.3.3 is due to Ziegler (personal communication).

3. *Characterizing* $\mathscr{L}_{\infty\omega}$

This section is devoted to characterizations of $\mathscr{L}_{\infty\omega}$ by means of model-theoretic properties.

The property of a logic $\mathscr{L}$ of being *bounded* is a weakening of the compactness property ($\mathscr{L}$ is bounded, if for any $\mathscr{L}$-sentence $\varphi(<, \ldots)$ having only models with well-ordered $<$, there is an ordinal α such that the order type of $<$ is always less than α). As has been already mentioned in Chapter II this property may be regarded as a model-theoretic substitute for compactness. In fact, for some bounded logics results on non-axiomatizability, preservation thorems, upward Löwenheim–Skolem theorems and so on may be obtained in a way similar to the corresponding results for first-order logic provided one replaces compactness arguments by suitable applications of the boundedness property. This is also illustrated by the proof of Theorem 3.1 given below—a proof the reader should compare with the proof of Lemma 1.1.2.

One may regard the almost-all Löwenheim–Skolem property—the so-called *Kueker property*, which is introduced below—as a substitute for the Löwenheim–Skolem property in this model theoretic sense. Based on an interesting set-theoretical notion of countable approximations to uncountable objects, the Kueker property acts symmetrically on models and sentences. The reader should examine Kueker [1977, 1978] for a more penetrating view of the role of this property in model theory.

$\mathscr{L}_{\infty\omega}$ is bounded and has the Kueker property; and if the compactness and Löwenheim–Skolem property in Lindström's theorem are replaced by these substitutes, we obtain a characterization of $\mathscr{L}_{\infty\omega}$ as a maximal logic. We will derive this result as a consequence of Theorem 3.1, a theorem which shows that $\mathscr{L}_{\infty\omega}$ is a maximal bounded logic with the Karp property. The reader should also consult Chapter XVII, where these results are discussed from a set-theoretical point of view and where further characterizations of $\mathscr{L}_{\infty\omega}$ are obtained.

First, we define $\mathscr{L}_{\infty\omega}$-sentences which characterize the "α-isomorphism type" of a structure: Given an arbitrary $\boldsymbol{\tau}$ and a $\boldsymbol{\tau}$-structure $\mathfrak{A}$, for each ordinal α, we introduce an $\mathscr{L}_{\infty\omega}[\boldsymbol{\tau}]$-sentence $\varphi^\alpha_{\mathfrak{A}}$ such that for any $\mathfrak{B}$ the following are equivalent (compare Chapter VIII or Section II.4.2, in which for finite α, the corresponding formulas are introduced for the logic $\mathscr{L}_{\omega\omega}(Q_{\mathfrak{R}})$ with monotone $Q_{\mathfrak{R}}$):

(i) $\mathfrak{B} \models \varphi^\alpha_{\mathfrak{A}}$.
(ii) $\mathfrak{A} \cong_\alpha \mathfrak{B}$ (that is, $\mathfrak{A}$ and $\mathfrak{B}$ are α-isomorphic).
(iii) $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same $\mathscr{L}_{\infty\omega}$-sentences of rank $\leq\alpha$.

To define $\varphi^\alpha_{\mathfrak{A}}$, we first introduce by induction on α, for each finite sequence $\mathbf{a} = a_1 \ldots a_n \in A$, an $\mathscr{L}_{\infty\omega}[\boldsymbol{\tau}]$-formula $\varphi^\alpha_{\mathbf{a}}(x_1, \ldots, x_n)$:

$$\begin{aligned}\varphi^0_{\mathbf{a}} = \bigwedge \{\psi(x_1, \ldots, x_n) \mid \mathfrak{A} \models \psi[\mathbf{a}] &\text{ and } \psi \text{ has the form}\\ &(\neg)\, Rx_{i_1} \ldots x_{i_j} \text{ or } (\neg) f(x_{i_1}, \ldots, x_{i_j}) = x_i \text{ or}\\ &(\neg)\, c = x_i \text{ or } (\neg)\, x_j = x_i\},\end{aligned}$$

$$\varphi^{\alpha+1}_{\mathbf{a}} = \bigwedge_{a \in A} \exists x_{n+1}\, \varphi^\alpha_{\mathbf{a}a} \wedge \forall x_{n+1} \bigvee_{a \in A} \varphi^\alpha_{\mathbf{a}a},$$

$$\varphi^\alpha_{\mathbf{a}} = \bigwedge_{\beta<\alpha} \varphi^\beta_{\mathbf{a}} \quad \text{for a limit ordinal } \alpha.$$

Now let $\varphi^\alpha_{\mathfrak{A}}$ be the sentence $\varphi^\alpha_{\varnothing}$, where $\varnothing$ denotes the empty sequence. An easy induction on α shows the following: in case either $|\tau|$ or α is infinite there are not more than $\beth_{\alpha+1}(|\tau|)$ sentences (pairwise non-equivalent) of the form $\varphi^\alpha_{\mathfrak{A}}$ and each such sentence $\varphi^\alpha_{\mathfrak{A}}$ belongs to $\mathscr{L}_{\beth_\alpha(|\tau|)^+\omega}[\tau]$. Otherwise their number is finite and each is a first-order sentence. Recall that the sequence of beth cardinals $\beth_\alpha(\kappa)$, where κ is a cardinal and α an ordinal, is defined by: $\beth_0(\kappa) = \kappa$, $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$ and $\beth_\alpha(\kappa) = \sup\{\beth_\beta(\kappa) \mid \beta < \alpha\}$, if α is a limit ordinal. Write $\beth_\alpha$ for $\beth_\alpha(0)$; in particular, $\beth_\omega = \omega$.

We adapt the proof methods used in Section 1.1 to show

3.1 Theorem. *Assume $\mathscr{L}$ is a regular logic with $\mathscr{L}_{\infty\omega} \leq \mathscr{L}$. If $\mathscr{L}$ is bounded and has the Karp property, then $\mathscr{L} \equiv \mathscr{L}_{\infty\omega}$.*

Proof. By contradiction suppose that φ is an $\mathscr{L}$-sentence not equivalent to an $\mathscr{L}_{\infty\omega}$-sentence. For an ordinal α, let

$$\chi^\alpha = \bigvee \{\varphi^\alpha_{\mathfrak{A}} \mid \mathfrak{A} \models \varphi\}.$$

Then, by the preceding remarks, χ^α is an $\mathscr{L}_{\infty\omega}$-sentence and $\models \varphi \to \chi^\alpha$. Therefore $\not\models \chi^\alpha \to \varphi$. That is, for some $\mathfrak{B}_\alpha$, $\mathfrak{B}_\alpha \models \chi^\alpha$, but $\mathfrak{B}_\alpha \models \neg\varphi$. By the definition of χ^α there exists $\mathfrak{A}_\alpha$ such that $\mathfrak{A}_\alpha \models \varphi$ and $\mathfrak{B}_\alpha \models \varphi^\alpha_{\mathfrak{A}_\alpha}$. Hence, $\mathfrak{A}_\alpha \cong_\alpha \mathfrak{B}_\alpha$. Summarizing, we thus have:

$$(*) \quad \begin{array}{l} \text{for each ordinal } \alpha \text{ there are } \mathfrak{A}_\alpha \text{ and } \mathfrak{B}_\alpha \text{ such that} \\ \mathfrak{A}_\alpha \models \varphi, \qquad \mathfrak{B}_\alpha \models \neg\varphi \quad \text{and} \quad \mathfrak{A}_\alpha \cong_\alpha \mathfrak{B}_\alpha. \end{array}$$

Coding partial isomorphisms (as in the proof of Lemma 1.1.2), we obtain an $\mathscr{L}$-sentence which contains among others relation symbols V, W (unary) and $\leq$, I (binary), and which expresses:

> "the V-part is a model of φ, the W-part a model of $\neg\varphi$; $<$ is an ordering, for each x in its field $Ix\cdot$ is a non-empty set of partial isomorphisms from the V-part to the W-part, and the sequence $Ix\cdot$ with x in the field of $<$ has the back and forth property."

By $(*)$, for each ordinal α, ψ has a model such that $<$ is well-ordered of order type $\geq\alpha$. Since $\mathscr{L}$ is bounded, ψ has a non-well-ordered model $\mathfrak{D}$. Then $V^{\mathfrak{D}}$ is a model of φ, $W^{\mathfrak{D}}$ a model of $\neg\varphi$. And, as in the preceding proofs (see Lemma 1.1.2), by choosing an element in the field of $<$ with an infinite descending sequence of predecessors, one shows that $V^{\mathfrak{D}}$ and $W^{\mathfrak{D}}$ are partially isomorphic. But this is a contradiction, since $\mathscr{L}$ was assumed to have the Karp property. $\square$

Observe that in case $\mathscr{L}$ has the finite occurrence property, we can omit the hypothesis $\mathscr{L}_{\infty\omega} \leq \mathscr{L}$ in the preceding theorem and obtain $\mathscr{L} \leq \mathscr{L}_{\infty\omega}$ as conclusion. We state some results that are obtained by slight changes in the last proof. For $\kappa = \omega$ the following theorem is essentially the characterization of $\mathscr{L}_{\omega\omega}$ as given in Theorem 2.1.1. (Consult Section II.5.2 for the definition and properties of the well-ordering number of a logic.)

3.2 Theorem. *Suppose κ is a cardinal and $\kappa = \beth_\kappa$. Assume also that $\mathscr{L}$ with $\mathscr{L}_{\kappa\omega} \leq \mathscr{L}$ is a regular logic with occurrence number $\leq \kappa$. If the well-ordering number of $\mathscr{L}$ is $\leq \kappa$ and $\mathscr{L}$ has the Karp property, then $\mathscr{L} \equiv \mathscr{L}_{\kappa\omega}$.*

Proof. We employ the notations used in the proof of Theorem 3.1 and note that in case the well-ordering number of $\mathscr{L}$ is $\leq \kappa$ this proof shows that any $\varphi \in \mathscr{L}[\tau]$ with $|\tau| < \kappa$ is equivalent to some χ^α with $\alpha < \kappa$. For any $\beta, \lambda < \kappa$ we have $\beth_\beta(\lambda) \leq \beth_{\lambda+\beta} < \beth_\kappa = \kappa$. Hence, $\chi^\alpha \in \mathscr{L}_{\kappa\omega}$ by the above remarks on the number of non-equivalent sentences of the form $\varphi^\alpha_{\mathfrak{A}}$. $\square$

3.3 Remarks. (a) Clearly, one can generalize Theorems 3.1 and 3.2 in the spirit of the "separation theorem" 1.1.3 and, for example, derive: *Assume that $\mathscr{L}$ with $\mathscr{L}_{\infty\omega} \leq \mathscr{L}$ is a logic with the relativization property and closed under (finitary) conjunctions and disjunctions. If $\mathscr{L}$ is bounded and has the Karp property, then any two disjoint $\mathscr{L}$-classes can be separated by an $\mathscr{L}_{\infty\omega}$-class* (the reader should consult Makowsky–Shelah–Stavi [1976], where this result is stated for $\mathscr{L} = \Delta(\mathscr{L}_{\infty\omega})$).

In fact, if Mod(φ) and Mod(ψ) are disjoint $\mathscr{L}$-classes not separable by an $\mathscr{L}_{\infty\omega}$-class, for each α define χ^α as above. Then there are $\mathfrak{A}_\alpha$ and $\mathfrak{B}_\alpha$ such that $\mathfrak{A}_\alpha \models \varphi$, $\mathfrak{B}_\alpha \models \psi$, and $\mathfrak{B}_\alpha \models \varphi^\alpha_{\mathfrak{A}_\alpha}$, and we obtain a contradiction as in Theorem 3.1.

(b) Suppose $\mathscr{L}$ is a regular logic with the Karp property. For an $\mathscr{L}$-sentence φ and an ordinal α, let $\chi^\alpha = \bigvee\{\varphi^\alpha_{\mathfrak{A}} \mid \mathfrak{A} \models \varphi\}$. Then $\models(\bigwedge_{\alpha\,\text{ordinal}} \chi^\alpha) \to \varphi$. In fact, suppose for the sake of argument that for all α, $\mathfrak{B} \models \chi^\alpha$ and $\mathfrak{B} \models \neg\varphi$. Let $\kappa = |B|^+$. Choose $\mathfrak{A} \models \varphi$ such that $\mathfrak{A} \cong_\kappa \mathfrak{B}$. We show that $\mathfrak{A} \cong_p \mathfrak{B}$ which—in view of $\mathfrak{B} \models \neg\varphi$ and $\mathfrak{A} \models \varphi$—contradicts the assumption "$\mathscr{L}$ has the Karp property". From $\mathfrak{A} \cong_\kappa \mathfrak{B}$, we obtain $\mathfrak{A} \equiv_{\mathscr{L}_{\kappa\omega}} \mathfrak{B}$, since each $\mathscr{L}_{\kappa\omega}$-sentence has quantifier rank $< \kappa$. Hence, $\mathfrak{A} \equiv_{\mathscr{L}_{\infty\omega}} \mathfrak{B}$, because each $\mathscr{L}_{\infty\omega}$-sentence is equivalent in $\mathrm{Th}_{\mathscr{L}_{\kappa\omega}}(\mathfrak{B})$ to an $\mathscr{L}_{\kappa\omega}$-sentence (see Flum [1971c]). Thus $\mathfrak{A} \cong_p \mathfrak{B}$. Summarizing, we have shown: *Assume $\mathscr{L}$ is a logic with the Karp property. Then for any $\mathscr{L}$-sentence φ we have $\models \varphi \leftrightarrow \bigwedge_{\alpha\,\text{ordinal}} \chi^\alpha$, where $\chi^\alpha = \bigvee\{\varphi^\alpha_{\mathfrak{A}} \mid \mathfrak{A} \models \varphi\}$.*

Since $\mathscr{L}_{\infty G}$ is a logic with the Karp property, this result applies to $\mathscr{L}_{\infty G}$ (see Keisler [1968a] and compare with Chapter XVII for a more general version).

(c) For a generalized quantifier Q one can extend the preceding results to logics $\mathscr{L}$ of the form $\mathscr{L} = \mathscr{L}_{\infty\omega}(Q)$ or $\mathscr{L} = \mathscr{L}_{\kappa\omega}(Q)$, if there is an appropriate characterization of $\mathscr{L}$-equivalence by means of partial isomorphisms and if there are $\mathscr{L}$-sentences which play the rôle of the formulas $\varphi^\alpha_{\mathfrak{A}}$. For example, if $Q = Q_1$, that is, in case Q is the quantifier "there are uncountable many" and if we define the "$\aleph_1$-Karp property" as suggested by the corresponding back and forth notions for $\mathscr{L}_{\omega\omega}(Q_1)$ (see Section II.4.2), we then obtain (the reader is referred to Caicedo [1981b] for further results in this direction)

> If $\mathscr{L}$ with $\mathscr{L}_{\infty\omega}(Q_1) \leq \mathscr{L}$ is bounded and has the "$\aleph_1$-Karp property", then $\mathscr{L} \equiv \mathscr{L}_{\infty\omega}(Q_1)$.

From now on in this section we will assume that all logics under consideration are built up by set-theoretical principles so that their sentences are sets.

We will quickly review some definitions and results concerning the notions of approximations of sets and of the closed unbounded filter, and ask the reader to consult Barwise [1974b] for details.

We work in a universe of sets and urelements and define for any sets x and s the *approximation* x^s *of* x *in* s by $\in$-recursion:

$$p^s = p \quad \text{if } p \text{ is an urelement,}$$

$$x^s = \{y^s \mid y \in x \cap s\}, \quad \text{if } x \text{ is a set.}$$

Let M be a transitive set and let I be the set $P_{\omega_1}(M)$ of all countable subsets of M. The closed unbounded filter on M consists of all $X \subset I$ such that for some $X^0 \subset X$:

(i) every $s \in I$ is a subset of some $s' \in X^0$; and
(ii) X^0 is closed under unions of countable chains.

Let $\mathfrak{R}$ be an n-ary predicate of sets and urelements. For given $x_1, \ldots, x_n$ in a transitive set M, we say that $\mathfrak{R}x_1^s \ldots x_n^s$ *holds for almost all countable* s, if the set

$$\{s \in P_{\omega_1}(M) \mid \mathfrak{R}x_1^s \ldots x_n^s\}$$

is a member of the closed unbounded filter on M. This notion is independent of the particular transitive set M containing $x_1, \ldots, x_n$.

We say that a predicte $\mathfrak{R}$ of sets and urelements is Σ, if it is definable by a Σ-formula of set theory. Barwise [1974b] generalized Levy's Absoluteness Lemma and showed:

3.4 Proposition. *Let* $\mathfrak{R}$ *be an n-ary* Σ*-predicate. If* $\mathfrak{R}x_1 \ldots x_n$*, then* $\mathfrak{R}x_1^s \ldots x_n^s$ *for almost all countable s.* $\square$

We assume that vocabularies and universes of structures consist of urelements only. Then for almost all countable s:

$$\mathfrak{A}^s \text{ is the } \tau^s\text{-substructure of } \mathfrak{A} \upharpoonright \tau^s \text{ with universe } A \cap s,$$

and for any $\mathscr{L}_{\infty\omega}$-sentence φ and almost all s,

$$\varphi^s = \varphi^{[s]},$$

where $\varphi^{[s]} = \varphi$, if φ is atomic;

$$(\neg\varphi)^{[s]} = \neg\varphi^{[s]};$$

$$(\exists x\varphi)^{[s]} = \exists x\, \varphi^{[s]};$$

and

$$(\bigvee \Phi)^{[s]} = \bigvee \{\varphi^{[s]} \mid \varphi \in \Phi \cap s\}.$$

Here—as also in the proof of Theorem 3.6 below—we assume that the operations $\varphi \mapsto \neg\varphi$, $\varphi \mapsto \exists x\varphi$, ... are "simple" operations, say Σ-operations. Thus, for example, $(\neg\varphi)^s$ and $\neg(\varphi^s)$ are equal for almost all countable s.

"$\mathfrak{A} \cong_p \mathfrak{B}$" and "$\mathfrak{A} \models \varphi$" for an $\mathscr{L}_{\infty\omega}$-sentence φ are Σ-predicates of $\mathfrak{A}$ and $\mathfrak{B}$, resp. $\mathfrak{A}$ and φ. Therefore, using Proposition 3.4, we obtain (see Barwise [1974b]):

3.5 Proposition. (a) *If $\mathfrak{A} \cong_p \mathfrak{B}$, then $\mathfrak{A}^s \cong \mathfrak{B}^s$ for almost all countable s.*

(b) *If φ is an $\mathscr{L}_{\infty\omega}$-sentence, then*

$$\mathfrak{A} \models \varphi \quad \textit{implies} \quad \mathfrak{A}^s \models \varphi^s \textit{ for almost all countable } s. \quad \square$$

We say that a logic $\mathscr{L}$ has the *Kueker property*, if for any $\mathscr{L}$-sentence φ, $\mathfrak{A} \models \varphi$ implies $\mathfrak{A}^s \models \varphi^s$ for almost all countable s. Thus, in particular, we assume that φ^s is an $\mathscr{L}$-sentence for almost all countable s.

In particular, $\mathscr{L}_{\infty\omega}$ is a logic with the Kueker property. Moreover—as was announced in the troduction to this section—this property together with the boundedness property characterize $\mathscr{L}_{\infty\omega}$.

3.6 Theorem. *Let $\mathscr{L}$ be a regular logic with $\mathscr{L}_{\infty\omega} \leq \mathscr{L}$. If $\mathscr{L}$ is bounded and has the Kueker property, then $\mathscr{L} \equiv \mathscr{L}_{\infty\omega}$.*

Proof. By Theorem 3.1 it suffices to show that $\mathscr{L}$ has the Karp property. So let φ be an $\mathscr{L}$-sentence and suppose that $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \cong_p \mathfrak{B}$. For the sake of argument suppose that $\mathfrak{B} \models \neg\varphi$. Then, by Proposition 3.5 and the Kueker property, we have for almost all countable s

$$\mathfrak{A}^s \cong \mathfrak{B}^s, \qquad \mathfrak{A}^s \models \varphi^s \quad \text{and} \quad \mathfrak{B}^s \models \neg\varphi^s,$$

a contradiction. $\square$

What is the corresponding separation property of $\mathscr{L}_{\infty\omega}$? Let φ and ψ be sentences of a logic with the Kueker property. Consider the following properties (i) and (ii) of φ and ψ:

(i) $\mathrm{Mod}(\varphi) \cap \mathrm{Mod}(\psi) = \varnothing$;
(ii) $\mathrm{Mod}(\varphi^s) \cap \mathrm{Mod}(\psi^s) = \varnothing$ for almost all countable s.

Clearly, (ii) implies (i). However, in general, (i) does not imply (ii); for otherwise the next theorem would show that $\mathscr{L}_{\infty\omega}$ has the interpolation property. This theorem contains the separation result corresponding to the maximality result of Theorem 3.6.

3.7 Theorem. *Suppose $\mathscr{L}$ with $\mathscr{L}_{\infty\omega} \leq \mathscr{L}$ is a logic closed under finitary conjunctions and disjunctions and has the relativization property. Assume that $\mathscr{L}$ is bounded and has the Kueker property, and let φ and ψ be $\mathscr{L}$-sentences. If* $\mathrm{Mod}(\varphi^s) \cap \mathrm{Mod}(\psi^s) = \varnothing$ *for almost all countable s then for some $\chi \in \mathscr{L}_{\infty\omega}$ and almost all countable s*

$$\mathrm{Mod}(\varphi^s) \subset \mathrm{Mod}(\chi^s) \quad \textit{and} \quad \mathrm{Mod}(\chi^s) \cap \mathrm{Mod}(\psi^s) = \varnothing,$$

and consequently,

$$\mathrm{Mod}(\varphi) \subset \mathrm{Mod}(\chi) \quad \textit{and} \quad \mathrm{Mod}(\chi) \cap \mathrm{Mod}(\psi) = \emptyset.$$

Proof. For an ordinal α, let $\chi^\alpha = \bigvee \{\varphi^\alpha_{\mathfrak{A}} \mid \mathfrak{A} \models \varphi\}$. Then χ^α is an $\mathscr{L}_{\infty\omega}$- sentence with $\models \varphi \to \chi^\alpha$. If $\mathrm{Mod}(\chi^\alpha) \cap \mathrm{Mod}(\psi) = \emptyset$ holds for some α, then we let $\chi = \chi^\alpha$. Otherwise, for each α there are structures $\mathfrak{A}_\alpha$ and $\mathfrak{B}_\alpha$ such that $\mathfrak{A}_\alpha \models \varphi$, $\mathfrak{B}_\alpha \models \psi$, and $\mathfrak{A}_\alpha \cong_\alpha \mathfrak{B}_\alpha$. Using the boundedness property of $\mathscr{L}$ and arguing as in the proof of Theorem 3.1, we obtain structures $\mathfrak{A}$ and $\mathfrak{B}$ such that

$$\mathfrak{A} \models \varphi, \qquad \mathfrak{B} \models \psi \quad \text{and} \quad \mathfrak{A} \cong_p \mathfrak{B}.$$

Hence, $\mathfrak{A}^s \models \varphi^s$, $\mathfrak{B}^s \models \psi^s$, and $\mathfrak{A}^s \cong \mathfrak{B}^s$ for almost all countable s—a contradiction. $\square$

Taking as $\mathscr{L}$ the Σ^1_1-sentences over $\mathscr{L}_{\infty\omega}$, Theorem 3.7 above is Theorem 2 in Kueker [1978].

3.8 Notes. Theorems 3.1 and 3.2 are due to Barwise [1974a]. $\mathscr{L}_{\omega_1\omega}$ is a well-behaved logic with a fruitful model theory. For the problem of characterizing $\mathscr{L}_{\omega_1\omega}$, the reader is referred to Barwise [1972a], Gostanian-Hrbacek [1980], and Harrington [1980].

4. *Characterizing Cardinality Quantifiers*

In this section we characterize the logics $\mathscr{L}_{\omega\omega}(Q_\alpha)$ with the quantifier "there are $\aleph_\alpha$-many" among the logics of the form $\mathscr{L}_{\omega\omega}(Q)$, where Q is a unary quantifier.

Given a unary Lindström quantifier Q and a non-empty set A, let $Q(A)$ be the set of "big" subsets of A,

$$Q(A) = \{X \subset A \mid (A, X) \models QyUy\}.$$

In the terminology of Chapter II, Q is the quantifier associated with the class $\mathfrak{K} = \{(A, X) \mid A \neq \emptyset, X \in Q(A)\}$. Clearly

(1) $\quad$ if $\;(A, X) \cong (B, Y)$, $\;$ then $\;(X \in Q(A)$ iff $Y \in Q(B))$.

Throughout this section all quantifiers are assumed to be unary. We call a quantifier Q monotone, if

$$X \in Q(A) \quad \text{and} \quad X \subset Y \subset A \quad \text{imply} \quad Y \in Q(A).$$

We now list some examples of monotone quantifiers:

the existential quantifier $\exists$, $\exists(A) = \{X \subset A \mid X \neq \varnothing\}$,

the quantifier Q_α, $Q_\alpha(A) = \{X \subset A \mid |X| \geq \aleph_\alpha\}$,

the Chang quantifier Q_c, $Q_c(A) = \{X \subset A \mid |X| \geq \aleph_0, |X| = |A|\}$,

the "non-cofinal complement" quantifier Q_{ncc}, where

$$Q_{\mathrm{ncc}}(A) = \{X \subset A \mid |A \backslash X| < \mathrm{cf}(|X|)\}.$$

The dual quantifier Q^{d} of a quantifier Q is defined by

$$Q^{\mathrm{d}}(A) = \{X \subset A \mid (A \backslash X) \notin Q(A)\}.$$

Observe that $Q^{\mathrm{d}} y\varphi(y)$ is equivalent to $\neg Qy \neg \varphi(y)$ and that Q^{d} is monotone, if Q is monotone. Clearly, we have

$$(2) \qquad \mathscr{L}_{\omega\omega}(Q^{\mathrm{d}}) \equiv \mathscr{L}_{\omega\omega}(Q).$$

The main result of this section is the following characterization of the logics of the form $\mathscr{L}_{\omega\omega}(Q_\alpha)$.

4.1 Theorem. *Suppose $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q)$ is a regular logic where Q is a monotone quantifier. Then*

$$\mathscr{L} \equiv \mathscr{L}_{\omega\omega} \quad \text{or} \quad \mathscr{L} \equiv \mathscr{L}_{\omega\omega}(Q_\alpha) \quad \text{for some } \alpha.$$

As an immediate consequence of this theorem, we obtain:

4.2 Corollary. *Suppose $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q)$ with $\mathscr{L}_{\omega\omega} < \mathscr{L}$ is a regular logic, where Q is a monotone quantifier. If $\mathscr{L}$ has Löwenheim number $\aleph_\alpha$, then $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}(Q_\alpha)$.* □

($\mathscr{L}$ has Löwenheim number κ, if any satisfiable $\mathscr{L}$-sentence has a model of power $\leq \kappa$, and κ is the least cardinal with this property.)

To prove Theorem 4.1, we must introduce some terminology and notation.

For $n \in \omega$ let $\exists^{\geq n}$, Q_α^n, and Q_α^{cn} be the monotone quantifiers definable in $\mathscr{L}_{\omega\omega}$ and $\mathscr{L}_{\omega\omega}(Q_\alpha)$, respectively, by:

$\exists^{\geq n} x\varphi :\leftrightarrow$ "there are at least n elements x satisfying φ";

$Q_\alpha^n x\varphi :\leftrightarrow (Q_\alpha x\, x = x \to Q_\alpha x\varphi) \wedge (\neg Q_\alpha x\, x = x \to \exists^{\geq n} x\varphi)$;

$Q_\alpha^{cn} x\varphi :\leftrightarrow (Q_\alpha x\, x = x \to Q_\alpha x\varphi) \wedge (\neg Q_\alpha x\, x = x \to \exists^{< n} x \neg \varphi)$;

where $\exists^{<n}$ means "there are less than n".

Clearly, $\exists^{\geq n}$, Q_α^n, and Q_α^{cn} are monotone and

(3) $$\mathscr{L}_{\omega\omega}(\exists^{\geq n}) \equiv \mathscr{L}_{\omega\omega}; \qquad \mathscr{L}_{\omega\omega}(Q_\alpha^n) \equiv \mathscr{L}_{\omega\omega}(Q_\alpha); \qquad \mathscr{L}_{\omega\omega}(Q_\alpha^{cn}) \equiv \mathscr{L}_{\omega\omega}(Q_\alpha).$$

(For example, that $\mathscr{L}_{\omega\omega}(Q_\alpha) \leq \mathscr{L}_{\omega\omega}(Q_\alpha^n)$ holds is shown by

$$\models Q_\alpha x\varphi \leftrightarrow (Q_\alpha^n x\varphi \wedge \forall x_1 \dots \forall x_n \neg Q_\alpha^n y\,(y = x_1 \vee \cdots \vee y = x_n)).)$$

Let Q be an arbitrary monotone quantifier. By the isomorphism condition (1) stated at the beginning of this section, whether $X \in Q(A)$ holds or not only depends on the cardinalities of the sets A, X and $A \backslash X$. We associate with Q a function g $(= g^Q)$ defined on the class of non-zero cardinals which maps each cardinal $\lambda \neq 0$ on a pair of cardinals, $g(\lambda) = (\mu, \nu)$, where for any A with $|A| = \lambda$,

$$\mu = \lambda \quad \text{and} \quad \nu = 0, \quad \text{if } Q(A) = \varnothing,$$

and otherwise

$$\mu = \inf\{|X| \mid X \in Q(A)\}, \qquad \nu = \sup\{|A\backslash X|^+ \mid X \in Q(A)\}.$$

Then, by monotonicity,

$$Q(A) = \{X \subset A \mid |X| \geq \mu, |A\backslash X| < \nu\},$$

and hence Q is uniquely determined by g. Moreover, note that $\mu \leq \lambda$, $\nu \leq \lambda^+$ and $\mu + \nu \leq \lambda^+$.

In particular,

$$g^{Q_\alpha}(\lambda) = \begin{cases} (\lambda, 0) & \text{for } \lambda < \aleph_\alpha, \\ (\aleph_\alpha, \lambda^+) & \text{for } \lambda \geq \aleph_\alpha. \end{cases}$$

Given monotone quantifiers Q and Q', we say that Q and Q' are *eventually equal*, if there is $n_0 \in \omega$ such that for all $\lambda \geq n_0$, $g^Q(\lambda) = g^{Q'}(\lambda)$. Clearly,

(4) if Q and Q' are eventually equal, then $\mathscr{L}_{\omega\omega}(Q) \equiv \mathscr{L}_{\omega\omega}(Q')$.

In view of (2)–(4), Theorem 4.1 is an immediate consequence of the following lemma.

4.3 Lemma. *Suppose $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q)$ is a regular logic with a monotone quantifier Q. Then for some ordinal α and some $n \in \omega$, Q or its dual is eventually equal to*

$$\exists^{\geq n} \quad \text{or} \quad Q_\alpha^n \quad \text{or} \quad Q_\alpha^{cn}.$$

Proof. Denote by g the function g^Q. We establish the lemma by showing the following claims (i)–(v):

(i) If $\omega \leq \lambda < \mu$ and $g(\lambda) \neq (\lambda, 0)$, then $g(\mu) \neq (\mu, 0)$.
(ii) Suppose $\lambda \geq \omega$ and $n \in \omega$.

If $g(\lambda) = (\lambda, n)$ then there is $m_0 \in \omega$ such that for all $m \geq m_0$
$g(m) = (m - n + 1, n)$.

If $g(\lambda) = (n, \lambda^+)$ then there is $m_0 \in \omega$ such that for all $m \geq m_0$
$g(m) = (n, m - n + 1)$.

By (i) and (ii) we see that in case there is no $\lambda \geq \omega$ such that $g(\lambda) = (\mu, \nu)$ with infinite μ and ν, then Q or Q^{d} is eventually equal to $\exists^{\geq n}$.

Now, let $\lambda_0 = \inf \Lambda$ where $\Lambda = \{\lambda \mid g(\lambda) = (\mu, \nu)$ for some infinite $\mu, \nu\}$ is assumed to be non-empty.

(iii) $g(\lambda_0) = (\lambda_0, \lambda_0^+)$ or $g(\lambda_0) = (\lambda_0, \lambda_0)$.
(iv) If $\lambda_0 = \omega$ then for some m_0 and $n \in \omega$ we have
for all $m \geq m_0$, $\quad g(m) = (n, m - n + 1)$ or
for all $m \geq m_0$, $\quad g(m) = (m - n + 1, n)$.
(v) If $g(\lambda_0) = (\lambda_0, \lambda_0^+)$ then for $\lambda \geq \lambda_0$, $g(\lambda) = (\lambda_0, \lambda^+)$.

Let us show, for example, for the case $\omega < \lambda_0$, $g(\lambda_0) = (\lambda_0, \lambda_0)$ and $g(\omega) = (n, \omega^+)$, how we obtain from (i)–(v) the assertion of the lemma. For the dual quantifier Q^{d}, we have $g^{\mathrm{d}}(\lambda_0) = (\lambda_0, \lambda_0^+)$ and $g^{\mathrm{d}}(\omega) = (\omega, n)$. Hence, by (v)

$$g^{\mathrm{d}}(\lambda) = (\lambda_0, \lambda^+) \quad \text{for } \lambda \geq \lambda_0,$$

and by (ii) there is $m_0 \in \omega$ such that

$$g^{\mathrm{d}}(m) = (m - n + 1, n) \quad \text{for } m \geq m_0.$$

Thus for α with $\aleph_\alpha = \lambda_0$ we have

Q^{d} is eventually equal to Q_α^{cn}.

The proofs of (i)–(v) make essential use of the relativization property. We sketch the idea underlying these proofs. Suppose, for example, that $g(\lambda) = (\mu, \lambda^+)$, where $\mu = \aleph_\alpha$; that is, Q is the quantifier Q_α in models of power λ. Then each $\mathscr{L}_{\omega\omega}(Q)$-sentence is equivalent to an $\mathscr{L}_{\omega\omega}(Q_\alpha)$-sentence in models of power λ. Now for unary relations symbols U and P let φ be the relativization of $QxPx$ to U; that is, we let $\varphi = (QxPx)^U$. Then for $\mathfrak{A} = (A, U^A, P^A)$ with $U^A \supset P^A$ we have

$$(*) \qquad (A, U^A, P^A) \models \varphi \quad \text{iff} \quad P^A \in Q(U^A).$$

Let ψ be an $\mathscr{L}_{\omega\omega}(Q_\alpha)$-sentence equivalent to φ in models of power λ. By $(*)$, we obtain the possible values of $g^Q(\rho)$ for $\rho < \lambda$—if we determine the expressive power of $\mathscr{L}_{\omega\omega}(Q_\alpha)$-sentences in structures of cardinality λ of the above form. This can be done with the back-and-forth methods of Chapter II. $\square$

4.4 Remark. (a) One can use the idea of the preceding proof to determine the logics with the relativization property in more general cases, for example, in the cases of logics of the form $\mathscr{L}_{\omega\omega}(Q^1, \ldots, Q^n)$ with unary monotone $Q^1, \ldots, Q^n$.

(b) Since the proof of Lemma 4.3 is given in a way that only unary relation symbols are used, we see that in case we restrict to logics for monadic vocabularies the statement corresponding to Theorem 4.1 is true.

We now state yet another immediate consequence of Theorem 4.1.

4.5 Theorem. *Suppose $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q)$ with $\mathscr{L}_{\omega\omega} < \mathscr{L}$ is a regular logic with a monotone quantifier. If Q is trivial for finite sets, that is, $Q(A) = \varnothing$ for finite A, then for some α*

$$Q = Q_\alpha \quad \textit{or} \quad Q = Q_\alpha^{\mathrm{d}}.$$

If, moreover,

$$X \cup Y \in Q(A) \quad \textit{implies} \quad X \in Q(A) \quad \textit{or} \quad Y \in Q(A),$$

then $Q = Q_\alpha$. $\square$

Caicedo [1981b] calls a monotone quantifier a *cofilter quantifier*, if for any A and X, $Y \subset A$

$$X \cup Y \in Q(A) \quad \text{implies} \quad X \in Q(A) \quad \text{or} \quad Y \in Q(A).$$

Then for finite A we have $Q(A) = P(A)$, $Q(A) = P(A)\backslash\{\varnothing\}$ or $Q(A) = \varnothing$. Denote by Card and Card_∞ the class of non-zero cardinals and the class of infinite cardinals, respectively. If $f\colon \mathrm{Card} \to \{0, 1\} \cup \mathrm{Card}_\infty$ is a function, let Q_f be the quantifier given by

$$Q_f(A) = \{X \subset A \mid |X| \geq f(|A|)\}.$$

Clearly, Q_f is a cofilter quantifier (observe that we do not require that $\mathscr{L}_{\omega\omega}(Q_f)$ has the relativization property). Moreover, we have

4.6 Theorem. If Q is a cofilter quantifier, then $Q = Q_f$ for some $f\colon \mathrm{Card} \to \{0, 1\} \cup \mathrm{Card}_\infty$.

Proof. Note that a function $f\colon \mathrm{Card} \to \{0, 1\} \cup \mathrm{Card}_\infty$ is well defined by

$$f(|A|) = \begin{cases} \inf\{|X| \mid X \in Q(A)\} & \text{if } Q(A) \neq \varnothing, \\ \sup\{\omega, |A|^+\} & \text{if } Q(A) = \varnothing. \end{cases}$$

We show that for arbitrary A

$$(*) \qquad Q(A) = \{X \subset A \mid |X| \geq f(|A|)\},$$

that is, we show that $Q = Q_f$.

Clearly, $(*)$ holds if $Q(A) = \emptyset$. Now suppose $Q(A) \neq \emptyset$. If $f(|A|)$ is finite, then $f(|A|)$ is either 0 or 1, and $(*)$ holds by monotonicity. Let $f(|A|) = \mu$ be infinite. Then, by monotonicity $(*)$ holds, once we have established:

$$(\overset{*}{*}) \qquad \text{There is an } X \in Q(A) \quad \text{such that} \quad |X| = \mu \quad \text{and} \quad |A \setminus X| \geq \mu.$$

Otherwise, by definition of μ, we have $|A| = \mu$. Take any $Y \subset A$ with $|Y| = \mu$ and $|A \setminus Y| = \mu$. Since $Y \cup (A \setminus Y) = A$ and $A \in Q(A)$, we must have, by the cofilter property, $Y \in Q(A)$ or $(A \setminus Y) \in Q(A)$. But then $X := Y$ or $X := A \setminus Y$ satisfies $(\overset{*}{*})$. □

4.7 Notes. Theorem 4.1 is new here. As is shown by its proof, the theorem tells us that relativization is a strong property. Theorem 4.6 is due to Caicedo [1981b].

5. *A Lindström-Type Theorem for Invariant Sentences*

Lindström's theorem tells us that for algebraic structures of the logics satisfying the compactness and the Löwenheim–Skolem theorem, first-order logic is a maximal logic. Are there maximal logics with these properties for other kinds of structures—for instance, for topological structures? By isolating the main assumptions and ideas of the proof of Lindström's theorem, we will be able to prove an abstract maximality theorem for ordinary structures. The general character of this theorem will enable us to obtain maximal logics for certain classes of structures, in particular, for the class of topological structures.

Let R be a binary relation between structures and φ a sentence of a logic $\mathscr{L}$. We say that φ is *R-invariant* if

$$\mathfrak{A} R \mathfrak{B} \quad \text{and} \quad \mathfrak{A} \models \varphi \quad \text{imply} \quad \mathfrak{B} \models \varphi.$$

Denote the class of R-invariant sentences of $\mathscr{L}$ by $\mathscr{L}^R$. In case $\mathscr{L} = \mathscr{L}^R$, we say that $\mathscr{L}$ is a *logic of R-invariant sentences.* In particular, if a logic $\mathscr{L}$ is given, then $\mathscr{L}^R$ is a logic of R-invariant sentences.

5.1. *Let $\mathscr{L}$ be a logic with the Löwenheim–Skolem property and suppose that R_1 and R_2 are binary relations between structures. If R_1 and R_2 are* PC *in $\mathscr{L}$ and agree on countable structures, then $\mathscr{L}^{R_1} = \mathscr{L}^{R_2}$.* □

Let R be the relation $\cong_p$ of partial isomorphism. Logics of $\cong_p$-invariant sentences are precisely logics with the Karp property. Thus, Theorem 2.1.1 can now be stated in the following form:

5.2. *Among the logics of $\cong_p$-invariant sentences $\mathscr{L}_{\omega\omega}^{\cong_p}$ is a maximal compact logic.*

(Observe that in Theorem 2.1.1 we needed only countable compactness, since we restricted to logics with the finite occurrence property). Since $\cong_p$ and the relation $\cong$ of isomorphism agree on countable structures, we obtain from 5.2 using 5.1:

5.3. *Among the logics of $\cong$-invariant sentences $\mathscr{L}^{\cong}_{\omega\omega}$ is a maximal logic with the Löwenheim–Skolem and the compactness property.*

But $\mathscr{L}^{\cong} = \mathscr{L}$ holds for any logic, hence the result in 5.3 is precisely Lindström's first theorem.

Similarly, Theorem 3.1 can be stated in the form:

5.4. *Among the logics of $\cong_p$-invariant sentences $\mathscr{L}^{\cong_p}_{\infty\omega}$ is a maximal bounded logic.*

$\cong_p$ is a relation between structures PC in $\mathscr{L}_{\omega\omega}$. For each ordinal α, the relation $\cong_\alpha$ of α-isomorphism is an "approximation" of $\cong_p$. For finite n, $\cong_n$ is explicitly definable in $\mathscr{L}_{\omega\omega}$ in the sense that for any structure $\mathfrak{A}$, there is a sentence $\varphi^n_{\mathfrak{A}} \in \mathscr{L}_{\omega\omega}$ such that for arbitrary $\mathfrak{B}$,

$$\mathfrak{B} \models \varphi^n_{\mathfrak{A}} \quad \text{iff} \quad \mathfrak{A} \cong_n \mathfrak{B}.$$

The following "abstract maximality theorem" is obtained from 5.2 replacing $\cong_p$ by an arbitrary relation R having all the properties of $\cong_p$ and its approximations $\cong_n$ that are used in the proof of Lindström's theorem. Essentially, Theorem 5.5 tells us that in case R is itself definable by R-invariant first-order sentences and has definable approximations, then $\mathscr{L}^R_{\omega\omega}$ is a maximal compact logic of R-invariant sentences.

Note that Theorem 5.5 deals with many-sorted logics. For the sake of simplicity, we restrict to finite vocabularies. In the following, the term "logic" will always mean "many-sorted logic" in the sense of Chapter II. Furthermore, if it is not otherwise stated, we will always assume closure under boolean operations.

5.5 Theorem. *Suppose there is given for any vocabulary τ, a set $\Phi^\tau \subset \mathscr{L}_{\omega\omega}[\tau]$ and let $\mathfrak{K}^\tau = \mathrm{Mod}(\Phi^\tau)$. Assume that R is a binary relation between structures such that $\mathfrak{A}R\mathfrak{B}$ implies $\mathfrak{A}, \mathfrak{B} \in \mathfrak{K}^\tau$ for some τ. Suppose that*

(1) *R (restricted to τ-structures) is an equivalence relation on $\mathfrak{K}^\tau$.*
(2) *If $\rho: \tau \to \bar{\tau}$ is an injective renaming, then for all $\bar{\tau}$-structures $\mathfrak{A}$ and $\mathfrak{B}$*

$$\mathfrak{A}R\mathfrak{B} \quad \textit{implies} \quad \mathfrak{A}^{-\rho}R\mathfrak{B}^{-\rho}.$$

(3) *("R is invariantly definable and has definable finite approximations.") Given τ there are for some $\tau^*, \tau \subset \tau^*$, $\mathscr{L}_{\omega\omega}[\tau^*]$-sentences $\varphi_0, \varphi_1, \varphi_2, \ldots$ such that for arbitrary τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ the following hold:*

$$\mathfrak{A}R\mathfrak{B} \quad \textit{iff} \quad (\mathfrak{A}, \mathfrak{B}, \ldots) \models \{\varphi_i \mid i \in \omega\} \quad \textit{for some choice of (the universes and relations in)} \ldots,$$

and for $n \in \omega$ the relation R_n on $\mathfrak{K}^\tau$ given by

$$\mathfrak{A} R_n \mathfrak{B} \quad \textit{iff} \quad (\mathfrak{A}, \mathfrak{B}, ---) \models \{\varphi_i \mid i \leq n\} \quad \textit{for some } ---$$

has the following two properties:

(i) *R_n is an equivalence relation on $\mathfrak{K}^\tau$.*
(ii) *For $\mathfrak{A} \in \mathfrak{K}^\tau$, there is $\psi^n_{\mathfrak{A}} \in \mathscr{L}_{\omega\omega}[\tau]$ such that for $\mathfrak{B} \in \mathfrak{K}^\tau$*

$$\mathfrak{A} R_n \mathfrak{B} \quad \textit{iff} \quad \mathfrak{B} \models \psi^n_{\mathfrak{A}}.$$

Then among the logics of R-invariant sentences and semantics restricted to structures in $\bigcup \{\mathfrak{K}^\tau \mid \tau$ vocabulary$\}$ the logic $\mathscr{L}^R_{\omega\omega}$ of R-invariant first-order sentences is a maximal compact logic.

Moreover, if $\mathscr{L}$ with $\mathscr{L}^R_{\omega\omega} \leq \mathscr{L}$ is a compact logic of R-invariant sentences which is closed under conjunctions and disjunctions (but not necessarily under negations), then any two $\mathscr{L}$-classes can be separated by an $\mathscr{L}^R_{\omega\omega}$-class.

Proof. Clearly $\mathscr{L}^R_{\omega\omega}$ with semantics restricted to $\mathfrak{K} := \bigcup \{\mathfrak{K}^\tau \mid \tau \text{ vocabulary}\}$ is compact. Moreover, $\mathscr{L}^R_{\omega\omega}$ is closed under boolean operations (since R is an equivalence relation) and has the reduct and renaming property (by (2)). Note that for $\mathfrak{A} \in \mathfrak{K}^\tau$, the sentence $\psi^n_{\mathfrak{A}}$ mentioned in assumption (3) (ii) is R-invariant. In fact, let $\mathfrak{B} R \mathfrak{C}$ and $\mathfrak{B} \models \psi^n_{\mathfrak{A}}$. Then $\mathfrak{A} R_n \mathfrak{B}$. Since $R \subset R_n$ and R_n is an equivalence relation, we obtain $\mathfrak{A} R_n \mathfrak{C}$. Hence, $\mathfrak{C} \models \psi^n_{\mathfrak{A}}$.

It suffices to prove the separation claim in the theorem, since this claim implies the maximality property of $\mathscr{L}^R_{\omega\omega}$. Let $\mathscr{L}$ be as above and choose $\varphi, \psi \in \mathscr{L}[\tau]$ such that $\mathrm{Mod}(\varphi) \cap \mathrm{Mod}(\psi) = \varnothing$, where $\mathrm{Mod}(\ldots)$ denotes the class of models of ... in $\mathfrak{K}^\tau$.

For $n \in \omega$ we have

$$(*) \qquad \models_{\mathfrak{K}} \varphi \to \bigvee \{\psi^n_{\mathfrak{A}} \mid \mathfrak{A} \in \mathfrak{K}^\tau, \mathfrak{A} \models \varphi\}.$$

By the preceding remark, $\psi^n_{\mathfrak{A}}$ is R-invariant. Hence, by $\mathscr{L}$-compactness it follows that the disjunction in $(*)$ can be replaced by a finite one. That is, there is such a finite disjunction $\chi^n \in \mathscr{L}^R_{\omega\omega}$ with $\models_{\mathfrak{K}} \varphi \to \chi^n$.

By $\mathscr{L}$-compactness it suffices to show that $\{\chi^n \mid n \in \omega\} \cup \{\psi\}$ has no model in $\mathfrak{K}$, for it will then follow that for some $n \in \omega$, $\mathrm{Mod}(\varphi) \subset \mathrm{Mod}(\chi^0 \wedge \cdots \wedge \chi^n)$ and $\mathrm{Mod}(\chi^0 \wedge \cdots \wedge \chi^n) \cap \mathrm{Mod}(\psi) = \varnothing$. By contradiction, suppose that $\mathfrak{B}$ in $\mathfrak{K}$ is a model of $\{\chi^n \mid n \in \omega\} \cup \{\psi\}$. Then, for each n, there is $\mathfrak{A}_n \in \mathfrak{K}$ with $\mathfrak{A}_n \models \varphi$ and $\mathfrak{B} \models \psi^n_{\mathfrak{A}_n}$. Whence $\mathfrak{A}_n R_n \mathfrak{B}$. By (3), we now have

$$(\mathfrak{A}_n, \mathfrak{B}, ---) \models \{\varphi_i \mid i \leq n\}$$

for appropriate ---. By $\mathscr{L}$-compactness, there are $\overline{\mathfrak{A}}$, $\overline{\mathfrak{B}}$ and appropriate ... such that

$$(\overline{\mathfrak{A}}, \overline{\mathfrak{B}}, \ldots) \models \{\varphi_i \mid i \in \omega\}$$

with $\overline{\mathfrak{A}} \models \varphi$ and $\overline{\mathfrak{B}} \models \psi$. But then $\overline{\mathfrak{A}}R\overline{\mathfrak{B}}$. Hence it must be that $\overline{\mathfrak{B}} \models \varphi$, since φ is R-invariant. Therefore, $\overline{\mathfrak{B}} \in \mathrm{Mod}(\varphi) \cap \mathrm{Mod}(\psi)$—a contradiction. □

5.6 Remarks. (a) Note that the preceding proof shows that each $\mathscr{L}^R_{\omega\omega}$-sentence is equivalent in $\mathfrak{K}$ to a disjunction of sentences $\psi^n_{\mathfrak{A}}$. Thus, if $\mathscr{L}$ is a compact logic of R-invariant sentences containing all first-order sentences $\psi^n_{\mathfrak{A}}$, then $\mathscr{L}^R_{\omega\omega} \equiv \mathscr{L}$.

(b) Theorem 5.5 also holds for $\mathscr{L}_{\infty\omega}$ instead of $\mathscr{L}_{\omega\omega}$, if for each ordinal α, we introduce the corresponding relations R_α and also assume that each R_α has set-many equivalence classes. The conclusions will then read as follows:

> Among the logics of R-invariant sentences, $\mathscr{L}^R_{\infty\omega}$ is a maximal bounded logic; and
>
> If $\mathscr{L}^R_{\infty\omega} \leq \mathscr{L}$ and $\mathscr{L}$ is a bounded logic of R-invariant sentences, closed under conjunctions and disjunctions, then any two disjoint $\mathscr{L}$-classes can be separated by an $\mathscr{L}^R_{\infty\omega}$-class.

(c) One can even prove a more general theorem that will cover the cases in Theorem 5.5 and in the preceding remark, replacing $\mathscr{L}_{\omega\omega}$ by an arbitrary logic $\mathscr{L}$ and explicitly using the well-ordering number of $\mathscr{L}$. This theorem would also include the corresponding results (indicated in Section 3) for the logic with the added quantifier "there are uncountably many".

We now give the applications of Theorem 5.5 to topological structures and to other types of structures as well.

A *topological structure* is a pair $(\mathfrak{A}, \mu)$ consisting of an (algebraic) structure $\mathfrak{A}$ and of a topology μ on A. Topological spaces and topological groups are examples of topological structures. Let $\underline{Top}$ denote the class of topological structures. We obtain a logic for $\underline{Top}$ which is neither compact nor has the Löwenheim–Skolem property, if we take the two-sorted first-order language corresponding to structures of the form $(\mathfrak{A}, \mu, \in)$, where $(\mathfrak{A}, \mu) \in \underline{Top}$ and where $\in$ is the membership relation between elements of A and open sets. In particular, quantified variables of the second sort range over open sets.

Now, consider arbitrary structures of the form $(\mathfrak{A}, \mu, E)$, where A and μ are the universes and E is a binary relation with $E \subset A \times \mu$. For $U \in \mu$, put

$$U_E = \{a \in A \mid aEU\} \text{ and } \mu_E = \{U_E \mid U \in \mu\}.$$

Let

$$\underline{Bas} = \{(\mathfrak{A}, \mu, E) \mid \mu_E \text{ is basis of a topology on } A\},$$

and, for $(\mathfrak{A}, \mu, E) \in \underline{Bas}$ denote by $\widetilde{(\mathfrak{A}, \mu, E)}$ the induced structure in $\underline{Top}$.

$\underline{Bas}$ consists precisely of the models of the following (two-sorted) first-order sentence

$$\varphi_{Bas} = \forall x\, \exists X\, x \in X \wedge \forall x\, \forall X\, \forall Y(x \in X \wedge x \in Y \rightarrow \exists Z(x \in Z \wedge \forall z(z \in Z \rightarrow (z \in X \wedge z \in Y)))).$$

Let $\cong^t$ be the relation of topological homeomorphism on $\underline{Bas}$. That is, $\cong^t$ is the relation given by the isomorphism relation of induced topological structures:

$$(\mathfrak{A}, \mu, E) \cong^t (\mathfrak{B}, \nu, F) \quad \text{iff} \quad \widetilde{(\mathfrak{A}, \mu, E)} \cong \widetilde{(\mathfrak{B}, \nu, F)}.$$

Observe that a sentence φ is $\cong^t$-invariant just in case

$$(1) \qquad (\mathfrak{A}, \mu, E) \models \varphi \quad \text{iff} \quad \widetilde{(\mathfrak{A}, \mu, E)} \models \varphi$$

holds for $(\mathfrak{A}, \mu, E) \in \underline{Bas}$. Therefore, we also speak of *basis-invariant* sentences instead of $\cong^t$-invariant sentences.

If $\mathscr{L}$ is any logic for topological structures, then, using (1) as definition, we obtain a logic for structures in $\underline{Bas}$ which will consist only of basis-invariant sentences. On the other hand, if $\mathscr{L}$ is a logic for structures in $\underline{Bas}$ which consists of basis-invariant sentences, then—using again (1) as definition—we obtain a logic for $\underline{Top}$. Because of this one-to-one correspondence, maximal logics for $\underline{Bas}$ "are" maximal logics for $\underline{Top}$.

We apply Theorem 5.5 to obtain maximal logics for $\underline{Bas}$—it being clear how the notion of one-sorted and many-sorted type and structure must be redefined in our case. For this, choose Φ^τ such that $\mathrm{Mod}(\Phi^\tau)$ is the class of structures in $\underline{Bas}$ of type τ, for example, $\Phi^\tau = \{\varphi_{Bas}\}$ for "one-sorted" τ. As R and R_n take the relation $\cong^t_p$ of *partial homeomorphism* and the relation $\cong^t_n$ of *n-homeomorphism*, respectively (they correspond to the relation of partial isomorphism and n-isomorphism of induced topological structures; the reader is referred to Chapter XV or to Flum–Ziegler [1980, p. 18]). By Theorem 5.5, $\mathscr{L}^{\cong^t_p}_{\omega\omega}$ is a maximal compact logic of $\cong^t_p$-invariant sentences. Since $\cong^t$ and $\cong^t_p$ are first-order definable relations which agree on countable structures, we obtain from this result and from 5.1:

5.7 Theorem. *The logic of basis-invariant first-order sentences is maximal among the logics for topological structures with the compactness property and the following Löwenheim–Skolem property: if φ has a topological model, then there is $(\mathfrak{A}, \mu) \in \underline{Top}$ such that $(\mathfrak{A}, \mu) \models \varphi$, A is countable and μ has a countable basis.* ☐

5.8 Remarks. (a) Since $\cong^t_p$ and $\cong^i$ agree on countable structures, one can get from the proof of Theorem 5.5, the interpolation theorem for the logic of basis-invariant first-order sentences in a way similar to that for first-order logic given in Example 1.1.7(a).

(b) By the preceding results and Remark 5.6(a), any logic containing sentences $\psi^n_{(\mathfrak{A}, \sigma)}$ characterizing the n-isomorphism type of any topological structure $(\mathfrak{A}, \sigma)$ already contains all basis-invariant first-order sentences. This result will be used in Chapter XV.

Similarly, one can obtain maximal logics for other types of structures. We will give two further examples.

A *uniform structure* is a pair $(\mathfrak{A}, \mu)$ where $\mu \subset A \times A$ is a uniformity on A. A *monotone structure* is a pair $(\mathfrak{A}, \mu)$ where $\mu \subset A$ is a monotone system on A, that is, a non-empty set of subsets of A such that $X \in \mu$ and $X \subset Y \subset A$ imply $Y \in \mu$.
Using in both cases the corresponding notions of basis and the corresponding Löwenheim–Skolem properties we obtain in the same way as for topological structures the following result:

5.9 Theorem. *Among the logics for uniform structures (monotone structures) with the compactness and the Löwenheim–Skolem property, the logic of basis-invariant first-order sentences is maximal.* □

5.10 Remarks. (a) In Chapter XV the reader can find syntactic characterizations of the basis-invariant sentences for the above cases.

(b) Observe that the result which we obtain from Remark 5.6(b) for the corresponding infinitary logics are not satisfactory. For example, Remark 5.6(b) tells us that among the logics for topological structures, $\mathscr{L}^{\cong^t_p}_{\infty\omega}$ is a maximal bounded logic of $\cong^t_p$-invariant sentences. And it is not hard to give a "syntactic" characterization of the sentences in $\mathscr{L}^{\cong^t_p}$. But is $\mathscr{L}^{\cong^t_p}_{\infty\omega} = \mathscr{L}^{\cong^t}_{\infty\omega}$? That is, is $\mathscr{L}^{\cong^t_p}_{\infty\omega}$ the class of basis-invariant $\mathscr{L}_{\infty\omega}$-sentences?

5.11 Notes. The Lindström-type results for topological structures, monotone structures, and so on are due to Ziegler [1976]. Theorem 5.5 is new here. The reader should compare our approach to maximal logics with that given by Sgro [1977b]. Sgro's main result—when translated into our terminology—reads as follows: Given a relation R between structures, the logic $\mathscr{L}^R_{\omega\omega}$ is, among the logics of R-invariant sentences, a maximal logic satisfying a "Łos ultraproduct theorem". Since the ultraproduct operation commutes with the operation which associates to each model in <u>*Bas*</u> the induced model in <u>*Top*</u>, we obtain: The logic of basis-invariant sentences is a maximal logic for topological structures satisfying a Łos ultraproduct theorem.