

# Special Notations

## Chapter I

$\text{Dm } \varphi$	domain of $\varphi$	7
$\text{Im } \varphi$	image of $\varphi$	7
$\varphi(x) \downarrow$	$\varphi(x)$ is defined, $x \in \text{Dm } \varphi$	7
$\varphi(x) \uparrow$	$\varphi(x)$ is undefined, $x \notin \text{Dm } \varphi$	7
$\approx$	strong equality	7
$\varphi \upharpoonright X$	restriction of $\varphi$ to $X$	7
$\varphi''X$	image of $X$ under $\varphi$	7
$\varphi : X \rightarrow Y$	function from $X$ into $Y$	7
${}^*XY$	total functions $X \rightarrow Y$	7
$x \mapsto y_x$	function which assigns $y_x$ to $x$ for each $x \in Z$	8
$\lambda x. y_x$		
$\langle y_x : x \in Z \rangle$		
$\omega$	set of natural numbers	8
$\text{lg}$	length of a finite sequence	8
$x \subseteq y$	$y$ extends $x$	8
$x * y$	$x$ concatenated with $y$	8
$x * \varphi$	$x$ concatenated with $\varphi$	8
$x \in Z$	$(\forall i < \text{lg}(x)) x_i \in Z$	8
$\varphi(x)$	$(\varphi(x_0), \dots, \varphi(x_{k-1}))$	8
${}^k \omega$	${}^k \omega \times {}^l (\omega)$	8
$F[\mathbf{m}, \alpha]$	$\lambda p. F(p, \mathbf{m}, \alpha)$	8
$\sim R$	complement	8
$K_{\mathbf{R}}$	characteristic functional	9
$\text{Gr}_F, \text{Gr}(F)$	graph	9
${}^{k,l} \omega$	${}^k \omega \times {}^l (\omega) \times {}^{l'} (\omega^\omega)$	9
$\wedge, \vee, \neg,$ $\rightarrow, \leftrightarrow, \forall, \exists$	logical symbols	9
$(\exists p < m),$ $(\forall \alpha \in A)$	bounded quantifier	10
$\exists! x$	exists exactly one $x$	10
$\langle \mathbf{m} \rangle, \langle \alpha \rangle$	codes for finite sequences	10
$( )_i$	$i$ -th component	10, 11
$\text{lg}$	length	10, 11
$*$	concatenation	10, 11
$\text{Sq}, \text{Sq}_i$	set of sequence codes	10, 11

$(\gamma)^n$	$n$ -th component of a coded infinite sequence	11
$\text{ZF(ZFC)}$	Zermelo-Fraenkel set theory (with axiom of choice)	11
AC	axiom of choice	11
DC	axiom of dependent choice	11
$\text{AC}_\omega$	axiom of countable choice	11
Or	class of ordinals	11
$\text{inf } X$	least element of $X$	11
$\text{sup } X$	least ordinal $\geq$ all elements of $X$	12
$\text{sup}^+ X$	least ordinal $>$ all elements of $X$	12
$\text{Lim } X$	limit points of $X$	12
$\text{Card}(X)$	cardinal of $X$	13
$\aleph_\sigma$	$\sigma$ -th infinite cardinal	13
$\mathbf{P}(X)$	power-set of $X$	13
$\text{Fld}(Z), \text{Fld}(\gamma)$	field of the relation $Z, \leq_\gamma$	13, 15
$\ Z\ , \ \gamma\ $	order-type of the (pre-)wellordering $Z, \leq_\gamma$	14, 15
$o(X)$	least ordinal not the type of a pre-wellordering of $X$	14
$\leq_\gamma$	binary relation coded by $\gamma$	14
$W$	codes for well-orderings of $\omega$	15, 81
$\gamma \upharpoonright p$	code for initial segment of $\leq_\gamma$	15
$ p _\gamma$	ordinal represented by $p$ in $\leq_\gamma$	15
$[\mathbf{m}]$	interval determined by $\mathbf{m}$	16
Bir	binary irrationals	19, 160
mes	Lebesgue measure	20
$\bar{\Gamma}$	set inductively defined by $\Gamma$	22
$\Gamma^{(\sigma)}, \Gamma^\sigma$	stages of an inductive definition	22
$ \Gamma $	closure ordinal	23

Chapter II

$sg^+, sg^-$  signum functions 29  
 "least"  $q < p$  bounded search 30  
 $\exists^0, \forall^0$  bounded number quantification 31  
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 $[a]$  primitive recursive functional indexed by  $a \in \text{Pri}$  34  
 $\{a\}$  partial recursive functional indexed by  $a$  38  
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 Sb, substitution functions 41  
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 $T, T$  normal form relations 46, 49  
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 $\exists^1, \forall^1$  type-1 (function) quantification 57  
 $cX$  set of complements of members of  $X$  59  
 $dg(\alpha)$  degree of  $\alpha$  63  
 $R \leq A$   $R$  is (many-one) reducible to  $A$  65  
 $A^\omega, \beta^\omega$  ordinary jump 65, 66

Chapter III

$\Sigma^0, \Pi^0, \Delta^0, \Delta^0_{(\omega)}$  arithmetical hierarchy 69, 77, 78  
 $U^0, U^0_r$  universal relations 73  
 $\Sigma^1, \Pi^1, \Delta^1, \Delta^1_{(\omega)}$  analytical hierarchy 80, 86, 87  
 $U^1, U^1_r$  universal relations 84  
 $\Delta^1$ - $dg(\alpha)$   $\Delta^1$ -degree of  $\alpha$  86  
 $\mathcal{A}$  Suslin operation/quantifier 88  
 $Z_\alpha$  zeros of  $\alpha$  89  
 $P_\Gamma$  relation which represents  $\Gamma$  89  
 $U^0_{(\omega)}$  universal set for  $\Delta^0_{(\omega)}$  93  
 $\omega_1$  least non-recursive ordinal 97  
 $\delta^1$  least non- $\Delta^1$  ordinal 97, 208  
 $\mathfrak{M}$  standard model for arithmetic 114  
 $\sigma[m, \alpha]$  value of  $\sigma$  at  $(m, \alpha)$  in  $\mathfrak{M}$  115  
 $\models \mathfrak{M}[m, \alpha]$   $\mathfrak{M}$  is true at  $(m, \alpha)$  in  $\mathfrak{M}$  115  
 $\exists^1, \forall^1$  classes of arithmetic formulas 116  
 $\mathcal{T} \vdash \mathfrak{M}$   $\mathfrak{M}$  is a theorem of  $\mathcal{T}$  118  
 $\mathcal{T} \vdash^\omega \mathfrak{M}$   $\mathfrak{M}$  is a theorem of  $\mathcal{T} + \omega$ -rule 121  
 $\Vdash \mathfrak{M}[m, s]$   $\mathfrak{M}$  is forced at  $(m, s)$  126

Chapter IV

$R \leq A$   $R$  is (many-one) reducible to

A 136

$\leq$  recursive dense linear ordering of  $Sq$  136  
 $\leq^p_{m, \alpha}$  restriction of  $\leq$  136  
 $\leq_{\Sigma}, <_{\Sigma}, \leq_{\Pi}, <_{\Pi}$  ordinal comparison on  $W$  138, 144  
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 $W$  (number) codes for recursive ordinals 140  
 $\omega_1[\beta]$  least ordinal not recursive in  $\beta$  140  
 $W[\beta]$  (number) codes for ordinals recursive in  $\beta$  140  
 $\leq_\beta$  reducible recursively in  $\beta$  141  
 $\ll$  reducible by a continuous functional 141  
 $\Sigma^1_{hyp}$   $(\exists \beta \in \Delta^1[\alpha])P(m, \alpha, \beta)$  147  
 $A \leq^1 B$   $A \in \Delta^1[B]$  149  
 $hdg(A)$  hyperdegree of  $A$  149  
 $A^{hj}$  hyperjump 155  
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 $V = L$  Hypothesis of Constructibility: all sets in  $X$  are constructible 215  
 $<_L$  well-ordering of constructible functions 215  
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 $\gamma \neq \delta$  play resulting from two strategies 222  
 $\text{Det}(X)$  all sets in  $X$  are determined 222  
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 $B(\Phi)$  canonical base of  $\Phi$  237  
 $\Phi^\circ$  dual operation 238  
 $\nabla(\Phi)$  relations generated by  $\Phi$  239  
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 $E_1$  Suslin-quantifier functional 263  
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