Part C

Generalized Recursion Theories

Each of the measures of complexity we have discussed in Parts A and B can be seen as an analysis of the means required to define a class of relations into two components — one constructive or effective and the other in some way non-constructive. Recursive relations are from our point of view entirely constructive. The arithmetical and analytical relations are generated from them by the non-constructive operation of quantification over an infinite set. For the relativized and boldface hierarchies, the parameters from ${}^{\omega}\omega$ are another non-constructive component. In the case of inductive definability, the operators are defined non-constructively, but we have implicitly extended our concept of constructivity to include the process of iteration over the ordinals.

The generalized recursion theories we study in this part are an elaboration of this dichotomy. In each case the "recursive" functionals are those which are computable by essentially the same constructive means as in Chapter II together with some non-constructive one. In Chapters VI and VII the non-constructive component consists in the introduction of various fixed functionals; in Chapter VIII it is the extension of the fundamental domain from ω to various ordinals κ or to the class of all ordinals.

As might be expected, there are many connections with the earlier theory. Let

$$\mathsf{E}(\alpha) = \begin{cases} 0, & \text{if } \exists m\,.\,\alpha(m) = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Then the relations (semi-) recursive in E are exactly the (Π^1_1) Δ^1_1 relations (Theorem VI.1.7). The relations recursive in a normal operation Φ or a jump operator J turn out to be exactly those encompassed by the hierarchies of § V.5. (Π^1_1) Δ^1_1 is also exactly the class of relations which are (semi-) recursive when the fundamental domain is enlarged from ω to ω_1 (Theorem VIII.3.4). If computations are carried out over the ordinal δ^1_2 (or over $\aleph_1$), the (semi-) recursive relations on numbers are exactly the (Σ^1_2) Δ^1_2 relations (Theorem VIII.3.7 and Corollary VIII.5.10).

Chapter VI
Recursion in a Type-2 Functional

In the notes to § II.5 we described an alternative approach to ordinary recursion theory in which the notion "F is partial recursive in β" is defined first for functions $F : {}^k\omega \to \omega$ and a fixed β and used to derive the notion of a recursive functional. For the development of the theory of recursion with type-2 arguments, we have the same choice "one type up". We may take as our primary notion either partial recursiveness of functionals $\mathbb{F} : {}^{k,l,l'}\omega \to \omega$ (with arguments of type $(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I})$) or the notion "$\mathsf{F}$ is partial recursive in I" for functionals $\mathsf{F} : {}^{k,l}\omega \to \omega$ and a fixed I. We choose the second course for two reasons. First, the technical details of the theory involving only one fixed type-2 parameter are considerably simpler. Second, the primary objects under study remain functionals and relations on ${}^{k,l}\omega$, and thus the theory forms a natural extension of that of earlier chapters. Partial recursiveness for functions $\mathbb{F} : {}^{k,l,l'}\omega \to \omega$ will be discussed in § 7.

In § 1 we establish the simplest properties of recursion relative to a functional and discuss the connections with the arithmetical and analytical hierarchies. The most important technical feature which distinguishes recursion relative to a functional from ordinary recursion is the absence of normal form theorems as in § II.3. This lack necessitates substantially longer and more involved proofs of many of the results corresponding to those in §§ II.3–4 and these occupy §§ 2–4 of this chapter. The fact that Δ_1^1 is the class of relations recursive in E suggests that the class of relations recursive in other functionals may share some of the properties of Δ_1^1. This is already evident in §§ 3 and 4 and is carried further by the discussion of hierarchies in § 5. In § 6 we discuss a flaw in this analogy and an alternative analogy related to the operator $*$ of § V.4.

1. Basic Properties

To avoid unecessary complications we shall restrict ourselves to the study of recursion relative to a single total functional $\mathsf{I} : {}^{\omega}\omega \to \omega$. Other cases may be reduced to this one as indicated at the end of this section. Let us consider first the intuitive notion of a functional F being mechanically calculable relative to I. As for recursion relative to a type-1 function, we imagine an idealized computer prepared to accept inputs of the form $(\mathbf{m}, \boldsymbol{\alpha})$ and connected to a memory device M which contains the graph of I. Of course, the graph of I is a set of power $2^{\aleph_0}$ so this M must be larger than the memory units we have previously considered. Somewhat more troublesome is the fact that in order to obtain from M a desired value $\mathsf{I}(\beta)$, the computer must in some way present M with the argument β. As β is an infinite object, it is not obvious how we should imagine this to be accomplished. We have, however, a precedent in the mechanism for presenting the computer with (infinite) inputs $(\mathbf{m}, \boldsymbol{\alpha})$. Accordingly, we imagine a subsidiary infinite memory unit M' large enough to store a single function β. Then to obtain a value $\mathsf{I}(\beta)$ during the course of a computation, the computer "loads" M' with the graph of β, whereupon M responds with the desired value $\mathsf{I}(\beta)$.

This model leads immediately to two observations. First, computations relative to I must in general be infinite, as the computer cannot be expected to load M' with the graph of β in a finite time, and it must in general be decided during the course of the computation for which β the value $\mathsf{I}(\beta)$ is needed, so that M' cannot be loaded before the start of the computation. Second, during any given computation the only values $\mathsf{I}(\beta)$ which may be obtained are those for which β itself is computable — in fact, the calculation of the values of β is a part of the computation. Thus if F is partial computable in I, the value of $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha})$ depends on the values $\mathsf{I}(\beta)$ at most for β which are computable in I and $\boldsymbol{\alpha}$ (cf. Exercise 1.13).

The formal definition is very similar to that of § II.2 with an additional clause to introduce values of I.

1.1 Definition. For any total functional $\mathsf{I} : {}^{\omega}\omega \to \omega$, the set $\Omega[\mathsf{I}]$ is the smallest set such that for all k, l, n, p, q, r, and s, all $i < k$ and $j < l$, and all $(\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$,

(0)
(1) identical to the corresponding clauses of Definition II.2.1;
(2)

(3) for any b and any β, if for all p, $(b, p, \mathbf{m}, \boldsymbol{\alpha}, \beta(p)) \in \Omega[\mathsf{I}]$, then $(\langle 3, k, l, b\rangle, \mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}(\beta)) \in \Omega[\mathsf{I}]$.

In contrast with Ω, $\Omega[\mathsf{I}]$ is not a closure under finitary functions. It is, however, a closure under functions of rank ω, as clause (3) is equivalent to the requirement that $\Omega[\mathsf{I}]$ be closed under all the functions φ_β, where

$$\varphi_\beta(\{(b, p, \mathbf{m}, \boldsymbol{\alpha}, \beta(p)) : p \in \omega\}) = (\langle 3, k, l, b\rangle, \mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}(\beta)).$$

Hence the inductive operator is $\aleph_1$-compact and, by the intended result of

Exercise I.3.10, this implies that the inductive definition has closure ordinal at most $\aleph_1$. If we associate the ordinal of the level at which a given sequence $(a, \mathbf{m}, \boldsymbol{\alpha}, n)$ occurs with the "length" of the corresponding computation $\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$, we see that although computations relative to I may be infinite, they are all countable.

The first part of the theory of recursion relative to a functional I now proceeds almost exactly as in § II.2. The proof that for all $a, \mathbf{m}$, and $\boldsymbol{\alpha}$ there is at most one n such that $(a, \mathbf{m}, \boldsymbol{\alpha}, n) \in \Omega[\mathsf{I}]$ differs from that of Lemma II.2.2 only in that the induction is now on all countable ordinals. We write

$$\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n \leftrightarrow (a, \mathbf{m}, \boldsymbol{\alpha}, n) \in \Omega[\mathsf{I}].$$

A functional F is *partial recursive in* I iff for some a, $\mathsf{F} = \{a\}^{\mathsf{I}}$; a is called an *index of* F *from* I. F is *recursive in* I iff F is partial recursive in I and total, and R is *recursive in* I iff $\mathsf{K_R}$ is recursive in I. R is *semi-recursive in* I iff R is the domain of some functional partial recursive in I, and *co-semi-recursive in* I iff $\sim\mathsf{R}$ is semi-recursive in I.

As in Remark II.2.4, it is easily verified that each clause in the Definition has its intended meaning. Thus

$$\{\langle 3, k, l, b\rangle\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{I}(\lambda p\,.\{b\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha})),$$

and the class of functionals partial recursive in I is closed under functional composition. It is clear from the definition that $\Omega \subseteq \Omega[\mathsf{I}]$. Hence if $\{a\}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$, then also $\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$. In particular, if $\{a\}$ is total, $\{a\} = \{a\}^{\mathsf{I}}$, so every recursive functional is recursive in I.

The functions Sb_i of Lemma II.2.5 also have the property that

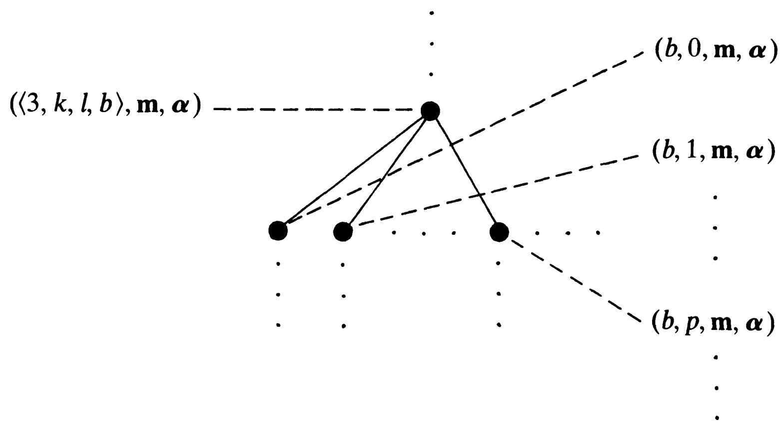
$$\{\mathrm{Sb}_i(a, m_0, \ldots, m_i)\}^{\mathsf{I}}(m_{i+1}, \ldots, m_{k-1}, \boldsymbol{\alpha}) \simeq \{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}),$$

and exactly the same proof as before establishes the I-*Recursion Theorem*: *for any functional* F *partial recursive in* I, *there exists an* $\bar{e} \in \omega$ *such that for all* $(\mathbf{m}, \boldsymbol{\alpha})$,

$$\{\bar{e}\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{F}(\bar{e}, \mathbf{m}, \boldsymbol{\alpha}).$$

From this follows exactly as in § II.2 that the class of functionals partial recursive in I is closed under primitive recursion, course-of-values recursion, and unbounded search. Hence by Corollary II.3.3, every partial recursive functional is also partial recursive in I. Definition by cases with relations recursive in I and functionals partial recursive in I is established exactly as for Theorem II.2.9. Similarly, the class of relations recursive in I is a Boolean algebra closed under composition with functionals recursive in I.

Computations relative to I may also be thought of as being arranged in labeled trees. Nodes corresponding to clauses (0)–(2) are as before, while for clause (3) we have nodes of the form:



Of course, branching is no longer finite, so the tree may be infinite without having an infinite branch.

Exactly as in Lemma II.4.2 we may prove that every relation recursive in I is also semi-recursive in I, but not conversely. Similarly, the class of relations semi-recursive in I is closed under finite intersection and bounded universal quantification — in fact,

1.2 Lemma. *For any* I, *the class of relations semi-recursive in* I *is closed under universal number quantification* ($\forall^0$).

Proof. Suppose $\mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \{a\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha})\downarrow$. Then

$$\forall p\, \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{I}(\lambda p\,.\{a\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha}))\downarrow$$

so $\forall^0 \mathsf{R} = \mathrm{Dm}\{\langle 3, k, l, a\rangle\}^{\mathsf{I}}$ and is thus semi-recursive in I. □

The remaining part of the theory of semi-recursive relations depends on the normal form theorems of § II.3. As we have no analogue for these for recursion relative to I, the corresponding results will be proved in a different way in §§ 2–4.

To illustrate the notion of recursion relative to a functional, we now consider some examples. Consider the functionals defined as follows:

$$\mathsf{E}(\alpha) = \begin{cases} 0, & \text{if } \exists m\,.\,\alpha(m) = 0; \\ 1, & \text{otherwise;} \end{cases}$$

$$\mathsf{E}_1(\alpha) = \begin{cases} 0, & \text{if} \quad \exists\beta\,\forall n.\,\alpha(\bar{\beta}(n)) = 0; \\ 1, & \text{otherwise}; \end{cases}$$

$$\mathsf{oJ}((\langle a, \mathbf{m}\rangle) * \alpha) = \begin{cases} 0, & \text{if} \quad \{a\}(\mathbf{m}, \alpha)\downarrow\,; \\ 1, & \text{otherwise}; \end{cases}$$

where (as defined in § I.1) $((\langle a, \mathbf{m}\rangle) * \alpha)(0) = \langle a, \mathbf{m}\rangle$ and $((\langle a, \mathbf{m}\rangle) * \alpha)(p+1) = \alpha(p)$. The functional oJ is of course just a natural encoding of the ordinary jump operator of the same name:

$$\mathsf{oJ}((\langle a, \mathbf{m}\rangle) * \alpha) = \alpha^{\mathsf{oJ}}(\langle a, \mathbf{m}\rangle).$$

1.3 Lemma. *For all* R,

(i) *if* R *is arithmetical, then* R *is recursive in* E;

(ii) *if* $\mathsf{R} \in \Sigma^1_1 \cup \Pi^1_1$, *then* R *is recursive in* E_1.

Proof. Consider the set X of all relations recursive in E. By the preceding remarks, X contains all recursive relations, so for (i) it suffices to show that X is closed under $\exists^0$ (as it is then closed under $\forall^0$ by complementation). Suppose R is recursive in E. Then

$$\begin{aligned} \mathsf{K}_{\exists^0\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}) &= \begin{cases} 0, & \text{if} \quad \exists p.\,\mathsf{K}_\mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}) = 0; \\ 1, & \text{otherwise}; \end{cases} \\ &= \mathsf{E}(\lambda p.\,\mathsf{K}_\mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha})). \end{aligned}$$

Hence if $\mathsf{K}_\mathsf{R} = \{b\}^\mathsf{E}$, then $\mathsf{K}_{\exists^0\mathsf{R}} = \{\langle 3, k, l, b\rangle\}^\mathsf{E}$, so $\exists^0\mathsf{R}$ is also recursive in E.

For (ii), if $\mathsf{R} \in \Sigma^1_1$, then for some recursive S,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists\beta\,\forall n\,\mathsf{S}(\bar{\beta}(n), \mathbf{m}, \boldsymbol{\alpha}).$$

Hence

$$\mathsf{K}_\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) = \mathsf{E}_1(\lambda p.\,\mathsf{K}_\mathsf{S}(p, \mathbf{m}, \boldsymbol{\alpha})),$$

so K_R is recursive in E_1. The result for Π^1_1 again follows by closure under complementation. □

Of course, it follows that E is recursive in E_1.

1.4 Lemma. *Each of* E *and* oJ *is recursive in the other.*

Proof. Let a be an index such that $\{a\}(\alpha) \simeq$ least $m . \alpha(m) = 0$. Then clearly $\mathsf{E}(\alpha) = {}^{\mathsf{o}}\mathsf{J}((\langle a \rangle) * \alpha)$, so if b is an index such that $\{b\}^{{}^{\mathsf{o}}\mathsf{J}}(0, \alpha) = a$ and $\{b\}^{{}^{\mathsf{o}}\mathsf{J}}(p+1, \alpha) = \alpha(p)$, then $\mathsf{E} = \{(3, 0, 1, b)\}^{{}^{\mathsf{o}}\mathsf{J}}$; so E is recursive in ${}^{\mathsf{o}}\mathsf{J}$. On the other hand from Theorem II.3.1 we have

$$ {}^{\mathsf{o}}\mathsf{J}((\langle a, \mathbf{m} \rangle) * \alpha) = \mathsf{E}(\lambda u . \mathsf{K}_{\mathsf{T}}(a, \langle \mathbf{m} \rangle, u, \langle \alpha \rangle)) $$

from which we similarly conclude that ${}^{\mathsf{o}}\mathsf{J}$ is recursive in E. □

We shall prove below that the relations recursive in E are exactly the Δ^1_1 relations, but we can already guess that more than the arithmetical relations are recursive in E. Recall the sets D_r of Theorem III.1.13:

$$ D_0 = \{0\} \quad \text{and} \quad D_{r+1} = (D_r)^{{}^{\mathsf{o}}\mathsf{J}}. $$

It is not hard to see that there is a primitive recursive function f such that for all r, $f(r)$ is an index of D_r from E. Then the set

$$ \{\langle r, m \rangle : m \in D_r\} = \{\langle r, m \rangle : \{f(r)\}^{\mathsf{E}}(m) = 0\} $$

is also recursive in E but, by III.1.13 and the Arithmetical Hierarchy Theorem, is not arithmetical. Similarly, Lemma 1.3(ii) is far from optimal. We shall return in §§5 and 6 to the class of relations recursive in E_1.

Upper bounds on the class of relations recursive in a given functional I again follow from the fundamental results on inductive definability of § III.3. Let

$$ \mathsf{V}^{\mathsf{I}}(a, \langle \mathbf{m} \rangle, n, \langle \boldsymbol{\alpha} \rangle) \leftrightarrow (a, \mathbf{m}, \boldsymbol{\alpha}, n) \in \Omega[\mathsf{I}]. $$

1.5 Theorem. *For any* I,
(i) *if* $\mathsf{I} \in \Delta^1_1$, *then* $\mathsf{V}^{\mathsf{I}} \in \Pi^1_1$;
(ii) *for all* $r \geq 2$, *if* $\mathsf{I} \in \Delta^1_r$, *then* $\mathsf{V}^{\mathsf{I}} \in \Delta^1_r$.

Proof. For each l and each $\boldsymbol{\alpha} \in {}^l({}^\omega\omega)$, we define operators $\Gamma_{\boldsymbol{\alpha},0}, \ldots, \Gamma_{\boldsymbol{\alpha},3}$ and $\Lambda_{\boldsymbol{\alpha},3}$ as follows. $\Gamma_{\boldsymbol{\alpha},q}$ corresponds to clause (q) of Definition 1.1. For any $A \subseteq \omega$,

$$ \begin{aligned} \Gamma_{\boldsymbol{\alpha},0}(A) = {} & \{\langle\langle 0, k, l, 0, n \rangle, \langle \mathbf{m} \rangle, n \rangle : k, n \in \omega \wedge \mathbf{m} \in {}^k\omega\} \\ & \cup \cdots \cup \{\langle\langle 0, k, l, 3, i, j \rangle, \langle \mathbf{m} \rangle, \alpha_j(m_i) \rangle : k \in \omega \wedge \mathbf{m} \in {}^k\omega \wedge \\ & \hfill i < k \wedge j < l\} \\ & \cup \cdots \cup \{\langle\langle 0, k+2, l, 5 \rangle, \langle p, q, \mathbf{m} \rangle, \mathrm{Sb}_0(p, q) \rangle : k, p, q \in \omega, \\ & \hfill \mathbf{m} \in {}^k\omega\}; \\ \Gamma_{\boldsymbol{\alpha},1}(A) = {} & \{\langle\langle 1, k, l, b, c_0, \ldots, c_{k'-1} \rangle, \langle \mathbf{m} \rangle, n \rangle : \text{for some } k', q_0, \ldots, \\ & q_{k'-1}, (\forall i < k')[\langle c_i, \langle \mathbf{m} \rangle, q_i \rangle \in A \wedge \langle b, \langle \mathbf{q} \rangle, n \rangle \in A]\}; \end{aligned} $$

$$\Gamma_{\alpha,2}(A) = \{\langle\langle 2, k+1, l\rangle, \langle b, \mathbf{m}\rangle, n\rangle : \langle b, \langle\mathbf{m}\rangle, n\rangle \in A\};$$

$$\Gamma_{\alpha,3}(A) = \{\langle\langle 3, k, l, b\rangle, \langle\mathbf{m}\rangle, n\rangle : \exists\beta\,(\forall p\,[\langle b, \langle p, \mathbf{m}\rangle, \beta(p)\rangle \in A] \wedge \mathsf{I}(\beta) = n)\};$$

$$\Lambda_{\alpha,3}(A) = \{\langle\langle 3, k, l, b\rangle, \langle\mathbf{m}\rangle, n\rangle : \forall p\, \exists q\,[\langle b, \langle p, \mathbf{m}\rangle, q\rangle \in A] \wedge \wedge \forall\beta\,(\forall p\,\forall q\,[\langle b, \langle p, \mathbf{m}\rangle, q\rangle \in A \rightarrow \beta(p) = q] \rightarrow \mathsf{I}(\beta) = n)\}.$$

First, note that all of these operators are monotone. Hence, if we set

$$\Gamma_{\langle\alpha\rangle}(A) = \Gamma_{\alpha,0}(A) \cup \Gamma_{\alpha,1}(A) \cup \Gamma_{\alpha,2}(A) \cup \Gamma_{\alpha,3}(A),$$

and

$$\Lambda_{\langle\alpha\rangle}(A) = \Gamma_{\alpha,0}(A) \cup \Gamma_{\alpha,1}(A) \cup \Gamma_{\alpha,2}(A) \cup \Lambda_{\alpha,3}(A),$$

then $\Gamma_{\langle\alpha\rangle}$ and $\Lambda_{\langle\alpha\rangle}$ are also monotone. Furthermore it is clear from the definition of $\Omega[\mathsf{I}]$ that

$$(a, \mathbf{m}, \boldsymbol{\alpha}, n) \in \Omega[\mathsf{I}] \leftrightarrow \langle a, \langle\mathbf{m}\rangle, n\rangle \in \bar{\Gamma}_{\langle\alpha\rangle}$$

that is, $\mathsf{V}^{\mathsf{I}} = \bar{\Gamma}$, where Γ is the decomposable monotone operator defined by the family $\{\Gamma_{\langle\alpha\rangle} : l \in \omega,\ \boldsymbol{\alpha} \in {}^{l}({}^{\omega}\omega)\}$.

Although it is not in general true that $\Gamma_{\alpha,3}(A) = \Lambda_{\alpha,3}(A)$, this is easily seen to be true whenever A has the property

$$(*) \qquad \langle a, \langle\mathbf{m}\rangle, n\rangle \in A \wedge \langle a, \langle\mathbf{m}\rangle, n'\rangle \in A \rightarrow n = n'.$$

An easy inductive argument shows that for all ordinals σ, $\Lambda^{\sigma}_{\langle\alpha\rangle}$ satisfies $(*)$ and hence coincides with $\Gamma^{\sigma}_{\langle\alpha\rangle}$. Thus $\mathsf{V}^{\mathsf{I}} = \bar{\Lambda}$ as well.

We complete the proof by evaluating the complexity of Γ and Λ. If $\mathsf{I} \in \Delta^1_r$, then $\Gamma \in \Sigma^1_r$ and $\Lambda \in \Pi^1_r$. For all $r \geqslant 1$ this implies $\mathsf{V}^{\mathsf{I}} \in \Pi^1_r$ and for all $r \geqslant 2$, $\mathsf{V}^{\mathsf{I}} \in \Sigma^1_r$. $\square$

1.6 Corollary. *For any* I,

(i) *if* $\mathsf{I} \in \Delta^1_1$, *then for all* R,

R *semi-recursive in* $\mathsf{I} \rightarrow \mathsf{R} \in \Pi^1_1$;

R *recursive in* $\mathsf{I} \rightarrow \mathsf{R} \in \Delta^1_1$;

(ii) *for any* $r \geqslant 2$, *if* $\mathsf{I} \in \Delta^1_r$, *then for all* R,

R *semi-recursive in* $\mathsf{I} \rightarrow \mathsf{R} \in \Delta^1_r$;

in particular, $\{\mathsf{R} : \mathsf{R}$ *recursive in* $\mathsf{I}\}$ *is a proper subset of* Δ^1_r.

Proof. If R is semi-recursive in I, then for some a,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists n\, \mathsf{V}^{\mathsf{I}}(a, \langle \mathbf{m} \rangle, n, \langle \boldsymbol{\alpha} \rangle),$$

and the implications are immediate from the theorem. $\square$

In particular, we have

$$\{\mathsf{R} : \mathsf{R} \text{ is recursive in } \mathsf{E}\} \subseteq \Delta^1_1 \quad \text{and} \quad \{\mathsf{R} : \mathsf{R} \text{ is recursive in } \mathsf{E}_1\} \subsetneqq \Delta^1_2.$$

To prove the converse of the first of these, we must "borrow" two results from later sections:

(†) (Corollary 2.11) for any I, the class of functionals partial recursive in I is closed under functional substitution — that is, for any functionals G and H partial recursive in I, if

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p \,.\, \mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha})),$$

then F is also partial recursive in I;

(††) (Corollary 4.3) the class of relations semi-recursive in E is closed under existential number quantification ($\exists^0$).

The basic tool is the following *Ordinal Comparison Lemma*:

1.7 Lemma. *There exists a functional* H *partial recursive in* E *such that for all* γ *and* δ,

(i) $\gamma \in \mathsf{W}$ *and* $\|\gamma\| \leq \|\delta\| \rightarrow \mathsf{H}(\gamma, \delta) \simeq 0$;

(ii) $\delta \in \mathsf{W}$ *and* $\|\delta\| < \|\gamma\| \rightarrow \mathsf{H}(\gamma, \delta) \simeq 1$.

Proof. Let the dual functional E° be defined by:

$$\mathsf{E}^\circ(\alpha) = 1 \dot{-} \mathsf{E}(\lambda p\, [1 \dot{-} \alpha(p)])$$

so that

$$\mathsf{E}^\circ(\alpha) = \begin{cases} 0, & \text{if } \forall p \,.\, \alpha(p) = 0; \\ 1, & \text{if } \exists p \,.\, \alpha(p) \neq 0. \end{cases}$$

Let

$$\mathsf{F}(e, \gamma, \delta) \simeq \begin{cases} 0, & \text{if } \|\gamma\| = 0; \\ 1, & \text{if } \|\gamma\| \neq 0 \wedge \|\delta\| = 0; \\ \mathsf{E}^\circ(\lambda p \,.\, \mathsf{E}(\lambda q \,.\, \{e\}^{\mathsf{E}}(\gamma \restriction p, \delta \restriction q))), & \text{otherwise.} \end{cases}$$

The relation $\|\gamma\| = 0$ is arithmetical, so by Lemma 1.3 and (†), F is partial recursive in E. Hence we may apply the E-Recursion Theorem to obtain an index $\bar{e}$ such that

$$\mathsf{F}(\bar{e}, \gamma, \delta) \simeq \{\bar{e}\}^{\mathsf{E}}(\gamma, \delta).$$

If we set $\mathsf{H} = \{\bar{e}\}^{\mathsf{E}}$, then H is partial recursive in E and

$$\mathsf{H}(\gamma, \delta) \simeq \begin{cases} 0, & \text{if } \|\gamma\| = 0; \\ 1, & \text{if } \|\gamma\| \neq 0 \wedge \|\delta\| = 0; \\ \mathsf{E}^\circ(\lambda p\,.\,\mathsf{E}(\lambda q\,.\,\mathsf{H}(\gamma \restriction p, \delta \restriction q))), & \text{otherwise.} \end{cases}$$

In other words, if both $\|\gamma\|$ and $\|\delta\|$ are non-zero, then $\mathsf{H}(\gamma, \delta)$ is defined just in case $\mathsf{H}(\gamma \restriction p, \delta \restriction q)$ is defined for all p and q, and if so, then

$$(1) \qquad \mathsf{H}(\gamma, \delta) \simeq 0 \leftrightarrow \forall p\, \exists q\,.\,\mathsf{H}(\gamma \restriction p, \delta \restriction q) \simeq 0;$$

$$(2) \qquad \mathsf{H}(\gamma, \delta) \simeq 1 \leftrightarrow \exists p\, \forall q\,.\,\mathsf{H}(\gamma \restriction p, \delta \restriction q) \simeq 1.$$

Clauses (i) and (ii) are vacuous unless one of γ and δ belongs to W. Hence we may prove (i) and (ii) by induction on $\sigma = \min\{\|\gamma\|, \|\delta\|\}$ for $\sigma < \aleph_1$. If $\sigma = 0$, then either $\|\gamma\| = 0$, so $\mathsf{H}(\gamma, \delta) \simeq 1$ in accord with (i), or $\|\delta\| = 0 < \|\gamma\|$, so $\mathsf{H}(\gamma, \delta) \simeq 1$ in accord with (ii). We assume now that $\sigma > 0$ and as induction hypothesis that (i) and (ii) hold for all γ_0 and δ_0 such that $\min\{\|\gamma_0\|, \|\delta_0\|\} < \sigma$.

Suppose first that $\gamma \in \mathsf{W}$ and $\|\gamma\| \leq \|\delta\|$. Then by (8) of § I.1, for all p, $\gamma \restriction p \in \mathsf{W}$ and $\|\gamma \restriction p\| < \|\gamma\| = \sigma$. Hence for all p and q, $\min\{\|\gamma \restriction p\|, \|\delta \restriction q\|\} < \sigma$ so from the induction hypothesis we have

$$(3) \qquad \|\gamma \restriction p\| \leq \|\delta \restriction q\| \to \mathsf{H}(\gamma \restriction p, \delta \restriction q) \simeq 0;$$

$$(4) \qquad \|\delta \restriction q\| < \|\gamma \restriction p\| \to \mathsf{H}(\gamma \restriction p, \delta \restriction q) \simeq 1.$$

As one of these holds for each p and q, we have $\mathsf{H}(\gamma \restriction p, \delta \restriction q)$ defined for all p, q. Since $\|\gamma\| \leq \|\delta\|$,

$$\forall p\, \exists q\,.\, \|\gamma \restriction p\| \leq \|\delta \restriction q\|,$$

so by (1) and (3), $\mathsf{H}(\gamma, \delta) \simeq 0$.

If $\delta \in \mathsf{W}$ and $\|\delta\| < \|\gamma\|$, then for all q, $\|\delta \restriction q\| < \|\delta\| = \sigma$, so again (3) and (4) hold for all p, q. Furthermore, there exists a $\bar{p}$ such that $\|\delta\| \leq \|\gamma \restriction \bar{p}\|$ and hence $\|\delta \restriction q\| < \|\gamma \restriction \bar{p}\|$ for all q. By (2) and (4), $\mathsf{H}(\gamma, \delta) \simeq 1$. □

1.8 Theorem. $\Delta^1_1 = \{\mathsf{R} : \mathsf{R}$ *is recursive in* $\mathsf{E}\}$.

Proof. The inclusion ($\supseteq$) is immediate from Corollary 1.6. Suppose $\mathsf{R} \in \Delta^1_1$, so both R and $\sim\mathsf{R}$ are Π_1^1. By Theorem IV.1.1 there exist recursive functionals F and G such that for all $(\mathbf{m}, \boldsymbol{\alpha})$,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}] \in \mathsf{W} \quad \text{and} \quad \sim\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{G}[\mathbf{m}, \boldsymbol{\alpha}] \in \mathsf{W}.$$

Then it is routine to check that for all $(\mathbf{m}, \boldsymbol{\alpha})$,

$$\mathsf{K}_{\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}) = \mathsf{H}(\mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}], \mathsf{G}[\mathbf{m}, \boldsymbol{\alpha}]),$$

which by (†) implies that K_{R} is recursive in E. □

1.9 Corollary. *For any $R \subseteq {}^k\omega$,*

$$R \in \Pi_1^1 \leftrightarrow R \textit{ is semi-recursive in } \mathsf{E}.$$

Proof. The implication ($\leftarrow$) is part of Corollary 1.6. Let $P(a) \leftrightarrow \{a\}^{\mathsf{E}}(a)\downarrow$. Clearly P is semi-recursive in E and thus is Π_1^1. However a diagonal argument shows P is not recursive in E and hence by the preceding theorem not Δ_1^1. By Theorem IV.1.8 there exists a recursive function f such that

$$P(a) \leftrightarrow f(a) \in W.$$

However, as $P \notin \Delta_1^1$, it follows from Theorem IV.2.2 that

$$\sup{}^+\{\|f(a)\| : P(a)\} = \omega_1.$$

Thus we have for all c,

$$\begin{aligned} c \in W &\leftrightarrow \exists a\,(P(a) \wedge \|c\| \leqslant \|f(a)\|) \\ &\leftrightarrow \{c\} \text{ is a total unary function and} \\ &\qquad\qquad \exists a\,(P(a) \wedge \mathsf{H}(\{c\}, \{f(a)\}) \simeq 0). \end{aligned}$$

The relation defined by the expression in parentheses is semi-recursive in E, so it follows from (††) that W is semi-recursive in E. Every $R \in \Pi_1^1$ is many-one reducible to W and thus is also semi-recursive in E. □

Exercise 1.16 provides an outline for extending Corollary 1.9 to all $\mathsf{R} \subseteq {}^{k,l}\boldsymbol{\omega}$:

1.10 Corollary. $\Pi_1^1 = \{\mathsf{R} : \mathsf{R}$ *is semi-recursive in* $\mathsf{E}\}$. □

It is worth noting that Lemma 1.7 (together with Corollary 1.6) provides a new proof of the pre-wellordering property for Π_1^1. Indeed, we have

$$\gamma \preccurlyeq_{\Pi}^{\mathsf{W}} \delta \leftrightarrow \gamma \in \mathsf{W} \wedge \mathsf{H}(\gamma, \delta) \simeq 0, \quad \text{and}$$
$$\gamma \preccurlyeq_{\Sigma}^{\mathsf{W}} \delta \leftrightarrow \delta \notin \mathsf{W} \vee \mathsf{H}(\gamma, \delta) \not\simeq 1.$$

We shall use a similar technique in § 3 to show that whenever E is recursive in I, then the class of relations semi-recursive in I has the pre-wellordering property.

Recursion relative to several functionals is defined by the usual coding. We call F *partial recursive in* $\mathbf{I} = (\mathsf{I}_0, \ldots, \mathsf{I}_n)$ iff F is partial recursive in the coded sequence $\langle \mathbf{I} \rangle$ where

$$\langle \mathbf{I} \rangle(\alpha) = \langle \mathsf{I}_0(\alpha), \ldots, \mathsf{I}_n(\alpha) \rangle.$$

Similarly, a functional F is *partial recursive in* $\mathbf{I}$ *and* β iff for some G partial recursive in $\mathbf{I}$, $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}, \beta)$.

Corresponding to the ordinary jump oJ on functions, we have for functionals the superjump sJ:

$$\mathsf{I}^{\mathsf{sJ}}((\langle a, \mathbf{m} \rangle) * \alpha) = \begin{cases} 0, & \text{if } \{a\}^{\mathsf{I}}(\mathbf{m}, \alpha)\downarrow; \\ 1, & \text{otherwise.} \end{cases}$$

The following is proved just as is Theorem II.5.7:

1.11 Theorem. *For any* I *and* $\boldsymbol{\beta}$ *and all* $\mathsf{R} \subseteq {}^{k,l}\omega$, *if* R *is semi-recursive in* I *(and* $\boldsymbol{\beta}$*), then* R *is recursive in* I^{sJ} *(and* $\boldsymbol{\beta}$*).* □

It follows that I^{sJ} is not recursive in I. Note that by Theorem 1.5, if $r \geqslant 2$ and $\mathsf{I} \in \Delta_r^1$, then also $\mathsf{I}^{\mathsf{sJ}} \in \Delta_r^1$.

1.12–1.20 Exercises

1.12. Show that the monotone operator which inductively defines $\Omega[\mathsf{I}]$ has closure ordinal exactly $\aleph_1$.

1.13 (Tugué [1960]). Show that if H and I are two functionals such that $\mathsf{H}(\beta) = \mathsf{I}(\beta)$ for all β recursive in I, then for all relations R on numbers, R is (semi-) recursive in I iff R is (semi-) recursive in H.

1.14. The class of functionals *primitive recursive in* I is the smallest class of total functionals which contains the initial functionals (of Definition II.1.1) and is closed under functional composition, primitive recursion, and I-Application, (I-$\mathrm{Ap}^{k,l}$), where for any functional G, I-$\mathrm{Ap}^{k,l}$(G) is the functional F of rank (k, l) such that

(a) if G is of rank $(k+1, l)$, then

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{I}(\lambda p \,.\, \mathsf{G}(p, \mathbf{m}, \boldsymbol{\alpha}));$$

(b) otherwise, $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq 0$.
Characterize the class of relations primitive recursive in E.

1.15. Give an alternative proof of Theorem 1.8 by constructing a primitive recursive function f such that for all $a \in N$ (Definition IV.4.1), $f(a)$ is an index of P_a from E, and applying Theorem IV.4.12.

1.16. Using the fact that the relation $\{a\}^{\mathsf{E}}(a, \boldsymbol{\alpha})\downarrow$ is $\Pi_1^1 \sim \Delta_1^1$, show that the relation $c \in W[\boldsymbol{\alpha}]$ is semi-recursive in E. Then apply the result of Exercise IV.1.24 to prove Corollary 1.10.

1.17. Suppose that F is partial recursive in E and $F: W \rightarrow W$. Show that there exists a function $G: W \rightarrow W$ partial recursive in E such that for all $c \in W$,

$$\|G(c)\| = \sup{}^{+}\{\|F(d)\| : \|d\| < \|c\|\}.$$

1.18. Show that for any relation R on numbers recursive in E_1,

$$\exists\beta\, \forall p\, R(\bar{\beta}(p), m) \leftrightarrow (\exists\beta \text{ recursive in } \mathsf{E}_1) \forall p\, R(\bar{\beta}(p), m).$$

Does this also hold with E in place of E_1? Does it hold with I in place of E_1 for any functional I such that E_1 is recursive in I?

1.19 (Hinman [1969]). For any I, let Iri (the set of I-recursive indices) be the smallest set such that for $u \in \mathsf{Iri}$ there exist functions $[u]$ which satisfy the following conditions:

(i) $0 \in \mathsf{Iri}$ and $[0](m) = 0$;

(ii) if $v \in \mathsf{Iri}$ and a is an index of a partial recursive functional $\{a\}$ such that for all m, $\{a\}(m, [v])\downarrow$, then $\langle 1, v, a\rangle \in \mathsf{Iri}$ and $[\langle 1, v, a\rangle](m) = \{a\}(m, [v])$;

(iii) if $v \in \mathsf{Iri}$ and for all m, $[v](m) \in \mathsf{Iri}$, then $\langle 2, v\rangle \in \mathsf{Iri}$ and $[\langle 2, v\rangle](m) = [[v](m)](m)$;

(iv) if $v \in \mathsf{Iri}$, then $\langle 3, v\rangle \in \mathsf{Iri}$ and $[\langle 3, v\rangle](m) = \mathsf{I}([v])$. Show that for any α, α is recursive in I iff for some $u \in \mathsf{Iri}$, $\alpha = [u]$.

1.20 (Gandy). Prove that each of $\mathsf{E}^{\#\mathsf{J}}$ and E_1 is recursive in the other.

1.21 Notes. The definition of recursive functionals with objects of types $\geqslant 2$ as arguments is due to Kleene [1959]. The formulation here is somewhat different from Kleene's. Kleene also proved Theorem 1.8 (by a method similar to that of Exercise 1.15). The only other basic treatment of recursion in higher types in print is Gandy [1967]. The superjump was invented and studied in Gandy [1967a]. Kleene also developed analogues of several of the other basic characterizations of ordinary recursion theory for higher types and showed them all equivalent (see Kleene [1963] for references).

2. Substitution Theorems

The first goal of this section is to establish that the class of functionals partial recursive in I is closed under functional substitution (($\dagger$) of the preceding section). Note that this is more than was true for ordinary recursion theory (cf. Exercise II.4.27 and Theorem II.3.9). As for ordinary recursion theory (cf. II.5.2), ($\dagger$) leads directly to:

(I) If β is recursive in I (and γ) and F is partial recursive in I and β, then F is partial recursive in I (and γ).

We are then led naturally to the following question — is it true also that

(II) if H is recursive in I and F is partial recursive in H, then F is partial recursive in I?

We shall see that although (II) is true, it becomes false if reference to I is omitted. Here we have only the weaker:

(III) if H is recursive and F is recursive in H, then F is recursive.

The proof of ($\dagger$) is substantially more complicated than the corresponding proof for ordinary recursion theory. This is due to the lack of simple normal forms for recursion relative to a functional. We must, rather, rely directly on the definition. A related complication is that we cannot prove the results as stated but must first prove effective versions. For example, for ($\dagger$) we show that there exists a primitive recursive function f such that if a and d are indices of G and H from I, respectively, then $f(a, d)$ is an index of F from I. The definitions are all by effective transfinite recursion.

First we see why (II) fails if the phrase "in I" is omitted:

2.1 Theorem. *There exists a recursive functional* H *and a functional* F *partial recursive in* H *such that* F *is not partial recursive.*

Proof. Let H be *any* (total) recursive functional, and R any Π_1^0 relation which is not recursive. By Lemma 1.2, R is semi-recursive in H, so there exists a functional F partial recursive in H such that R is the domain of F. Then $\mathrm{Dm}\,\mathsf{F}$ is not semi-recursive so F is not partial recursive. □

We shall approach ($\dagger$) by way of a series of approximations. We first treat one part of the special case in which H depends only on the function arguments $\boldsymbol{\alpha}$:

2.2 Lemma. *There exists a primitve recursive function f_0 such that for all* I, *a, d,* $\mathbf{m}$, $\boldsymbol{\alpha}$, *and n,*

$$\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p . \{d\}^{\mathsf{I}}(p, \boldsymbol{\alpha})) \simeq n \rightarrow \{f_0(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n.$$

Proof. We shall define first an auxiliary primitive recursive function h and then apply the Primitive Recursion Theorem (II.2.6) to obtain an index $\bar{e}$ such that for all a and d,

$$\{\bar{e}\}(a, d) \simeq h(\bar{e}, a, d).$$

$\{\bar{e}\}$ is the desired f_0.

We define h by course-of-values recursion on a and by cases according to the clauses of Definition 1.1 as follows: for any a, d, e, k, and l,

(0) if $a = \langle 0, k, l+1, i, \dots\rangle$ for $i = 0, 1, 2, 4$ or 5, then $h(e, a, d) = \langle 0, k, l, i, \dots\rangle$;

if $a = \langle 0, k, l+1, 3, i, j\rangle$ with $j < l$, then $h(e, a, d) = \langle 0, k, l, 3, i, j\rangle$;

if $a = \langle 0, k, l+1, 3, i, l\rangle$, then $h(e, a, d)$ is an index such that $\{h(e, a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \{d\}^{\mathsf{I}}(m_i, \boldsymbol{\alpha})$;

(1) if for some $k', b, c_0, \dots, c_{k'-1}$, $a = \langle 1, k, l+1, b, c_0, \dots, c_{k'-1}\rangle$, then $h(e, a, d) = \langle 1, k, l, h(e, b, d), h(e, c_0, d), \dots, h(e, c_{k'-1}, d)\rangle$;

(2) if $a = \langle 2, k+1, l+1\rangle$, let c be the natural index such that

$$\{c\}^{\mathsf{I}}(e, a, d, b, \mathbf{m}, \boldsymbol{\alpha}) \simeq \{\{e\}(b, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha})$$

and set $h(e, a, d) = \mathrm{Sb}_2(c, e, a, d)$;
(by a *natural index* for a given computation we mean the index which codes instructions to perform the computation in the manner indicated — in this case to compute first $\{e\}(b, d)$ and then use the result as an index applied to $(\mathbf{m}, \boldsymbol{\alpha})$. For the record we want here

$$c = \langle 1, k+4, l, \langle 2, k+1, l\rangle, e', \langle 0, k+4, l, 1, 4\rangle, \dots, \langle 0, k+4, l, 1, 3+k\rangle\rangle$$

where

$$e' = \langle 1, k+4, l, e, \langle 0, k+4, l, 1, 3\rangle, \langle 0, k+4, l, 1, 2\rangle\rangle).$$

(3) if for some b, $a = \langle 3, k, l+1, b\rangle$, then $h(e, a, d) = \langle 3, k, l, h(e, b, d)\rangle$;

(4) if a is none of these forms, $h(e, a, d) = 0$.

If $\lambda p . \{d\}^{\mathsf{I}}(p, \boldsymbol{\alpha})$ is not a total function, the implication of the theorem is trivial, so let d and $\boldsymbol{\alpha}$ be fixed such that $\lambda p . \{d\}^{\mathsf{I}}(p, \boldsymbol{\alpha})$ is a total function β. We prove by induction over $\Omega[\mathsf{I}]$ that for all a, $\mathbf{m}$, and n,

$$\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq n \rightarrow \{f_0(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n.$$

Let X denote the class of sequences $(a, \mathbf{m}, \boldsymbol{\alpha}, \beta, n)$ in $\Omega[\mathsf{I}]$ such that $\{f_0(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$. We must show that X satisfies the closure conditions which define $\Omega[\mathsf{I}]$. If a is of the form $\langle 0, k, l+1, \dots \rangle$ and $\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq n$, then it is obvious by inspection that for any e, $\{h(e, a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$ so in particular, $\{f_0(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$. Suppose next that $a = \langle 1, k, l+1, b, c_0, \dots, c_{k'-1} \rangle$ and for each $i < k'$ there exists a q_i such that

$$(c_i, \mathbf{m}, \boldsymbol{\alpha}, \beta, q_i) \in X \quad \text{and} \quad (b, \mathbf{q}, \boldsymbol{\alpha}, \beta, n) \in X.$$

Because $X \subseteq \Omega[\mathsf{I}]$ by definition, we have then

$$\{c_i\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq q_i \quad \text{and} \quad \{b\}^{\mathsf{I}}(\mathbf{q}, \boldsymbol{\alpha}, \beta) \simeq n,$$

so

$$\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n.$$

Furthermore, by the definition of X,

$$\{f_0(c_i, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq q_i \quad \text{and} \quad \{f_0(b, d)\}^{\mathsf{I}}(\mathbf{q}, \boldsymbol{\alpha}) \simeq n.$$

Hence, since

$$f_0(a, d) = \langle 1, k, l, f_0(b, d), f_0(c_0, d), \dots, f_0(c_{k'-1}, d) \rangle,$$

also

$$\{f_0(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$$

so $(a, \mathbf{m}, \boldsymbol{\alpha}, \beta, n) \in X$.

If $a = \langle 2, k+1, l+1 \rangle$ and $(b, \mathbf{m}, \boldsymbol{\alpha}, \beta, n) \in X$, then

$$\{a\}^{\mathsf{I}}(b, \mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq \{b\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq n$$

and

$$\{f_0(b, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n.$$

In this case, $f_0(a, d) = \mathrm{Sb}_2(c, \bar{e}, a, d)$ with c chosen exactly to make the following true:

$$\begin{aligned} \{f_0(a, d)\}^{\mathsf{I}}(b, \mathbf{m}, \boldsymbol{\alpha}) &\simeq \{c\}^{\mathsf{I}}(\bar{e}, a, d, b, \mathbf{m}, \boldsymbol{\alpha}) \\ &\simeq \{\{\bar{e}\}(b, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \\ &\simeq \{f_0(b, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n. \end{aligned}$$

Hence $(a, b, \mathbf{m}, \boldsymbol{\alpha}, \beta, n) \in X$. (In an attempt to rescue the reader from total confusion here we make two observations. First, the naturalness of c is not used here and will be needed only in the proof of Theorem 2.8. Second, the reader may be wondering why we did not use the simpler definition $h(e, a, d) = h(e, b, d)$ in this case. This is not possible because b comes from the argument list and not from the index, as in case (3), so it is not necessarily true that $b < a$ as would be required for the course-of-values recursion.)

Finally, suppose $a = \langle 3, k, l+1, b\rangle$ and γ is a function such that for all p, $(b, p, \mathbf{m}, \boldsymbol{\alpha}, \beta, \gamma(p)) \in X$. Then for all p,

$$\{b\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha}, \beta) = \gamma(p) \quad \text{and} \quad \{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \beta) = \mathsf{I}(\gamma).$$

Furthermore, for all p,

$$\{f_0(b, d)\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha}) = \gamma(p).$$

Hence, as $f_0(a, d) = \langle 3, k, l, f_0(b, d)\rangle$,

$$\begin{aligned}\{f_0(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) &= \mathsf{I}(\lambda p\,.\{f_0(b, d)\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha}))\\ &= \mathsf{I}(\gamma) = n,\end{aligned}$$

so again $(a, \mathbf{m}, \boldsymbol{\alpha}, \beta, n) \in X$. $\square$

2.3 Corollary. *There exists a primitive recursive function f_1 such that for all* I, *a, d, $\mathbf{m}$, $\boldsymbol{\alpha}$, and n,*

$$\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p\,.\{d\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha})) \simeq n \rightarrow \{f_1(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n.$$

Proof. Let h be a primitive recursive function such that for all d, p, $\mathbf{m}$, and $\boldsymbol{\alpha}$,

$$\{d\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \{h(d, \langle \mathbf{m}\rangle)\}^{\mathsf{I}}(p, \boldsymbol{\alpha}).$$

Then we take $f_1(a, d)$ to be an index such that

$$\{f_1(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \{f_0(a, h(d, \langle \mathbf{m}\rangle))\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}).$$

The result now follows from Lemma 2.2 by a direct computation. $\square$

With this corollary, we have in a sense completed half of the proof of (†): if G and H are partial recursive in I, say with indices a and d from I, and

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p\,.\mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha})),$$

then we have shown the existence of a functional F' partial recursive in I (with index $f_1(a, d)$ from I) such that $\mathsf{F} \subseteq \mathsf{F}'$. In fact F' may be properly larger than F. We shall complete the proof by showing that for $\mathbf{m}$ and $\boldsymbol{\alpha}$ such that $\lambda p . \mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha})$ is total, $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{F}'(\mathbf{m}, \boldsymbol{\alpha})$, and then observing that

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n \leftrightarrow \mathsf{I}(\lambda p . \mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha}))\downarrow \wedge \mathsf{F}'(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$$

so that F itself is partial recursive.

2.4 Definition. For each I and each $x \in \Omega[\mathsf{I}]$, the set $\mathrm{Sbc}(x)$ of *subcomputations of* x is defined recursively as follows: for any k, l, $\mathbf{m}$, $\boldsymbol{\alpha}$, and n,

(0) if $x = (\langle 0, k, l, \ldots \rangle, \mathbf{m}, \boldsymbol{\alpha}, n)$, then $\mathrm{Sbc}(x) = \varnothing$;

(1) if for some $k', b, c_0, \ldots, c_{k'-1}$, $x = (\langle 1, k, l, b, c_0, \ldots, c_{k'-1} \rangle, \mathbf{m}, \boldsymbol{\alpha}, n)$, and $q_0, \ldots, q_{k'-1}$ are the unique numbers such that for all $i < k'$ $y_i = (c_i, \mathbf{m}, \boldsymbol{\alpha}, q_i) \in \Omega[\mathsf{I}]$ and $z = (b, \mathbf{q}, \boldsymbol{\alpha}, n) \in \Omega[\mathsf{I}]$, then

$$\mathrm{Sbc}(x) = \bigcup \{\mathrm{Sbc}(y_i) : i < k'\} \cup \mathrm{Sbc}(z) \cup \{y_i : i < k'\} \cup \{z\};$$

(2) if for some b, $x = (\langle 2, k+1, l \rangle, b, \mathbf{m}, \boldsymbol{\alpha}, n)$, then

$$y = (b, \mathbf{m}, \boldsymbol{\alpha}, n) \in \Omega[\mathsf{I}] \quad \text{and} \quad \mathrm{Sbc}(x) = \mathrm{Sbc}(y) \cup \{y\};$$

(3) if for some b and β, $x = (\langle 3, k, l, b \rangle, \mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}(\beta))$ and for all p, $y_p = (b, p, \mathbf{m}, \boldsymbol{\alpha}, \beta(p)) \in \Omega[\mathsf{I}]$, then

$$\mathrm{Sbc}(x) = \bigcup \{\mathrm{Sbc}(y_p) : p \in \omega\} \cup \{y_p : p \in \omega\}.$$

This recursive definition is justified by the monomorphic character of the definition of $\Omega[\mathsf{I}]$ as in Theorem I.3.5 (cf. Exercise I.3.13).

We use the notion of subcomputation to derive a new principle of proof by induction over $\Omega[\mathsf{I}]$ which is related to ordinary proofs by induction over $\Omega[\mathsf{I}]$ as course-of-values induction over ω is related to ordinary induction over ω (cf. Exercise I.3.12).

2.5 Theorem. *For any $X \subseteq \Omega[\mathsf{I}]$, if for all $x \in \Omega[\mathsf{I}]$,*

$$\mathrm{Sbc}(x) \subseteq X \rightarrow x \in X,$$

then $X = \Omega[\mathsf{I}]$.

Proof. Suppose X satisfies the hypothesis of the theorem and set

$$Y = \{x : x \in X \wedge \mathrm{Sbc}(x) \subseteq X\}.$$

We show $Y = \Omega[\mathsf{I}]$ by ordinary induction over $\Omega[\mathsf{I}]$. If $x \in \Omega[\mathsf{I}]$ is of the form $x = (\langle 0, k, l, \ldots \rangle, \mathbf{m}, \boldsymbol{\alpha}, n)$, then $\mathrm{Sbc}(x) = \varnothing \subseteq X$ so $x \in X$ and thus also $x \in Y$.

The three inductive clauses are treated similarly and we do only (3). Suppose $x = (\langle 3, k, l, b \rangle, \mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}(\beta))$ and for all p, $y_p = (b, p, \mathbf{m}, \boldsymbol{\alpha}, \beta(p)) \in Y$. By the definition of Y, $\mathrm{Sbc}(y_p) \subseteq X$ for all p. Hence as $y_p \in X$, $\mathrm{Sbc}(x) \subseteq X$ and thus $x \in X$ and $x \in Y$. $\square$

For a given $x = (a, \mathbf{m}, \boldsymbol{\alpha}, n) \in \Omega[\mathsf{I}]$, the set $\{x\} \cup \mathrm{Sbc}(x)$ consists of exactly those sequences $(b, \mathbf{p}, \boldsymbol{\beta}, q)$ such that $\{b\}^{\mathsf{I}}(\mathbf{p}, \boldsymbol{\beta}) \simeq q$ and a node labeled $(b, \mathbf{p}, \boldsymbol{\beta})$ occurs in the tree for the computation $\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$. If we denote by $\mathrm{ISbc}(x)$ the *immediate* subcomputations of x — that is, those which appear in the tree immediately below x, then ordinary induction over $\Omega[\mathsf{I}]$ may be seen as deriving the conclusion $X = \Omega[\mathsf{I}]$ from the hypothesis $\mathrm{ISbc}(x) \subseteq X \rightarrow x \in X$.

Another way to view Theorem 2.5 is in terms of the levels of the inductive definition of $\Omega[\mathsf{I}]$. If Γ_{I} is the monotone operator implicit in Definition 1.1 such that $\Omega[\mathsf{I}] = \bar{\Gamma}_{\mathsf{I}}$, then for any σ and x,

$$x \in \Gamma_{\mathsf{I}}^{\sigma} \rightarrow \mathrm{Sbc}(x) \subseteq \Gamma_{\mathsf{I}}^{(\sigma)}.$$

Furthermore, if $X \subseteq \Omega[\mathsf{I}]$ satisfies $\Gamma_{\mathsf{I}}^{(\sigma)} \subseteq X \rightarrow \Gamma_{\mathsf{I}}^{\sigma} \subseteq X$, then clearly $X = \Omega[\mathsf{I}]$. The hypothesis of 2.5 is a refinement or "localization" of this condition in that membership of a given $x \in \Gamma_{\mathsf{I}}^{\sigma}$ in X requires only that the "relevant part" of $\Gamma_{\mathsf{I}}^{(\sigma)}$, $\mathrm{Sbc}(x)$, be included in X.

Before establishing the rest of (†), we need two technical lemmas whose proofs we leave to the reader (Exercise 2.20).

2.6 Lemma. *For all x and $y \in \Omega[\mathsf{I}]$,*

$$y \in \mathrm{Sbc}(x) \rightarrow \mathrm{Sbc}(y) \subseteq \mathrm{Sbc}(x). \quad \square$$

2.7 Lemma. *For any k, l, $i < k$, a, n, and $(\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$, if $\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$, then*

$$(a, \mathbf{m}, \boldsymbol{\alpha}, n) \in \mathrm{Sbc}(\mathrm{Sb}_i(a, m_0, \ldots, m_i), m_{i+1}, \ldots, m_{k-1}, \boldsymbol{\alpha}, n). \quad \square$$

2.8 Theorem. *For all I, a, d, $\mathbf{m}$, $\boldsymbol{\alpha}$, and n, if $\lambda p . \{d\}^{\mathsf{I}}(p, \boldsymbol{\alpha})$ is a total function, then*

$$\{f_0(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n \rightarrow \{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p . \{d\}^{\mathsf{I}}(p, \boldsymbol{\alpha})) \simeq n.$$

Proof. We shall apply Theorem 2.5 to the set

$$X = \{x : x \in \Omega[\mathsf{I}], \text{ and for all } a, d, \mathbf{m}, \boldsymbol{\alpha}, \text{ and } n, \text{ if } \lambda p . \{d\}^{\mathsf{I}}(p, \boldsymbol{\alpha}) \text{ is total}$$
$$\text{and } x = (f_0(a, d), \mathbf{m}, \boldsymbol{\alpha}, n), \text{ then } \{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p . \{d\}^{\mathsf{I}}(p, \boldsymbol{\alpha})) \simeq n\}.$$

We need thus to prove that for all $x \in \Omega[\mathrm{l}]$, if $\mathrm{Sbc}(x) \subseteq X$, then $x \in X$. If x is not of the form $x = (f_0(a, d), \mathbf{m}, \boldsymbol{\alpha}, n)$ with $\lambda p . \{d\}^{\mathrm{l}}(p, \boldsymbol{\alpha})$ total, there is nothing to prove, so we assume that it is of this form, that $\mathrm{Sbc}(x) \subseteq X$, and let β be the total function $\lambda p . \{d\}^{\mathrm{l}}(p, \boldsymbol{\alpha})$. We consider the cases (0)–(4) under which a may fall.

(0) If $a = \langle 0, k, l+1, \ldots \rangle$, the result is clear.

(1) If for some $k', b, c_0, \ldots, c_{k'-1}$, $a = \langle 1, k, l+1, b, c_0, \ldots, c_{k'-1} \rangle$, then $f_0(a, d) = \langle 1, k, l, f_0(b, d), f_0(c_0, d), \ldots, f_0(c_{k'-1}, d) \rangle$.
Since by assumption $\{f_0(a, d)\}^{\mathrm{l}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$, there exist $q_0, \ldots, q_{k'-1}$ such that for all $i < k'$,

$$\{f_0(c_i, d)\}^{\mathrm{l}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq q_i \quad \text{and} \quad \{f_0(b, d)\}^{\mathrm{l}}(\mathbf{q}, \boldsymbol{\alpha}) \simeq n.$$

Furthermore, for all $i < k'$,

$$(f_0(c_i, d), \mathbf{m}, \boldsymbol{\alpha}, q_i) \in \mathrm{Sbc}(x) \subseteq X,$$

and

$$(f_0(b, d), \mathbf{q}, \boldsymbol{\alpha}, n) \in \mathrm{Sbc}(x) \subseteq X.$$

Hence by the definition of X we have

$$\{c_i\}^{\mathrm{l}}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq q_i \quad \text{and} \quad \{b\}^{\mathrm{l}}(\mathbf{q}, \boldsymbol{\alpha}, \beta) \simeq n$$

and hence $\{a\}^{\mathrm{l}}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq n$, so $x \in X$.

(2) Suppose now that $a = \langle 2, k+1, l+1 \rangle$,

$$x = (f_0(a, d), b, \mathbf{m}, \boldsymbol{\alpha}, n) \in \Omega[\mathrm{l}], \quad \text{and} \quad \mathrm{Sbc}(x) \subseteq X.$$

Let c be as in the definition of f_0 so that

$$f_0(a, d) = \mathrm{Sb}_2(c, \bar{e}, a, d).$$

Thus

$$\begin{aligned} n \simeq \{f_0(a, d)\}^{\mathrm{l}}(b, \mathbf{m}, \boldsymbol{\alpha}) \simeq \{c\}^{\mathrm{l}}(\bar{e}, a, d, b, \mathbf{m}, \boldsymbol{\alpha}) \\ \simeq \{\{\bar{e}\}(b, d)\}^{\mathrm{l}}(\mathbf{m}, \boldsymbol{\alpha}) \\ \simeq \{f_0(b, d)\}^{\mathrm{l}}(\mathbf{m}, \boldsymbol{\alpha}). \end{aligned}$$

It follows from Lemma 2.7 that

$$(c, \bar{e}, a, d, b, \mathbf{m}, \boldsymbol{\alpha}, n) \in \mathrm{Sbc}(x).$$

The choice of c as the natural index for the indicated computation ensures that

$$(f_0(b, d), \mathbf{m}, \boldsymbol{\alpha}, n) \in \mathrm{Sbc}((c, \bar{e}, a, d, b, \mathbf{m}, \boldsymbol{\alpha}, n)).$$

Thus by Lemma 2.6, we have

$$(f_0(b, d), \mathbf{m}, \boldsymbol{\alpha}, n) \in \mathrm{Sbc}(x) \subseteq X.$$

Hence, by the definition of X, $\{b\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq n$ and thus also $\{a\}^{\mathsf{I}}(b, \mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq n$, so $x \in X$.

(3) If $a = \langle 3, k, l+1, b\rangle$ for some b, then $f_0(a, d) = \langle 3, k, l, f_0(b, d)\rangle$, and

$$n \simeq \{f_0(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{I}(\lambda p\,.\,\{f_0(b, d)\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha})).$$

Thus $\lambda p\,.\,\{f_0(b, d)\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha})$ is a total function γ (because I of it is defined) and for each p,

$$(f_0(b, d), p, \mathbf{m}, \boldsymbol{\alpha}, \gamma(p)) \in \mathrm{Sbc}(x) \subseteq X,$$

so for each p, $\{b\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq \gamma(p)$. Thus

$$\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq \mathsf{I}(\lambda p\,.\,\{b\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha}, \beta)) \simeq \mathsf{I}(\gamma) \simeq n.$$

(4) If a is none of these forms, $f_0(a, d) = 0$, so $x \notin \Omega[\mathsf{I}]$. $\square$

2.9 Corollary. *For all* I, a, d, $\mathbf{m}$, $\boldsymbol{\alpha}$, *and* n, *if* $\lambda p\,.\,\{d\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha})$ *is total, then*

$$\{f_1(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n \rightarrow \{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p\,.\,\{d\}^{\mathsf{I}'}(p, \mathbf{m}, \boldsymbol{\alpha})) \simeq n.$$

Proof. Suppose $\lambda p\,.\,\{d\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha})$ is a total function β and $\{f_1(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$. Then with notation as in Corollary 2.3,

$$\{f_0(a, h(d, \langle \mathbf{m}\rangle))\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$$

and by the preceding theorem,

$$\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p\,.\,\{h(d, \langle \mathbf{m}\rangle)\}^{\mathsf{I}}(p, \boldsymbol{\alpha})) \simeq n.$$

But h was chosen just so that for all p,

$$\{h(d, \langle \mathbf{m}\rangle)\}^{\mathsf{I}}(p, \boldsymbol{\alpha}) \simeq \{d\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \beta(p)$$

so that $\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq n$. $\square$

Now to establish (†) it remains only to remove the hypothesis that $\lambda p.\{d\}^{\mathsf{I}}(p,\mathbf{m},\boldsymbol{\alpha})$ be total. This we achieve by a simple trick.

2.10 Theorem. *There exists a primitive recursive function f such that for all* I, *a, d,* $\mathbf{m}$ *and* $\boldsymbol{\alpha}$,

$$\{f(a,d)\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}) \simeq \{a\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha},\lambda p.\{d\}^{\mathsf{I}}(p,\mathbf{m},\boldsymbol{\alpha})).$$

Proof. We want $f(a,d)$ to be an index such that

$$\{f(a,d)\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}) \simeq 0\cdot \mathsf{I}(\lambda p.\{d\}^{\mathsf{I}}(p,\mathbf{m},\boldsymbol{\alpha}))+\{f_1(a,d)\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}).$$

Clearly such an f can be defined explicitly from f_1 and indices for $+$ and $\cdot$ from I. Then

$$\begin{aligned}\{f(a,d)\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}) \simeq n &\leftrightarrow \mathsf{I}(\lambda p.\{d\}^{\mathsf{I}}(p,\mathbf{m},\boldsymbol{\alpha}))\downarrow \wedge \{f_1(a,d)\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}) \simeq n\\ &\leftrightarrow \lambda p.\{d\}^{\mathsf{I}}(p,\mathbf{m},\boldsymbol{\alpha}) \text{ is total} \wedge \{f_1(a,d)\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}) \simeq n\\ &\leftrightarrow \{a\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha},\lambda p.\{d\}^{\mathsf{I}}(p,\mathbf{m},\boldsymbol{\alpha})) \simeq n.\end{aligned}$$

The last equivalence uses Corollary 2.9 and the fact that computations are defined only for total arguments. □

2.11 Corollary (†). *For every* I, *the class of functionals partial recursive in* I *is closed under functional substitution.* □

The reader should ontrast this with the result of Exercise II.4.27 to see that the use of I in the proof of Theorem 2.10 is really essential.

To derive (I) from this we need that the class of functionals partial recursive in I is closed under expansion. This could have been proved earlier, but the method of proof is similar to that of Lemma 2.2 and at this point becomes an exercise.

2.12 Lemma. *There exists a primitive recursive function* f_2 *such that for all* I, *d, k, l,* $(\mathbf{m},\boldsymbol{\alpha})\in {}^{k,l}\omega$, $\mathbf{p}$, *and* $\boldsymbol{\beta}$,

$$\{f_2(d,k,l)\}^{\mathsf{I}}(\mathbf{p},\mathbf{m},\boldsymbol{\alpha},\boldsymbol{\beta}) \simeq \{d\}^{\mathsf{I}}(\mathbf{p},\boldsymbol{\beta}).$$

Proof. Exercise 2.21. □

2.13 Corollary (I). *For any* I, F, β, *and* γ, *if* β *is recursive in* I *(and* γ*) and* F *is partial recursive in* I *and* β, *then* F *is partial recursive in* I *(and* γ*).*

Proof. We prove the version without γ. Suppose $\mathsf{F}(\mathbf{m},\boldsymbol{\alpha}) \simeq \mathsf{G}(\mathbf{m},\boldsymbol{\alpha},\beta)$ and a and d are indices of G and β, respectively, from I. Then

$$\begin{aligned}\mathsf{F}(\mathbf{m},\boldsymbol{\alpha}) &\simeq \{a\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha},\lambda p.\{d\}^{\mathsf{I}}(p))\\ &\simeq \{a\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha},\lambda p.\{f_2(d,k,l)\}^{\mathsf{I}}(p,\mathbf{m},\boldsymbol{\alpha}))\\ &\simeq \{f(a,f_2(d,k,l))\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}). \quad \square\end{aligned}$$

We are now in a position to prove (the effective version of) (II). The technique is an extension of that used above.

2.14 Theorem. *There exists a primitive recursive function g such that for all* I, a, d, $\mathbf{m}$, *and* $\boldsymbol{\alpha}$, *if* $\lambda\beta.\{d\}^{\mathsf{I}}(\beta)$ *is a total functional* H, *then*

$$\{g(a,d)\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}) \simeq \{a\}^{\mathsf{H}}(\mathbf{m},\boldsymbol{\alpha}).$$

Proof. We define g as in the proof of Lemma 2.2 via the Recursion Theorem and an auxiliary primitive recursive function h, which is defined by cases as follows: for any a, d, e, k, and l,

(0) if $a = \langle 0,k,l,\ldots\rangle$, then $h(e,a,d) = a$;

(1) if for some k', b, $c_0,\ldots,c_{k'-1}$, $a = \langle 1,k,l,b,c_0,\ldots,c_{k'-1}\rangle$, then $h(e,a,d) = \langle 1,k,l,h(e,b,d),h(e,c_0,d),\ldots,h(e,c_{k'-1},d)\rangle$;

(2) if $a = \langle 2,k+1,l\rangle$, then h is defined exactly as in the corresponding case in the proof of Lemma 2.2;

(3) if for some b, $a = \langle 3,k,l,b\rangle$, then $h(e,a,d)$ is the natural index such that

$$\begin{aligned}\{h(e,a,d)\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}) \simeq{}& 0\cdot \mathsf{I}(\lambda p.\{h(e,b,d)\}^{\mathsf{I}}(p,\mathbf{m},\boldsymbol{\alpha}))\\ &+\{f(f_2(d,k,l),h(e,b,d))\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}),\end{aligned}$$

where f and f_2 are from 2.10 and 2.12, respectively;

(4) if a is of none of these forms, then $h(e,a,d) = 0$.

It is clear that h is primitve recursive, so let $\bar{e}$ be an index such that $\{\bar{e}\}(a,d) = h(\bar{e},a,d)$ and take $g = \{\bar{e}\}$. We need to prove that for all a, $\mathbf{m}$, $\boldsymbol{\alpha}$, and n,

$$\{g(a,d)\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}) \simeq n \leftrightarrow \{a\}^{\mathsf{H}}(\mathbf{m},\boldsymbol{\alpha}) \simeq n$$

under the assumption that $\mathsf{H} = \lambda\beta.\{d\}^{\mathsf{I}}(\beta)$ is total.

The implication ($\leftarrow$) is proved by induction over $\Omega[\mathsf{H}]$ much as in Lemma 2.2. Let X denote the set of sequences $(a,\mathbf{m},\boldsymbol{\alpha},n) \in \Omega[\mathsf{H}]$ such that $\{g(a,d)\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}) \simeq n$. We must show that X satisfies the closure conditions which define $\Omega[\mathsf{H}]$.

(0) if $a = \langle 0,k,l,\ldots\rangle$ and $\{a\}^{\mathsf{H}}(\mathbf{m},\boldsymbol{\alpha}) \simeq n$, then $g(a,d) = a$, so $\{g(a,d)\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}) \simeq \{a\}^{\mathsf{I}}(\mathbf{m},\boldsymbol{\alpha}) \simeq \{a\}^{\mathsf{H}}(\mathbf{m},\boldsymbol{\alpha}) \simeq n$. Thus $(a,\mathbf{m},\boldsymbol{\alpha},n) \in X$.

$\left.\begin{array}{l}(1)\\(2)\end{array}\right\}$ These are treated exactly as in the proof of Lemma 2.2.

(3) If for some b, $a = \langle 3, k, l, b\rangle$, then for some β, $(b, p, \mathbf{m}, \boldsymbol{\alpha}, \beta(p)) \in X$ for all p. Then for all p,

$$\{g(b, d)\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \beta(p)$$

so

$$\begin{aligned}\{g(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) &\simeq 0 \cdot \mathsf{I}(\beta) + \{f(f_2(d, k, l), g(b, d))\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha})\\ &\simeq \{f_2(d, k, l)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, \beta)\\ &\simeq \{d\}^{\mathsf{I}}(\beta)\\ &\simeq \mathsf{H}(\beta)\\ &\simeq \mathsf{H}(\lambda p\,.\{b\}^{\mathsf{H}}(p, \mathbf{m}, \boldsymbol{\alpha})) \simeq n.\end{aligned}$$

Thus $(a, \mathbf{m}, \boldsymbol{\alpha}, n) \in X$.

For the implication $(\rightarrow)$ we apply Theorem 2.5 to the set $X = \{x : x \in \Omega[\mathsf{I}]$, and for all a, d, $\mathbf{m}$, $\boldsymbol{\alpha}$, and n, if

$$x = (g(a, d), \mathbf{m}, \boldsymbol{\alpha}, n) \quad \text{then} \quad \{a\}^{\mathsf{H}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n\}.$$

We need to prove that for all $x \in \Omega[\mathsf{I}]$, if $\mathrm{Sbc}(x) \subseteq X$, then $x \in X$. If x is not of the form $x = (g(a, d), \mathbf{m}, \boldsymbol{\alpha}, n) \in \Omega[\mathsf{I}]$, then there is nothing to prove, so we assume that it is of this form and that $\mathrm{Sbc}(x) \subseteq X$. We consider the case (0)–(4) under which a may fall.

$\left.\begin{array}{l}(0)\\(1)\\(2)\end{array}\right\}$ These cases are treated as in the proof of Theorem 2.8.

(3) If for some b, $a = \langle 3, k, l, b\rangle$, then since $\{g(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha})\downarrow$, it follows from the definition of g in this case that $\mathsf{I}(\lambda p\,.\{g(b, d)\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha}))\downarrow$ and thus that for some β, $\{g(b, d)\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \beta(p)$ for all p. Furthermore, for all p,

$$(g(b, d), p, \mathbf{m}, \boldsymbol{\alpha}, \beta(p)) \in \mathrm{Sbc}(x) \subseteq X$$

(this is why $h(e, a, d)$ must be a natural index in this case). Thus, for all p, $\{b\}^{\mathsf{H}}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \beta(p)$ and

$$\begin{aligned}\{a\}^{\mathsf{H}}(\mathbf{m}, \boldsymbol{\alpha}) &\simeq \mathsf{H}(\lambda p\,.\{b\}^{\mathsf{H}}(p, \mathbf{m}, \boldsymbol{\alpha}))\\ &\simeq \mathsf{H}(\beta)\\ &\simeq \{g(a, d)\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n,\end{aligned}$$

by the computation in the first half of the proof.

(4) If a is of none of these forms, $g(a, d) = 0$ so $x \notin \Omega[\mathsf{I}]$. □

2.15 Corollary (II). *For any* F, H, *and* I, *if* H *is recursive in* I *and* F *is partial recursive in* H, *then* F *is partial recursive in* I. □

Of course, we used the same trick in the induction step (3) of the definition of g as we did in Theorem 2.11 and it is clear that the proof will not work without I.

2.16 Corollary. *The relation "recursive in" is transitive among total functionals.* □

2.17 Corollary. *For any* H *and* I, *if* H *is recursive in* I, *then* H^{sJ} *is recursive in* I^{sJ}.

Proof. Suppose H is recursive in I with index d. Then with g as in Theorem 2.14

$$\begin{aligned} \mathsf{H}^{sJ}((\langle a, \mathbf{m}\rangle) * \alpha) = 0 &\leftrightarrow \{a\}^{\mathsf{H}}(\mathbf{m}, \alpha) \text{ is defined} \\ &\leftrightarrow \{g(a, d)\}^{\mathsf{I}}(\mathbf{m}, \alpha) \text{ is defined} \\ &\leftrightarrow \mathsf{I}^{sJ}((\langle g(a, d)\rangle, \langle \mathbf{m}\rangle) * \alpha) \text{ is defined.} \quad \square \end{aligned}$$

2.18 Theorem. *There exists a primitive recursive function g' such that for all a, d, $\mathbf{m}$, $\boldsymbol{\alpha}$, and n, if $\lambda\beta.\{d\}(\beta)$ is a total recursive functional* H, *then*

$$\{a\}^{\mathsf{H}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n \rightarrow \{g'(a, d)\}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n.$$

Proof. Exercise 2.22. □

2.19 Corollary (III). *If* H *is recursive and* F *is recursive in* H, *then* F *is recursive.*

Proof. If F has index a from H and H is recursive with index d, then $\mathsf{F} \subseteq \{g'(a, d)\}$ by the preceding theorem. Since F is total, these functionals are equal. □

2.20–2.29 Exercises

2.20. Prove Lemmas 2.6 and 2.7. Along the same lines, show that if a is any I-index for a functional F partial recursive in I and $\bar{e}$ is the natural index given by the proof of the I-Recursion Theorem such that

$$\{\bar{e}\}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{F}(\bar{e}, \mathbf{m}, \boldsymbol{\alpha})$$

then

$$(a, \bar{e}, \mathbf{m}, \boldsymbol{\alpha}, \mathsf{F}(\bar{e}, \mathbf{m}, \boldsymbol{a})) \in \mathrm{Sbc}(\bar{e}, \mathbf{m}, \boldsymbol{\alpha}, \mathsf{F}(\bar{e}, \mathbf{m}, \boldsymbol{\alpha})).$$

2.21. Prove Lemma 2.12.

2.22. Prove Theorem 2.18.

2.23. Let I be any functional such that E_1 is recursive in I and denote by $\omega_1[\mathsf{I}]$ the least ordinal not the order-type of a well-ordering of ω recursive in I. Show that

(i) for any β recursive in I, $\omega_1[\beta] < \omega_1[\mathsf{I}]$;

(ii) for any $\sigma < \omega_1[\mathsf{I}]$, there exists a β recursive in I such that $\sigma < \omega_1[\beta]$.

2.24. Show that for any I such that E is recursive in I and any relation P semi-recursive in I, if

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall\beta\, \exists p\; \mathsf{P}(\bar{\beta}(p), \mathbf{m}, \boldsymbol{\alpha})$$

then also R is semi-recursive in I.

2.25. (Cf. Exercise II.4.31). For any I and any partial function f of rank 1, let $\Omega[\mathsf{I}, f]$ be defined as is $\Omega[\mathsf{I}]$ with the following additional clause:

$$\text{if} \quad f(p) \simeq n, \quad \text{then} \quad (\langle 0, k+1, l, 6\rangle, p, \mathbf{m}, \boldsymbol{\alpha}, n) \in \Omega[\mathsf{I}, f].$$

We write

$$\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}, f) \simeq n \leftrightarrow (a, \mathbf{m}, \boldsymbol{\alpha}, n) \in \Omega[\mathsf{I}, f].$$

We have thus defined the class of functionals partial recursive in I with arguments of type $(\mathbf{m}, \boldsymbol{\alpha}, f)$. Show that for any such functional F partial recursive in I,

(i) there exists an ordinary functional G partial recursive in I such that

$$\mathsf{G}(e, \mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}, \{e\}^{\mathsf{I}});$$

(ii) (*First* I*-Recursion Theorem*) there exists a function $\bar{f}$ partial recursive in I such that for all p, $\mathsf{F}(p, \bar{f}) \simeq \bar{f}(p)$ and for any h, if also for all p, $\mathsf{F}(p, h) \simeq h(p)$, then $\bar{f} \subseteq h$.

(iii) Formulate and prove a version of (ii) which allows for the presence of parameters.

Hint. (i) is immediate from a minor modification of Lemma 2.9. For (ii), show that if G is chosen correctly in (i) with, say, index b from I, then if $\mathsf{G}(e, p) \simeq n$ and g is a partial function defined by:

$$g(q) \simeq r \leftrightarrow (e, q, r) \in \mathrm{Sbc}((b, e, p, n)),$$

then $\mathsf{F}(p, g) \simeq n$ (i.e. all of the values of $\{e\}$ which are "necessary" in order that $\mathsf{F}(p, \{e\}) \simeq n$ are computed as subcomputations of the computation $\mathsf{G}(e, p) \simeq n$). Then show that if $\bar{e}$ is defined as in the proof of the I-Recursion Theorem such that $\{\bar{e}\}^{\mathsf{I}}(m) \simeq \mathsf{G}(\bar{e}, m)$, then $\bar{f} = \{\bar{e}\}^{\mathsf{I}}$ is the required solution.

2.26. For any I and any inductive operator Γ on ω, Γ is *positive semi-recursive* in I iff there exists a functional F partial recursive in I in the sense of the preceding Exercise such that for any m and f

$$\mathsf{F}(m, f) \simeq 0 \leftrightarrow m \in \Gamma(\{p : f(p) \simeq 0\}).$$

(i) Use the First I-Recursion Theorem to show that for any Γ which is positive semi-recursive in I, $\bar{\Gamma}$ is semi-recursive in I.

(ii) Prove that every Π^1_1 set of numbers is semi-recursive in I (for arbitrary I!).

(iii) Discuss the relationship of this to Theorem 1.8 and Corollary 1.9. Does it follow that every Δ^1_1 set is recursive in an arbitrary I?

2.27. Characterize the classes of relations recursive and semi-recursive in the functional $\lambda\alpha\,.\,0$.

2.28. A functional I is called *effectively discontinuous* iff there exists a function F recursive in I such that the sequence of functions $F_p = \lambda m\,.\,F(p, m)$ converges to a function G, but the sequence of values $\mathsf{I}(F_p)$ does not converge to $\mathsf{I}(G)$. Show that I is effectively discontinuous iff E is recursive in I.

2.29. Show that for any I, E is recursive in I iff $\{\alpha : \alpha$ is recursive in I$\}$ is closed under the ordinary jump oJ. (Suppose E is not recursive in I so by the preceding exercise I is not effectively discontinuous. For each s let $\beta_s = ((s)_0, \ldots, (s)_{\mathrm{lg}(s)-1}) * (\lambda m\,.\,0)$ and $\gamma(s) = \mathsf{I}(\beta_s)$. Show that every α recursive in I is recursive in γ^{oJ}.)

2.30 Notes. (I), (II) for total F, and (III) and their effective versions are due to Kleene [1963]. The improved versions 2.10 and (II) (2.14) first appear in Hinman [1966]. Kleene [1963] gave the counterexample 2.1. Exercises 2.28 and 2.29 are due to Grilliot [1971].

3. *Ordinal Comparison*

This section is devoted to the proof of a technical result, the Ordinal Comparison Theorem, which is the key to the theory of relations semi-recursive in a type-2 functional. This theorem is closely related, in both statement and proof, to

Lemma 1.7. There we were able, using E, to compare the ordinals assigned to members of W, a set to which all relations semi-recursive in E can be reduced. For an arbitrary functional I such that E is recursive in I, we shall compare the ordinals assigned to members of U^{I}, a relation universal for the class of relations semi-recursive in I.

3.1 Definition. For any I and $\boldsymbol{\alpha}$,

(i) $\mathsf{U}^{\mathsf{I}}(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) \leftrightarrow \{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha})\downarrow$;

(ii) $U^{\mathsf{I}}_{\boldsymbol{\alpha}} = \{\langle a, \mathbf{m}\rangle : \mathsf{U}^{\mathsf{I}}(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle)\}$;

(iii) $U^{\mathsf{I}} = U^{\mathsf{I}}_{\varnothing}$.

To each element of U^{I} we assign an ordinal which intuitively measures how "long" the corresponding computation is:

$$|(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle)|^{\mathsf{I}} = \text{least } \sigma.\, (a, \mathbf{m}, \boldsymbol{\alpha}, \{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha})) \in \Omega[\mathsf{I}]^{\sigma},$$

where, as usual, $\Omega[\mathsf{I}]^{\sigma}$ is the σ-th stage of the inductive definition of $\Omega[\mathsf{I}]$. If $\sim\mathsf{U}^{\mathsf{I}}(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle)$, we set $|(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle)|^{\mathsf{I}} = \aleph_1$. Similarly, for any $\boldsymbol{\alpha}$,

$$|\langle a, \mathbf{m}\rangle|^{\mathsf{I}}_{\boldsymbol{\alpha}} = |(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle)|^{\mathsf{I}}.$$

We often write $|a, \mathbf{m}, \boldsymbol{\alpha}|^{\mathsf{I}}$ and $|a, \mathbf{m}|^{\mathsf{I}}_{\boldsymbol{\alpha}}$ and omit the sub- and superscripts when they are clear from the context.

3.2 Lemma. *For any* I, a, k, l, *and* $(\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$,

(0) *if* $a = \langle 0, k, l, \ldots\rangle$ *and* $\langle a, \mathbf{m}\rangle \in U^{\mathsf{I}}_{\boldsymbol{\alpha}}$, *then* $|a, \mathbf{m}|^{\mathsf{I}}_{\boldsymbol{\alpha}} = 0$;

(1) *if for some* $b, c_0, \ldots, c_{k'-1}$, $a = \langle 1, k, l, b, c_0, \ldots, c_{k'-1}\rangle$ *and* $\langle a, \mathbf{m}\rangle \in U^{\mathsf{I}}_{\boldsymbol{\alpha}}$, *then*

$$|a, \mathbf{m}|^{\mathsf{I}}_{\boldsymbol{\alpha}} = \max\{|c_i, \mathbf{m}|^{\mathsf{I}}_{\boldsymbol{\alpha}} + 1 : i < k'\} \cup \{|b, \mathbf{q}|^{\mathsf{I}}_{\boldsymbol{\alpha}} + 1\},$$

where for all $i < k'$, $\{c_i\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq q_i$;

(2) *if* $a = \langle 2, k+1, l\rangle$, *and* $\langle a, b, \mathbf{m}\rangle \in U^{\mathsf{I}}_{\boldsymbol{\alpha}}$, *then*

$$|a, b, \mathbf{m}|^{\mathsf{I}}_{\boldsymbol{\alpha}} = |b, \mathbf{m}|^{\mathsf{I}}_{\boldsymbol{\alpha}} + 1;$$

(3) *if for some* b, $a = \langle 3, k, l, b\rangle$ *and* $\langle a, \mathbf{m}\rangle \in U^{\mathsf{I}}_{\boldsymbol{\alpha}}$, *then*

$$|a, \mathbf{m}|^{\mathsf{I}}_{\boldsymbol{\alpha}} = \sup{}^{+}\{|b, p, \mathbf{m}|^{\mathsf{I}}_{\boldsymbol{\alpha}} : p \in \omega\}.$$

Proof. Immediate from the definitions. $\square$

3.3 Ordinal Comparison Theorem. *For any* I *such that* E *is recursive in* I, *there exists a functional* H *partial recursive in* I *such that for all* u^0, u^1, *and* $\boldsymbol{\alpha}$,

(i) *if* $u^0 \in U^{\mathsf{I}}_{\boldsymbol{\alpha}}$ *and* $|u^0|^{\mathsf{I}}_{\boldsymbol{\alpha}} \leqslant |u^1|^{\mathsf{I}}_{\boldsymbol{\alpha}}$, *then* $\mathsf{H}(u^0, u^1, \langle \boldsymbol{\alpha} \rangle) \simeq 0$;
(ii) *if* $u^1 \in U^{\mathsf{I}}_{\boldsymbol{\alpha}}$ *and* $|u^1|^{\mathsf{I}}_{\boldsymbol{\alpha}} < |u^0|^{\mathsf{I}}_{\boldsymbol{\alpha}}$, *then* $\mathsf{H}(u^0, u^1, \langle \boldsymbol{\alpha} \rangle) \simeq 1$.

Proof. To simplify things slightly, we shall give the proof for the case $\boldsymbol{\alpha} = \varnothing$; the general case merely requires including $\langle \boldsymbol{\alpha} \rangle$ as a parameter throughout. The basic intuition behind the proof is the same as for Lemma 1.7: to compare the ordinals $|u^0|$ and $|u^1|$, H compares the ordinals assigned to the "immediate subcomputations" of u^0 and u^1. If at least one of u^0 and u^1 belongs to U^{I}, some of these will be smaller than $\min\{|u^0|, |u^1|\}$ and a recursion will be established. Of course, the partial recursiveness of H is ensured by means of the I-Recursion Theorem: we define first a functional F partial recursive in I, choose $\bar{e}$ such that $\{\bar{e}\}^{\mathsf{I}}(u^0, u^1) \simeq \mathsf{F}(\bar{e}, u^0, u^1)$, and set $\mathsf{H} = \{\bar{e}\}^{\mathsf{I}}$.

The definition of F is divided into 16 cases labeled (r, s), with $0 \leqslant r, s \leqslant 3$ plus two "otherwise" cases. In case (r, s) we define $\mathsf{F}(e, u^0, u^1)$ for all u^0 and u^1 such that $u^i = \langle a^i, \mathbf{m}^i \rangle$, a^0 is an index appropriate for the arguments $\mathbf{m}^0$ under clause (r) of the definition of $\Omega[\mathsf{I}]$, and a^1 is an index appropriate for the arguments $\mathbf{m}^1$ under clause (s). "Appropriate" means merely that $(a^0)_0 = r$, $(a^0)_1 = \lg(\mathbf{m}^0)$, and $(a^0)_2 = 0$, etc., not that the corresponding computation is defined. If u^i is not of the form $\langle a^i, \mathbf{m}^i \rangle$ with a^i an appropriate index, we say that u^i is not of the *proper form*.

First, if u^1 is not of the proper form, we set $\mathsf{F}(e, u^0, u^1) \simeq 0$, while if u^1 is of the proper form but u^0 is not $\mathsf{F}(e, u^0, u^1) \simeq 1$.

Cases $(0, s)$ *with* $0 \leqslant s \leqslant 3$. $\mathsf{F}(e, u^0, u^1) \simeq 0$.

Cases $(r, 0)$ *with* $1 \leqslant r \leqslant 3$. $\mathsf{F}(e, u^0, u^1) \simeq 1$.

Case (1,2). The assumption of proper form for this case means that for some b^0, c, and b^1, $k^0 = \lg(\mathbf{m}^0)$, and $k^1 = \lg(\mathbf{m}^1)$, $u^0 = \langle\langle 1, k^0, 0, b^0, c \rangle, \mathbf{m}^0 \rangle$ and $u^1 = \langle\langle 2, k^1 + 1, 0 \rangle, b^1, \mathbf{m}^1 \rangle$ (for simplicity we have taken $k^{0\prime} = 1$). Our aim is that the computation of H should proceed according to the following "flow diagram":

$$\begin{tikzcd} \boxed{\mathsf{H}(\langle c^0, \mathbf{m}^0 \rangle, \langle b^1, \mathbf{m}^1 \rangle)} \arrow[r, "0"] \arrow[d, "1"] & \boxed{\mathsf{H}(\langle b^0, q^0 \rangle, \langle b^1, \mathbf{m}^1 \rangle)} \arrow[d, "0"] \arrow[d, shift left=6, "1"] \\ {} & {} \end{tikzcd}$$

where q^0 denotes the (possibly undefined) value of $\{c\}^{\mathsf{I}}(\mathbf{m}^0)$. To effect this, we set

$$\mathsf{F}(e, u^0, u^1) \simeq \mathsf{G}(\{e\}^{\mathsf{I}}(\langle c^0, \mathbf{m}^0 \rangle, \langle b^1, \mathbf{m}^1 \rangle), e, u^0, u^1)$$

where

$$G(n, e, u^0, u^1) \simeq \begin{cases} \{e\}^{\mathsf{I}}(\langle b^0, q^0 \rangle, \langle b^1, \mathbf{m}^1 \rangle), & \text{if} \quad n = 0; \\ 1, & \text{if} \quad n = 1. \end{cases}$$

Case (1,1). In this and succeeding cases we shall give only the recursive conditions we want H to satisfy — in some cases by means of a flow diagram as above — and leave it to the reader to verify that F may be defined so that the resulting H does indeed satisfy these conditions. Here again we assume for simplicity that $k^{0'} = k^{1'} = 1$. The computation of H (u^0, u^1) is to proceed according to the diagram:

$$\begin{tikzcd}
\boxed{\mathsf{H}(\langle c^0, \mathbf{m}^0\rangle, \langle c^1, \mathbf{m}^1\rangle)} \arrow[r, "0"] \arrow[d, "1"] & \boxed{\mathsf{H}(\langle b^0, q^0\rangle, \langle c^1, \mathbf{m}^1\rangle)} \arrow[r, "0"] \arrow[d, "1"] & {} \\
\boxed{\mathsf{H}(\langle c^0, \mathbf{m}^0\rangle, \langle b^1 q^1\rangle)} \arrow[r, "0"] \arrow[d, "1"] & \boxed{\mathsf{H}(\langle b^0, q^0\rangle, \langle b^1, q^1\rangle)} \arrow[r, "0"] \arrow[d, "1"] & {} \\
{} & {} &
\end{tikzcd}$$

where q^i denotes the (possibly undefined) value of $\{c^i\}^1(\mathbf{m}^i)$.

Case (2,1).

$$\begin{tikzcd}
\boxed{\mathsf{H}(\langle b^0, \mathbf{m}^0\rangle, \langle c^1, \mathbf{m}^1\rangle)} \arrow[r, "1"] \arrow[d, "0"] & \boxed{\mathsf{H}(\langle b^0, \mathbf{m}^0\rangle, \langle b^1, q^1\rangle)} \arrow[r, "0"] \arrow[d, "1"] & {} \\
{} & {} &
\end{tikzcd}$$

Case (2,2). $\mathsf{H}(\langle b^0, \mathbf{m}^0\rangle, \langle b^1, \mathbf{m}^1\rangle)$.

Case (1,3).

$$\begin{tikzcd}
\boxed{\mathsf{E}(\lambda q . \mathsf{H}(\langle c^0, \mathbf{m}^0\rangle, \langle b^1, q, \mathbf{m}^1\rangle))} \arrow[r, "1"] \arrow[d, "0"] & {} \\
\boxed{\mathsf{E}(\lambda q . \mathsf{H}(\langle b^0, q^0\rangle, \langle b^1, q, \mathbf{m}^1\rangle))} \arrow[r, "1"] \arrow[d, "0"] & {} \\
{} &
\end{tikzcd}$$

Case (3,1).

$$\begin{tikzcd}
\boxed{\mathsf{E}^\circ(\lambda p\,.\,\mathsf{H}(\langle b^0, p, \mathbf{m}^0\rangle, \langle c^1, \mathbf{m}^1\rangle))} \arrow[r, "0"] \arrow[d, "1"] & {} \\
\boxed{\mathsf{E}^\circ(\lambda p\,.\,\mathsf{H}(\langle b^0, p, \mathbf{m}^0\rangle, \langle b^1, q^1\rangle))} \arrow[r, "0"] \arrow[d, "1"] & {} \\
{} &
\end{tikzcd}$$

Case (2,3). $\mathsf{E}(\lambda q\,.\,\mathsf{H}(\langle b^0, \mathbf{m}^0\rangle, \langle b^1, q, \mathbf{m}^1\rangle))$.
Case (3,2). $\mathsf{E}^\circ(\lambda p\,.\,\mathsf{H}(\langle b^0, p, \mathbf{m}^0\rangle, \langle b^1, \mathbf{m}^1\rangle))$.
Case (3,3). $\mathsf{E}^\circ(\lambda p\,.\,\mathsf{E}(\lambda q\,.\,H(\langle b^0, p, \mathbf{m}^0\rangle, \langle b^1, q, \mathbf{m}^1\rangle)))$.

We first observe that F is indeed partial recursive in I as all case distinctions are recursive and the indicated computations require I, E, or E° and are thus recursive in I by the assumption that E is recursive in I. Thus the I-Recursion Theorem applies and there exists an H which satisfies the stated conditions.

Clauses (i) and (ii) of the theorem are vacuous unless either u^0 or u^1 belongs to U^{I}. Hence, it suffices to prove (i) and (ii) by induction on $\sigma = \min\{|u^0|, |u^1|\}$ for $\sigma < \aleph_1$. If u^1 is not of the proper form, then $|u^0| \leqslant |u^1| = \aleph_1$ and $\mathsf{H}(u^0, u^1) \simeq 0$ in accord with (i). If u^1 is of the proper form but u^0 is not, then $\mathsf{H}(u^0, u^1) \simeq 1$ in accord with the fact that at most clause (ii) can apply. Hence, for the induction we may assume that (u^0, u^1) falls under one of the cases (r, s) with $0 \leqslant r, s \leqslant 3$.

If $\sigma = 0$, then either $|u^0| = 0$ or $|u^1| = 0$ or both. If $|u^0| = 0$ (cases $(0, s)$, $0 \leqslant s \leqslant 3$), then $\mathsf{H}(u^0, u^1) \simeq 0$ in accord with (i). If $|u^1| = 0$ while $|u^0| \neq 0$ (cases $(r, 0)$, $1 \leqslant r \leqslant 3$), then $\mathsf{H}(u^0, u^1) \simeq 1$ in accord with (ii).

Suppose now that $0 < \sigma = \min\{|u^0|, |u^1|\} < \aleph_1$ and that for any v^0 and v^1, clauses (i) and (ii) hold for v^0 and v^1 provided that $\min\{|v^0|, |v^1|\} < \sigma$. We treat three of the nine remaining cases in some detail and leave it to the reader to check the others.

Case (1,2). Let u^0 and u^1 be as in the definition of F for this case. Suppose first that $u^0 \in U^{\mathsf{I}}$ and $|u^0| \leqslant |u^1|$. Then for some q^0, $\{c^0\}^{\mathsf{I}}(\mathbf{m}^0) \simeq q^0$ and $\{b^0\}^{\mathsf{I}}(q^0)\downarrow$. Furthermore, by Lemma 3.2 (1),

(1) $$|c^0, \mathbf{m}^0| < |u^0| = \sigma \quad \text{and} \quad |b^0, q^0| < |u^0| = \sigma.$$

From (1) and Lemma 3.2(2) follows

(2) $$|c^0, \mathbf{m}^0| \leqslant |b^1, \mathbf{m}^1| \quad \text{and} \quad |b^0, q^0| \leqslant |b^1, \mathbf{m}^1|,$$

and from (1) it is obvious that

(3) $$\min\{|c^0, \mathbf{m}^0|, |b^1, \mathbf{m}^1|\} < \sigma \quad \text{and} \quad \min\{|b^0, q^0|, |b^1, \mathbf{m}^1|\} < \sigma.$$

Then from (2), (3), and the induction hypothesis we have

$$(4)\qquad \mathsf{H}(\langle c^0, \mathbf{m}^0\rangle, \langle b^1, \mathbf{m}^1\rangle) \simeq 0 \quad \text{and} \quad \mathsf{H}(\langle b^0, q^0\rangle, \langle b^1, \mathbf{m}^1\rangle) \simeq 0.$$

From (4) and the flow diagram for H in this case it is evident that $\mathsf{H}(u^0, u^1) \simeq 0$ in accord with (i).

Suppose now that $u^1 \in U^I$ and $|u^1| < |u^0|$ so that $|b^1, \mathbf{m}^1| < |u^1| = \sigma$. If $|b^1, \mathbf{m}^1| < |c^0, \mathbf{m}^0|$, then we see as above that

$$\mathsf{H}(u^0, u^1) \simeq \mathsf{H}(\langle c^0, \mathbf{m}^0\rangle, \langle b^1, \mathbf{m}^1\rangle) \simeq 1,$$

in accord with (ii). Otherwise $|c^0, \mathbf{m}^0| \leqslant |b^1, \mathbf{m}^1|$. Since $|b^1, \mathbf{m}^1| < \aleph_1$, also $|c^0, \mathbf{m}^0| < \aleph_1$, so $\langle c^0, \mathbf{m}^0\rangle \in U^I$ and there exists a number $q^0 \simeq \{c^0\}^I(\mathbf{m}^0)$. Note that in this case there is no necessity that $\{b^0\}^I(q^0)$ be defined. If it were the case that $|b^0, q^0| \leqslant |b^1, \mathbf{m}^1|$, we would have by Lemma 3.2(1,2),

$$|u^0| = \max\{|c^0, \mathbf{m}^0| + 1, |b^0, q^0| + 1\} \leqslant |b^1, \mathbf{m}^1| + 1 = u^1,$$

contrary to the present assumption. Hence $|b^1, \mathbf{m}^1| < |\mathrm{b}^0, \mathrm{q}^0|$, and by the induction hypothesis we have

$$\mathsf{H}(\langle c^0, \mathbf{m}^0\rangle, \langle b^1, \mathbf{m}^1\rangle) \simeq 0 \quad \text{and} \quad \mathsf{H}(\langle b^0, q^0\rangle, \langle b^1, \mathbf{m}^1\rangle) \simeq 1.$$

Referring again to the flow diagram, we see that $\mathsf{H}(u^0, u^1) \simeq 1$ in accord with (ii).

Case (1,1). Let $u^i = \langle\langle 1, k^i, 0, b^i, c^i\rangle, \langle \mathbf{m}^i\rangle\rangle$ and suppose first that $u^0 \in U^I$ and $|u^0| \leqslant |u^1|$. Then there exists a $q^0 \simeq \{c^0\}^I(\mathbf{m}^0)$, (1) holds, and by the induction hypothesis, for any v,

$$(5)\qquad |c^0, \mathbf{m}^0| \leqslant |v| \to \mathsf{H}(\langle c^0, \mathbf{m}^0\rangle, v) \simeq 0;$$

$$(6)\qquad |v| < |c^0, \mathbf{m}^0| \to \mathsf{H}(\langle c^0, \mathbf{m}^0\rangle, v) \simeq 1;$$

$$(7)\qquad |b^0, q^0| \leqslant |v| \to \mathsf{H}(\langle b^0, q^0\rangle, v) \simeq 0;$$

$$(8)\qquad |v| < |b^0, q^0| \to \mathsf{H}(\langle b^0, q^0\rangle, v) \simeq 1.$$

Suppose first that $|c^0, \mathbf{m}^0| \leqslant |c^1, \mathbf{m}^1|$ so that by (5), $\mathsf{H}(\langle c^0, \mathbf{m}^0\rangle, \langle c^1, \mathbf{m}^1\rangle) \simeq 0$. If also $|b^0, q^0| \leqslant |c^1, \mathbf{m}^1|$, then by (7) and the flow diagram,

$$\mathsf{H}(u^0, u^1) \simeq \mathsf{H}(\langle b^0, q^0\rangle, \langle c^1, \mathbf{m}^1\rangle) \simeq 0$$

in accord with (i). If, on the other hand, $|c^1, \mathbf{m}^1| < |b^0, q^0|$, then $\langle c^1, \mathbf{m}^1\rangle \in U^I$ so there exists a $q^1 \simeq \{c^1\}^I(\mathbf{m}^1)$. If it were the case that $|b^1, q^1| < |b^0, q^0|$, then

$$|u^1| = \max\{|c^1, \mathbf{m}^1| + 1, |b^1, q^1| + 1\} < |b^0, q^0| + 1 \leqslant |u^0|$$

contrary to assumption. Hence $|b^0, q^0| \leqslant |b^1, q^1|$, and by (8) and (7) we have

$$\mathsf{H}(\langle b^0, q^0\rangle, \langle c^1, \mathbf{m}^1\rangle) \simeq 1 \quad \text{and} \quad \mathsf{H}(\langle b^0, q^0\rangle, \langle b^1, q^1\rangle) \simeq 0,$$

so again $\mathsf{H}(u^0, u^1) \simeq 0$ in accord with (i).

The other possibility under (i) is that $|c^1, \mathbf{m}^1| < |c^0, \mathbf{m}^0|$, so that by (6), $\mathsf{H}(\langle c^0, \mathbf{m}^0\rangle, \langle c^1, \mathbf{m}^1\rangle) \simeq 1$ and there exists a $q^1 \simeq \{c^1\}^{\mathsf{I}}(\mathbf{m}^1)$. In this case it is impossible that $|b^1, q^1| < |c^0, \mathbf{m}^0|$ or $|b^1, q^1| < |b^0, q^0|$, as either leads to the conclusion $|u^1| < |u^0|$, contrary to assumption. Hence by (5) and (7) we have

$$\mathsf{H}(\langle c^0, \mathbf{m}^0\rangle, \langle b^1, q^1\rangle) \simeq 0 \quad \text{and} \quad \mathsf{H}(\langle b^0, q^0\rangle, \langle b^1, q^1\rangle) \simeq 0,$$

whence $\mathsf{H}(u^0, u^1) \simeq 0$ as required.

The argument for clause (ii) is similar and is omitted.

Case (3,3). This case is similar to the main case encountered in the proof of Lemma 1.7. By the case hypothesis we have $u^i = \langle\langle 3, k^i, 0, b^i\rangle, \mathbf{m}^i\rangle$. The definition of H in this case is such that $\mathsf{H}(u^0, u^1)$ is defined just in case for all p and q, $\mathsf{H}(\langle b^0, p, \mathbf{m}^0\rangle, \langle b^1, q, \mathbf{m}^1\rangle)$ is defined, and if so, then for all p and q,

(9) $$\mathsf{H}(u^0, u^1) \simeq 0 \leftrightarrow \forall p\, \exists q\,.\, \mathsf{H}(\langle b^0, p, \mathbf{m}^0\rangle, \langle b^1, q, \mathbf{m}^1\rangle) \simeq 0;$$

(10) $$\mathsf{H}(u^0, u^1) \simeq 1 \leftrightarrow \exists p\, \forall q\,.\, \mathsf{H}(\langle b^0, p, \mathbf{m}^0\rangle, \langle b^1, q, \mathbf{m}^1\rangle) \simeq 1.$$

Suppose first that $u^0 \in U^{\mathsf{I}}$ and $|u^0| \leqslant |u^1|$. Then

$$\{\langle 3, k^0, 0, b^0\rangle\}^{\mathsf{I}}(\mathbf{m}^0) \simeq \mathsf{I}(\lambda p\,.\,\{b^0\}^{\mathsf{I}}(p, \mathbf{m}^0))$$

is defined, so for all p, $\langle b^0, p, \mathbf{m}^0\rangle \in U^{\mathsf{I}}$. Furthermore, by Lemma 3.2(3), for all p, $|b^0, p, \mathbf{m}^0| < |u^0| = \sigma$. Hence, for all p and q,

$$\min\{|b^0, p, \mathbf{m}^0|, |b^1, q, \mathbf{m}^1|\} < \sigma,$$

so from the induction hypothesis we have for all p and q,

(11) $$|b^0, p, \mathbf{m}^0| \leqslant |b^1, q, \mathbf{m}^1| \rightarrow \mathsf{H}(\langle b^0, p, \mathbf{m}^0\rangle, \langle b^1, q, \mathbf{m}^1\rangle) \simeq 0;$$

(12) $$|b^1, q, \mathbf{m}^1| < |b^0, p, \mathbf{m}^0| \rightarrow \mathsf{H}(\langle b^0, p, \mathbf{m}^0\rangle, \langle b^1, q, \mathbf{m}^1\rangle) \simeq 1.$$

As one of these holds for each p and q, $\mathsf{H}(\langle b^0, p, \mathbf{m}^0\rangle, \langle b^1, q, \mathbf{m}^1\rangle)$ is defined for all p and q and thus $\mathsf{H}(u^0, u^1)$ is defined. If $u^1 \notin U^{\mathsf{I}}$, then for some q, $\{b^1\}^{\mathsf{I}}(q, \mathbf{m}^1)$ is undefined and thus $|b^1, q, \mathbf{m}^1| = \aleph_1$. In this case, $\mathsf{H}(u^0, u^1) \simeq 0$ by (9) and (11). If $u^1 \in U^{\mathsf{I}}$, then by the assumption and Lemma 3.2(3), for each p,

$$|b^0, p, \mathbf{m}^0| < |u^0| \leqslant |u^1| = \sup{}^+\{|b^1, q, \mathbf{m}^1| : q \in \omega\}.$$

Hence for each p, there is a q such that $|b^0, p, \mathbf{m}^0| \leq |b^1, q, \mathbf{m}^1|$ and thus again $\mathsf{H}(u^0, u^1) \simeq 0$ by (9) and (11).

The subcase (ii) is handled similarly by use of (10) and (12). □

At this point it may come as a shock to the reader to learn that the preceding theorem is in fact a simplified version of the full Ordinal Comparison Theorem. Fortunately, the proof of the latter is an obvious translation of the preceding proof. To state it, we return to the notation of § V.1 and set $x^i = (a^i, \langle \mathbf{m}^i \rangle, \langle \boldsymbol{\alpha}^i \rangle)$. Similarly, we write

$$\mathsf{H}(x^0, x^1) \quad \text{for} \quad \mathsf{H}(\langle a^0, \mathbf{m}^0 \rangle, \langle a^1, \mathbf{m}^1 \rangle, \langle \boldsymbol{\alpha}^0 \rangle, \langle \boldsymbol{\alpha}^1 \rangle).$$

3.4 Theorem. *For any* I *such that* E *is recursive in* I, *there exists a functional* H *partial recursive in* I *such that for all* x^0 *and* x^1,

(i) *if* $\mathsf{U}^{\mathsf{I}}(x^0)$ *and* $|x^0|^{\mathsf{I}} \leq |x^1|^{\mathsf{I}}$, *then* $\mathsf{H}(x^0, x^1) \simeq 0$;

(ii) *if* $\mathsf{U}^{\mathsf{I}}(x^1)$ *and* $|x^1|^{\mathsf{I}} < |x^0|^{\mathsf{I}}$, *then* $\mathsf{H}(x^0, x^1) \simeq 1$.

Proof. Follows the proof of Theorem 3.3 including everywhere $\boldsymbol{\alpha}^0$ and $\boldsymbol{\alpha}^1$ as parameters. (Cf. Exercise 3.7). □

3.5–3.7 Exercises

3.5. Write out explicitly the definition of F for cases (1,1) and (1,3) in the proof of Theorem 3.3.

3.6. Work out the flow diagram for arbitrary $k^{0'}$ and $k^{1'}$ for cases (1,2) and (1,1) in the proof of Theorem 3.3.

3.7. Sketch the proof of Theorem 3.4 by indicating what modifications are necessary to include the parameters $\boldsymbol{\alpha}$.

3.8 Notes. The method of ordinal comparison and its application to the Selection Theorem (4.1 below) were announced in Gandy [1967a]. Moschovakis [1967] contains the first published proof.

4. *Relations Semi-Recursive in a Type-2 Functional*

The theory of relations semi-recursive in a type-2 functional I (such that E is recursive in I) is a blend of the theory of (absolutely) semi-recursive relations and the theory of classes of relations which have the pre-wellordering property. We develop first that part of the theory which parallels the theory of semi-recursive relations of § II.4 and then study the effect of the pre-wellordering property. Both parts depend heavily on the Ordinal Comparison Theorem of the preceding section and we must therefore assume throughout that I is a fixed functional such that E is recursive in I.

4.1 Selection Theorem (Gandy). *There exists a functional* $\mathrm{Sel}^{\mathsf{I}}$ *partial recursive in* I *such that for all* $a, \mathbf{m}$, *and* $\boldsymbol{\alpha}$, *the following are equivalent*:

(i) $\exists p . \{a\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha})\downarrow$;

(ii) $\{a\}^{\mathsf{I}}(\mathrm{Sel}^{\mathsf{I}}(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle), \mathbf{m}, \boldsymbol{\alpha})\downarrow$.

Proof. For any a, let a^+ denote an index such that

$$\{a^+\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \{a\}^{\mathsf{I}}(p+1, \mathbf{m}, \boldsymbol{\alpha}).$$

There is a primitive recursive function $a \mapsto a^+$. Let H be the functional defined in the Ordinal Comparison Theorem (3.3) and F the functional computed according to the following flow diagram:

$$\begin{tikzcd}
\boxed{\mathsf{H}(\langle a, 0, \mathbf{m}\rangle, \langle e, a^+, \langle\mathbf{m}\rangle\rangle, \langle\boldsymbol{\alpha}\rangle)} \arrow[r, "0"] \arrow[d, "1"] & {} \\
\boxed{\mathsf{H}(\langle a, 0, \mathbf{m}\rangle, \langle a, \{e\}^{\mathsf{I}}(a^+, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) + 1, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle)} \arrow[r, "0"] \arrow[d, "1"] & {} \\
\boxed{\{e\}^{\mathsf{I}}(a^+, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) + 1} \arrow[r, "q+1"] & {}
\end{tikzcd}$$

By the I-Recursion Theorem there is an $\bar{e}$ such that

$$\{\bar{e}\}^{\mathsf{I}}(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) \simeq \mathsf{F}(\bar{e}, a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle),$$

and we set $\mathrm{Sel}^{\mathsf{I}} = \{\bar{e}\}^{\mathsf{I}}$.

Let $p_{a,\mathbf{m},\boldsymbol{\alpha}}$ denote the least p such that $\{a\}^{\mathsf{I}}(p, \mathbf{m}, \boldsymbol{\alpha})$ is defined if there is such a p. We prove the implication (i)$\rightarrow$(ii) by induction on $p_{a,\mathbf{m},\boldsymbol{\alpha}}$; that (ii)$\rightarrow$(i) is evident.

Suppose first that $p_{a,\mathbf{m},\boldsymbol{\alpha}} = 0$ — that is, $\{a\}^{\mathsf{I}}(0, \mathbf{m}, \boldsymbol{\alpha})$ is defined. Then $\langle a, 0, \mathbf{m}\rangle \in U^{\mathsf{I}}_{\boldsymbol{\alpha}}$, so for $n = 0$ or 1,

$$\mathsf{H}(\langle a, 0, \mathbf{m}\rangle, \langle\bar{e}, a^+, \langle\mathbf{m}\rangle\rangle, \langle\boldsymbol{\alpha}\rangle) \simeq n.$$

If $n = 0$, then

$$\mathrm{Sel}^{\mathrm{I}}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \simeq 0$$

as is appropriate. If $n = 1$, then $|\bar{e}, a^+, \langle \mathbf{m} \rangle| < |a, 0, \mathbf{m}|$ and thus for some q, $\{\bar{e}\}^{\mathrm{I}}(a^+, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \simeq q$. Again because $\langle a, 0, \mathbf{m} \rangle \in U^{\mathrm{I}}_{\boldsymbol{\alpha}}$, we have that for $r = 0$ or 1,

$$\mathsf{H}(\langle a, 0, \mathbf{m} \rangle, \langle a, q+1, \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \simeq r.$$

If $r = 0$, we have again $\mathrm{Sel}^{\mathrm{I}}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \simeq 0$ as is appropriate. If $r = 1$, it follows that $|a, q+1, \mathbf{m}| < |a, 0, \mathbf{m}|$ and thus that $\{a\}^{\mathrm{I}}(q+1, \mathbf{m}, \boldsymbol{\alpha})$ is defined. Furthermore, in this case

$$\begin{aligned}\mathrm{Sel}^{\mathrm{I}}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) &\simeq \{\bar{e}\}^{\mathrm{I}}(a^+, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) + 1 \\ &\simeq q + 1\end{aligned}$$

which is thus a suitable value. Note that in this case it is *not* the least p which is selected.

Now suppose that $p_{a,\mathbf{m},\boldsymbol{\alpha}} > 0$. By definition, $p_{a,\mathbf{m},\boldsymbol{\alpha}} = p_{a^+,\mathbf{m},\boldsymbol{\alpha}} + 1$, so by the induction hypothesis there exists a q such that

$$\{\bar{e}\}^{\mathrm{I}}(a^+, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \simeq \mathrm{Sel}^{\mathrm{I}}(a^+, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \simeq q$$

and

$$\{a\}^{\mathrm{I}}(q+1, \mathbf{m}, \boldsymbol{\alpha}) \simeq \{a^+\}^{\mathrm{I}}(q, \mathbf{m}, \boldsymbol{\alpha}),$$

which is defined. In particular, $\langle \bar{e}, a^+, \langle \mathbf{m} \rangle \rangle \in U^{\mathrm{I}}_{\boldsymbol{\alpha}}$. By the assumption, $\langle a, 0, \mathbf{m} \rangle \notin U^{\mathrm{I}}_{\boldsymbol{\alpha}}$, so $|\bar{e}, a^+, \langle \mathbf{m} \rangle| < |a, 0, \mathbf{m}|$ and thus

$$\mathsf{H}(\langle a, 0, \mathbf{m} \rangle, \langle \bar{e}, a^+, \langle \mathbf{m} \rangle \rangle, \langle \boldsymbol{\alpha} \rangle) \simeq 1.$$

Similarly, $|a, q+1, \mathbf{m}| < |a, 0, \mathbf{m}|$, so

$$\mathsf{H}(\langle a, 0, \mathbf{m} \rangle, \langle a, q+1, \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \simeq 1.$$

Then

$$\begin{aligned}\mathrm{Sel}^{\mathrm{I}}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) &\simeq \{\bar{e}\}^{\mathrm{I}}(a^+, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) + 1 \\ &\simeq q + 1,\end{aligned}$$

which is a suitable value. $\square$

The proofs of the following corollaries are exactly the same as the proofs of the corresponding results in § II.4.

4.2 Corollary. *For any relation* R *semi-recursive in* I, *there exists a functional* $\mathrm{Sel}_{\mathsf{R}}$ *partial recursive in* I *such that for all* $\mathbf{m}$ *and* $\boldsymbol{\alpha}$,

$$\exists p\,.\,\mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathrm{Sel}_{\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha}). \quad \square$$

4.3 Corollary. *The class of relations semi-recursive in* I *is closed under finite union and existential number quantification* $(\exists^0)$, *hence also under bounded existential quantification* $(\exists^0_<)$. $\square$

4.4 Corollary. *The class of functionals partial recursive in* I *and relations semi-recursive in* I *is closed under definition by positive cases.* $\square$

4.5 Corollary. *A relation is recursive in* I *iff it is both semi-recursive in* I *and co-semi-recursive in* I. $\square$

4.6 Corollary. *For any partial functional* F, F *is partial recursive in* I *iff* Gr_{F} *is semi-recursive in* I *and* F *is recursive in* I *iff* F *is total and* Gr_{F} *is recursive in* I. $\square$

In § V.1 we defined the pre-wellordering property only for the classes Σ^1_r and Π^1_r, but essentially the same definition applies to any indexable class of relations:

4.7 Definition. For any class X of relations, X has the *pre-wellordering property* iff there exists a relation V universal for X and relations $\preccurlyeq$, $\preccurlyeq_+$, and $\preccurlyeq_-$ such that:

(i) $\preccurlyeq$ is a pre-wellordering with field ${}^{2,1}\omega$ such that for all x and y,
  (a) $\sim\mathsf{V}(y) \to x \preccurlyeq y$, and
  (b) $\mathsf{V}(y) \wedge x \preccurlyeq y \to \mathsf{V}(x)$;
(ii) $\preccurlyeq_+$ belongs to X and $\preccurlyeq_-$ is the complement of a member of X;
(iii) for any x and y such that either $\mathsf{V}(x)$ or $\mathsf{V}(y)$,

$$(x \preccurlyeq_+ y) \leftrightarrow (x \preccurlyeq y) \leftrightarrow (x \preccurlyeq_- y).$$

4.8 Theorem. *The class of relations semi-recursive in* I *has the pre-wellordering property.*

Proof. With H as in Theorem 3.4, we set $\mathsf{V} = \mathsf{U}^{\mathsf{I}}$;

$$x \preccurlyeq y \leftrightarrow |x|^{\mathsf{I}} \leqslant |y|^{\mathsf{I}};$$

$$x \preccurlyeq_+ y \leftrightarrow \mathsf{U}^{\mathsf{I}}(x) \wedge \mathsf{H}(x, y) \simeq 0;$$

$$x \preccurlyeq_- y \leftrightarrow \sim\mathsf{U}^{\mathsf{I}}(y) \vee \mathsf{H}(x, y) \not\simeq 1.$$

Conditions (i) and (ii) of Definition 4.7 are immediate, and condition (iii) is easy to verify from the properties of H expressed in Theorem 3.4. □

Although we cannot now directly apply the results of § V.1, as they are stated only for Σ_r^1 and Π_r^1, in many cases the proofs depend only on properties shared by the class of relations semi-recursive in I. In particular, the classes of relations semi-recursive in I and recursive in I are closed under both kinds of number quantification ($\exists^0$ and $\forall^0$) (we continue to assume throughout that E is recursive in I). Of course, for E itself, the results are just those of §§ IV.1–2.

4.9 Theorem. (i) *The class of relations semi-recursive in* I *has the reduction property but not the separation property*;

(ii) *the class of relations co-semi-recursive in* I *has the separation property but not the reduction property.*

Proof. Exactly as for Theorem V.1.4. This could also be proved directly from the Selection Theorem as in Exercise II.4.33. □

As in § V.1, let $|\ \ |_0^{\mathsf{I}}$ be the norm induced on $\omega \times \omega$ by the restriction of the pre-wellordering $\preccurlyeq$ to sequences of type $(a, \langle \mathbf{m} \rangle, \langle\ \ \rangle)$, so that

$$|a, \langle \mathbf{m} \rangle|_0^{\mathsf{I}} = \sup{}^+\{|b, \langle \mathbf{n} \rangle|_0^{\mathsf{I}} : (b, \langle \mathbf{n} \rangle, \langle\ \ \rangle) < (a, \langle \mathbf{m} \rangle, \langle\ \ \rangle)\},$$

and set

$$\kappa^{\mathsf{I}} = \sup{}^+\{|a, \langle \mathbf{m} \rangle|_0^{\mathsf{I}} : U^{\mathsf{I}}(a, \langle \mathbf{m} \rangle)\}.$$

(In fact, $|a, \langle \mathbf{m} \rangle|_0^{\mathsf{I}} = |a, \langle \mathbf{m} \rangle, \langle\ \ \rangle|^{\mathsf{I}}$ (Exercise 4.27)).

4.10 Boundedness Theorem. *For any set A co-semi-recursive in* I, *if $A \subseteq U^{\mathsf{I}}$, then* $\sup^+\{|u|^{\mathsf{I}} : u \in A\} < \kappa^{\mathsf{I}}$.

Proof. Similar to that of Theorem V.1.5. □

For any ρ, let $U_\rho^{\mathsf{I}} = \{u : u \in U^{\mathsf{I}} \wedge |u|^{\mathsf{I}} < \rho\}$.

4.11 Hierarchy Theorem. *For all relations R on numbers,*

$$R \textit{ is recursive in } \mathsf{I} \leftrightarrow R \leqslant U_\rho^{\mathsf{I}} \textit{ for some } \rho < \kappa^{\mathsf{I}}$$
$$\leftrightarrow R = \{\mathbf{m} : \langle a, \mathbf{m} \rangle \in U_\rho^{\mathsf{I}}\} \textit{ for some } \rho < \kappa^{\mathsf{I}} \textit{ and } a \in \omega.$$

Proof. As for Theorem V.1.6. □

4.12. Upper Classification Theorem. $\{\alpha : \alpha$ *is recursive in* $\mathsf{I}\}$ *is semi-recursive in* I.

Proof. As for Theorem V.1.8 using the second part of Corollary 4.6. A different proof is obtained from the equivalence

$$\alpha \text{ is recursive in } \mathsf{I} \leftrightarrow \exists a\, \forall mn\, [\alpha(m) = n \to \{a\}^{\mathsf{I}}(m) \simeq n]. \quad \square$$

4.13. Corollary. *The set of functions recursive in* I *is not a basis for the class of sets co-semi-recursive in* I. $\square$

There are, of course, some properties of the class of semi-recursive relations which are not shared by the class of relations semi-recursive in I. For example it is not true that every relation semi-recursive in I is of the form $\exists^0\mathsf{P}$ with P recursive in I, as every such relation is itself recursive in I (remember, we are assuming that E is recursive in I). All of the equivalences of Theorem II.4.15 fail if we replace (semi-, partial) recursive by (semi-, partial) recursive in I. If *a* function $f : {}^k\omega \to \omega$ is recursive in I, then so is $\mathrm{Im}(f)$. On the other hand, from the assumption that a functional F is recursive in I we can conclude only that $\mathrm{Im}(\mathsf{F})$ is a non-empty set of the form $\exists^1\mathsf{P}$ for P a relation recursive in I:

$$n \in \mathrm{Im}(\mathsf{F}) \leftrightarrow \exists \boldsymbol{\alpha}\, [\exists \mathbf{m}\, .\, \mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n].$$

This estimate cannot be improved, for if A is any non-empty set of the form $\exists^1\mathsf{P}$ with P recursive in I, then $A = \mathrm{Im}(\mathsf{F})$, where

$$\mathsf{F}(m, \alpha) = \begin{cases} m, & \text{if } \mathsf{P}(m, \alpha); \\ \bar{m}, & \text{otherwise}; \end{cases}$$

where $\bar{m}$ is some fixed element of A. Thus, for example, we have by Theorem 1.8, for any $A \neq \varnothing$,

$$A \in \Sigma_1^1 \leftrightarrow A = \mathrm{Im}(\mathsf{F}) \text{ for some functional } \mathsf{F} \text{ recursive in } \mathsf{E}.$$

By a similar argument we see that the images of functionals partial recursive in I are exactly those of the form $\exists^1\mathsf{P}$ with P semi-recursive in I. Hence by Corollary 1.10,

$$A \in \Sigma_2^1 \leftrightarrow A = \mathrm{Im}(\mathsf{F}) \text{ for some function } \mathsf{F} \text{ partial recursive in } \mathsf{E}.$$

One result of this sort remains:

4.14 Theorem. *For any* $A \subseteq \omega$, *A is semi-recursive in* I *iff* $\mathsf{A} = \mathrm{Im}(f)$ *for some function f partial recursive in* I.

Proof. If A is semi-recursive in I and

$$f(m) \simeq \begin{cases} m, & \text{if } m \in A; \\ \text{undefined}, & \text{otherwise}; \end{cases}$$

then f is partial recursive in I by Corollary 4.4 and $\mathrm{Im}(f) = A$. Conversely, if f is partial recursive in I, then

$$n \in \mathrm{Im}(f) \leftrightarrow \exists m \,.\, f(m) \simeq n,$$

so $\mathrm{Im}(f)$ is semi-recursive in I by Corollaries 4.3 and 4.6. □

The astute reader will also have noticed that there is no result here corresponding to the Lower Classification Theorem V.1.13. Instead, we have

4.15 Theorem. *There exists a functional* I *such that* E *is recursive in* I *and* $\{\alpha : \alpha$ *is recursive in* $\mathsf{I}\}$ *is recursive in* I.

Proof. Let I be the functional defined by:

$$\mathsf{I}(\alpha) = \begin{cases} \mathsf{E}(\alpha), & \text{if } \alpha \in \Delta_1^1; \\ \mathsf{E}(\alpha)+2, & \text{otherwise}. \end{cases}$$

That E is recursive in I is evident from the fact that

$$\mathsf{E}(\alpha) = 0 \quad \text{iff} \quad (\mathsf{I}(\alpha) = 0 \text{ or } \mathsf{I}(\alpha) = 2).$$

For all α recursive in E (i.e. in Δ_1^1), $\mathsf{E}(\alpha) = \mathsf{I}(\alpha)$, so by the result of Exercise 1.13

$$\{\alpha : \alpha \text{ is recursive in } \mathsf{I}\} = \{\alpha : \alpha \text{ is recursive in } \mathsf{E}\}.$$

Since $\alpha \in \Delta_1^1$ iff $\mathsf{I}(\alpha) \leq 1$, this set is recursive in I. □

Corresponding to the result of Exercises IV.2.25 and V.1.27, we have the following *effective choice principle*:

4.16 Theorem. *For any relation* R *semi-recursive in* I,

(i) *if* $\forall \mathbf{m} \forall \boldsymbol{\alpha} \exists p\, \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha})$, *then there is an* F *recursive in* I *such that* $\forall \mathbf{m} \forall \boldsymbol{\alpha}\, \mathsf{R}(\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha})$;

(ii) *if* $\forall \mathbf{m} \forall \boldsymbol{\alpha}\, (\exists \beta$ recursive in I and $\boldsymbol{\alpha})\, \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta)$, *then there is a* G *recursive in* I *such that* $\forall \mathbf{m} \forall \boldsymbol{\alpha}\, \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \lambda q \,.\, \mathsf{G}(q, \mathbf{m}, \boldsymbol{\alpha}))$.

Proof. For (i) we take simply $\mathsf{F} = \mathrm{Sel}_{\mathsf{R}}$. F is partial recursive in I and is total by the hypothesis, so F is recursive in I. For (ii), let

$$\mathsf{S}(a, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \lambda q . \{a\}^{\mathsf{I}}(q, \mathbf{m}, \boldsymbol{\alpha})).$$

S is semi-recursive in I by Theorem 2.10 and we may set

$$\mathsf{G}(q, \mathbf{m}, \boldsymbol{\alpha}) \simeq \{\mathrm{Sel}_{\mathsf{S}}(\mathbf{m}, \boldsymbol{\alpha})\}^{\mathsf{I}}(q, \mathbf{m}, \boldsymbol{\alpha}).$$

We turn next to characterizing the ordinal κ^{I} (analogously to Corollary V.1.17). Let $\omega_1[\mathsf{I}]$ denote the least ordinal not the order-type of a well-ordering of ω recursive in I.

4.17 Theorem. $\kappa^{\mathsf{I}} = \omega_1[\mathsf{I}]$.

Proof. To establish $\kappa^{\mathsf{I}} \leqslant \omega_1[\mathsf{I}]$, we show that to every $u \in U^{\mathsf{I}}$ there corresponds a well-ordering of type $|u|^{\mathsf{I}}$ recursive in I. First note that for each $u \in U^{\mathsf{I}}$, the sets

$$\{v : v \in U^{\mathsf{I}} \wedge |v|^{\mathsf{I}} < |u|^{\mathsf{I}}\} \quad \text{and} \quad \{v : v \in U^{\mathsf{I}} \wedge |v|^{\mathsf{I}} = |u|^{\mathsf{I}}\}$$

are recursive in I (use the pre-wellordering property). For each $u \in U^{\mathsf{I}}$ the first of these sets is pre-wellordered in type $|u|^{\mathsf{I}}$ by a relation which is recursive in I. We obtain a *well*-ordering of the same type by a refinement. Let

$$A_u = \{v : v \in U^{\mathsf{I}} \wedge |v|^{\mathsf{I}} < |u|^{\mathsf{I}} \wedge \forall w\, (|w|^{\mathsf{I}} = |v|^{\mathsf{I}} \rightarrow v \leqslant w)\}.$$

A_u is recursive in I and contains a unique notation for each ordinal less than $|u|^{\mathsf{I}}$. Hence if

$$R_u(v, w) \leftrightarrow v,\ w \in A_u \wedge v \leqslant_+ w,$$

then R_u is the required well-ordering.

For the converse, let γ be any element of W recursive in I. We recall that $|p|_\gamma$ denotes the ordinal $\|\gamma \restriction p\|$. Let $\bar{p}$ be such that $|\bar{p}|_\gamma = 0$. We shall find an index $\bar{e}$ such that for all q, $\langle \bar{e}, q\rangle \in U^{\mathsf{I}}$ and $|\bar{e}, q|^{\mathsf{I}} \geqslant |q|_\gamma$. From this it follows that $\kappa^{\mathsf{I}} \geqslant \|\gamma\|$. As $\|\gamma\|$ is an arbitrary ordinal less than $\omega_1[\mathsf{I}]$, the result follows.

Let f and g be functions partial recursive in I defined as follows:

$$f(p, q) = \begin{cases} p, & \text{if } p <_\gamma q; \\ \bar{p}, & \text{otherwise}; \end{cases}$$

$$g(e, q) = \begin{cases} 0, & \text{if } q = \bar{p}; \\ \mathsf{I}(\lambda p . \{e\}^{\mathsf{I}}(f(p, q))), & \text{otherwise.} \end{cases}$$

By the I-Recursion Theorem there exists an index $\bar{e}$ such that $g(\bar{e}, q) \simeq \{\bar{e}\}^{\mathsf{I}}(q)$. We prove by induction on $|q|_\gamma$ that $\langle \bar{e}, q\rangle \in U^{\mathsf{I}}$ and $|\bar{e}, q|^{\mathsf{I}} \geq |q|_\gamma$.

If $|q|_\gamma = 0$, then $q = \bar{p}$ so

$$\{\bar{e}\}(q) \simeq g(\bar{e}, \bar{p}) \simeq 0$$

and thus $\langle \bar{e}, q\rangle \in U^{\mathsf{I}}$ and $|\bar{e}, q| \geq 0 = |q|_\gamma$.

Now suppose $|q|_\gamma > 0$. Note that for all p, $|f(p, q)|_\gamma < |q|_\gamma$ so by the induction hypothesis, $\langle \bar{e}, f(p, q)\rangle \in U^{\mathsf{I}}$. But then $\lambda p . \{\bar{e}\}^{\mathsf{I}}(f(p, q))$ is total so $g(\bar{e}, q)$ is defined and thus $\langle \bar{e}, q\rangle \in U^{\mathsf{I}}$. Let b be a natural index for g from I — that is, one such that

$$(\bar{e}, f(p, q), \{\bar{e}\}^{\mathsf{I}}(f(p, q))) \in \mathrm{Sbc}(b, \bar{e}, q, g(\bar{e}, q))$$

and thus

$$|\bar{e}, f(p, q)|^{\mathsf{I}} < |b, \bar{e}, q|^{\mathsf{I}}.$$

From the proof of the Recursion Theorem it is easy to verify (Exercise 2.20) that we may assume that $\bar{e}$ is chosen in such a way that

$$(b, \bar{e}, q, g(\bar{e}, q)) \in \mathrm{Sbc}(\bar{e}, q, g(\bar{e}, q))$$

and thus

$$|b, \bar{e}, q|^{\mathsf{I}} < |\bar{e}, q|^{\mathsf{I}}.$$

By the induction hypothesis we have for all $p <_\gamma q$,

$$|p|_\gamma \leq |\bar{e}, p|^{\mathsf{I}} = |\bar{e}, f(p, q)|^{\mathsf{I}} < |\bar{e}, q|^{\mathsf{I}},$$

from which it follows that $|q|_\gamma \leq |\bar{e}, q|^{\mathsf{I}}$. □

Of course, this, as well as most of the other results of this section, may be relativized to yield theorems about the class of relations semi-recursive in I and $\boldsymbol{\alpha}$. In particular, we obtain from the preceding theorem that $\kappa^{\mathsf{I}}_{\boldsymbol{\alpha}} = \omega_1[\mathsf{I}, \boldsymbol{\alpha}]$ — that is, $\sup^+\{|u|^{\mathsf{I}} : u \in U^{\mathsf{I}}_{\boldsymbol{\alpha}}\}$ is the least ordinal not recursive in I and $\boldsymbol{\alpha}$. From this it follows that

$$\sup^+\{|x|^{\mathsf{I}} : x \in \mathrm{U}^{\mathsf{I}}\} = \aleph_1.$$

In other words, the inductive definition of $\Omega[\mathsf{I}]$ has exactly $\aleph_1$ stages.

We conclude this section by establishing the analogue for recursion in I of the

Spector–Gandy Theorem (IV.2.9). We denote by $\Sigma_1^{1,\mathsf{I}}$ the class of relations R such that for some relation P recursive in I,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow (\exists \beta \text{ recursive in } \mathsf{I}, \boldsymbol{\alpha})\, \mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta).$$

4.18 Theorem. $\Sigma_1^{1,\mathsf{I}} = \{\mathsf{R} : \mathsf{R} \text{ is semi-recursive in } \mathsf{I}\}$.

Proof. If $\mathsf{R} \in \Sigma_1^{1,\mathsf{I}}$ and satisfies the preceding equivalence for some relation P recursive in I, then

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists a\, \mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p . \{a\}^{\mathsf{I}}(p, \boldsymbol{\alpha}))$$

and R is semi-recursive in I by Corollaries 2.11 and 4.3.

We shall prove the converse inclusion only for relations on numbers. For this it will clearly suffice to show $U^{\mathsf{I}} \in \Sigma_1^{1,\mathsf{I}}$. For any v, let

$$\beta_v(u^0, u^1) = \begin{cases} 0, & \text{if } |u^0| \leqslant |u^1| \text{ and } |u^0| < |v|; \\ 1, & \text{if } |u^1| < |u^0| \text{ and } |u^1| < |v|; \\ 2, & \text{otherwise}; \end{cases}$$

$$\gamma_v(\langle a, \mathbf{m}\rangle) = \begin{cases} \{a\}^{\mathsf{I}}(\mathbf{m}), & \text{if } |a, \mathbf{m}| < |v|; \\ 0, & \text{otherwise}. \end{cases}$$

We aim to define a relation P recursive in I such that for any v,

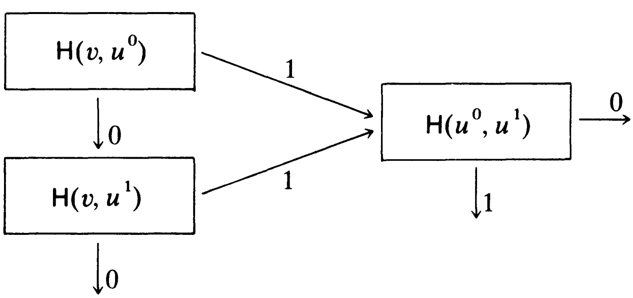
(i) if $v \in U^{\mathsf{I}}$, then $\mathsf{P}(v, \beta_v, \gamma_v)$ holds, and

(ii) if $v \notin U^{\mathsf{I}}$, then for any β and γ such that $\mathsf{P}(v, \beta, \gamma)$, and any $\langle a, \mathbf{m}\rangle \in U^{\mathsf{I}}$, $\{a\}^{\mathsf{I}}(\mathbf{m}) \simeq \gamma(\langle a, \mathbf{m}\rangle)$.

From (i) and (ii) we can conclude that for all v,

(iii) $v \in U^{\mathsf{I}} \leftrightarrow (\exists \beta, \gamma \text{ recursive in } \mathsf{I}) \mathsf{P}(v, \beta, \gamma)$

as follows. For the implication $(\rightarrow)$ it suffices to show that for all $v \in U^{\mathsf{I}}$, β_v and γ_v are recursive in I. Using the ordinal comparison functional H of Theorem 3.3, $\beta_v(u^0, u^1)$ may be computed according to the following flow diagram:



Since $v \in U^{\mathsf{I}}$, the values in the left-hand boxes are defined, and if the right-hand box is reached, also one of u^0 and u^1 belongs to U^{I}. Hence β_v is recursive in I. Then

$$\gamma_v(\langle a, \mathbf{m}\rangle) = \begin{cases} \{a\}^{\mathsf{I}}(\mathbf{m}), & \text{if} \quad \beta_v(\langle a, \mathbf{m}\rangle, \langle a, \mathbf{m}\rangle) = 0; \\ 0, & \text{otherwise;} \end{cases}$$

so γ_v is also recursive in I.

For the implication $(\leftarrow)$ of (iii), suppose $v \notin U^{\mathsf{I}}$ but that β and γ are functions recursive in I such that $\mathsf{P}(v, \beta, \gamma)$ holds. Then there exists an index a such that for all m, $\{a\}^{\mathsf{I}}(m) = \gamma(\langle m, m\rangle) + 1$. In particular, $\langle a, a\rangle \in U^{\mathsf{I}}$ so by (ii),

$$\gamma(\langle a, a\rangle) = \{a\}^{\mathsf{I}}(a) = \gamma(\langle a, a\rangle) + 1,$$

a contradiction.

To define P and establish (i) and (ii) we return to the proof of Theorem 3.3. For $\mathsf{P}(v, \beta, \gamma)$ to hold will require, roughly speaking, that β satisfy the recursive conditions imposed on H in that proof and that γ satisfy the recursive conditions inherent in the definition of $\Omega[\mathsf{I}]$, in each case only for arguments which precede v in the pre-wellordering of U^{I}. In order that P be recursive in I and not merely semi-recursive in I, we replace occurrences of $\{c\}^{\mathsf{I}}(\mathbf{m})$ in the recursive conditions on H by $\gamma(\langle c, \mathbf{m}\rangle)$.

First, for $0 \leqslant r, s \leqslant 4$ we define relations $\mathsf{Q}_{r,s}$ as follows. $\mathsf{Q}_{r,s}(u^0, u^1, \beta, \gamma)$ holds only if (u^0, u^1) falls under case (r, s), with "4" signifying "not of the proper form". Then $\mathsf{Q}_{r,s}(u^0, u^1, \beta, \gamma)$ is to hold when, with β replacing H, the answer 0 is obtained from the flow diagram for the corresponding case of Theorem 3.3. Thus, we set:

$$\mathsf{Q}_{r,4}(u^0, u^1, \beta, \gamma) \leftrightarrow 0 = 0 \qquad (0 \leqslant r \leqslant 4);$$
$$\mathsf{Q}_{4,s}(u^0, u^1, \beta, \gamma) \leftrightarrow 0 = 1 \qquad (0 \leqslant s \leqslant 3);$$
$$\mathsf{Q}_{0,s}(u^0, u^1, \beta, \gamma) \leftrightarrow 0 = 0 \qquad (0 \leqslant s \leqslant 3);$$
$$\mathsf{Q}_{r,0}(u^0, u^1, \beta, \gamma) \leftrightarrow 0 = 1 \qquad (1 \leqslant r \leqslant 3);$$
$$\mathsf{Q}_{1,2}(u^0, u^1, \beta, \gamma) \leftrightarrow \beta(\langle c^0, \mathbf{m}^0\rangle, \langle b^1, \mathbf{m}^1\rangle) = 0 \quad \text{and} \quad \beta(\langle b^0, q^0\rangle, \langle b^1, \mathbf{m}^1\rangle) = 0,$$

where $q^0 = \gamma(\langle c^0, \mathbf{m}^0\rangle)$;

$$\mathsf{Q}_{2,1}(u^0, u^1, \beta, \gamma) \leftrightarrow \beta(\langle b^0, \mathbf{m}^0\rangle, \langle c^1, \mathbf{m}^1\rangle) = 0 \quad \text{or} \quad [\beta(\langle b^0, \mathbf{m}^0\rangle, \langle c^1, \mathbf{m}^1\rangle) = 1 \text{ and } \beta(\langle b^0, \mathbf{m}^0\rangle, \langle b^1, q^1\rangle) = 0],$$

where $q^1 = \gamma(\langle c^1, \mathbf{m}^1\rangle)$;

$$\mathsf{Q}_{2,3}(u^0, u^1, \beta, \gamma) \leftrightarrow \exists q\,[\beta(\langle b^0, \mathbf{m}^0\rangle, \langle b^1, q, \mathbf{m}^1\rangle) = 0];$$

$$\mathsf{Q}_{3,3}(u^0, u^1, \beta, \gamma) \leftrightarrow \forall p\,\exists q\,[\beta(\langle b^0, p, \mathbf{m}^0\rangle, \langle b^1, q, \mathbf{m}^1\rangle) = 0].$$

The relations $\mathsf{Q}_{r,s}$ for the remaining cases are obtained similarly from the corresponding cases in the proof of Theorem 3.3. Let

$$\mathsf{Q}(u^0, u^1, \beta, \gamma) \leftrightarrow (\exists r \leqslant 4)(\exists s \leqslant 4)\mathsf{Q}_{r,s}(u^0, u^1, \beta, \gamma).$$

The conditions on γ are expressed by a relation S which may be thought of as saying "γ is *locally correct for* β":

$$\mathsf{S}(\beta, \gamma) \leftrightarrow \text{for all } u, k, \quad \text{and} \quad m \in {}^k\omega, \quad \text{if} \quad \beta(u, u) = 0,$$

then the following hold:

(0) if $u = \langle\langle 0, k, 0, \ldots\rangle, \mathbf{m}\rangle$ and $\langle 0, k, 0, \ldots\rangle$ is an index of the proper form for $\mathbf{m}$, then $\gamma(u) = \{\langle 0, k, 0, \ldots\rangle\}^{\mathsf{I}}(\mathbf{m})$;

(1) if for some $b, c_0, \ldots, c_{k'-1}$, $u = \langle\langle 1, k, 0, b, c_0, \ldots, c_{k'-1}\rangle, \mathbf{m}\rangle$, then $\gamma(u) = \gamma(\langle b, \gamma(\langle c_0, \mathbf{m}\rangle), \ldots, \gamma(\langle c_{k'-1}, \mathbf{m}\rangle)\rangle)$;

(2) if for some b, $u = \langle\langle 2, k+1, 0\rangle, b, \mathbf{m}\rangle$, then $\gamma(u) = \gamma(\langle b, \mathbf{m}\rangle)$;

(3) if for some b, $u = \langle\langle 3, k, 0, b\rangle, \mathbf{m}\rangle$, then $\gamma(u) = \mathsf{I}(\lambda p\,.\,\gamma(\langle b, p, \mathbf{m}\rangle))$.

Finally, we set

$\mathsf{P}(v, \beta, \gamma) \leftrightarrow \mathsf{S}(\beta, \gamma)$ and for all u^0 and u^1,

$$[\beta(u^0, u^1) = 0 \leftrightarrow \mathsf{Q}(u^0, u^1, \beta, \gamma) \wedge \sim\mathsf{Q}(v, u^0, \beta, \gamma)]$$

and

$$[\beta(u^0, u^1) = 1 \leftrightarrow \sim\mathsf{Q}(u^0, u^1, \beta, \gamma) \wedge \sim\mathsf{Q}(v, u^1, \beta, \gamma)].$$

Since Q and S are each recursive in I, so is P.

Towards (i), suppose $v \in U^{\mathsf{I}}$. Then it is obvious from the definitions that $\mathsf{S}(\beta_v, \gamma_v)$ holds. We need thus only show that for all u^0 and u^1,

$$\text{(iv)} \qquad \beta_v(u^0, u^1) = 0 \leftrightarrow \mathsf{Q}(u^0, u^1, \beta_v, \gamma_v) \wedge \sim\mathsf{Q}(v, u^0, \beta_v, \gamma_v),$$

and

$$\text{(v)} \qquad \beta_v(u^0, u^1) = 1 \leftrightarrow \sim\mathsf{Q}(u^0, u^1, \beta_v, \gamma_v) \wedge \sim\mathsf{Q}(v, u^1, \beta_v, \gamma_v).$$

These will, in turn, be derived from:

(vi) $\quad \min\{|u^0|,|u^1|\} \leqslant |v| \rightarrow [|u^0| \leqslant |u^1| \leftrightarrow \mathsf{Q}(u^0, u^1, \beta_v, \gamma_v)]$.

The proof of (vi) is by induction on $\sigma = \min\{|u^0|,|u^1|\}$ and is closely parallel to the corresponding part of the proof of Theorem 3.3. We examine only one case to indicate the relationship between the proofs. Suppose $u^0 = \langle\langle 1, k^0, 0, b^0, c^0\rangle, \mathbf{m}^0\rangle$ and $u^1 = \langle\langle 2, k'+1, 0\rangle, b^1, \mathbf{m}^1\rangle$, $\min\{|u^0|,|u^1|\} \leqslant |v|$, and first that $|u^0| \leqslant |u^1|$. Then $|u^0| \leqslant |v|$, for some q^0, $\{c^0\}^1(\mathbf{m}^0) \simeq q^0$, and

$$|c^0, \mathbf{m}^0| < |u^0| \leqslant |v| \quad \text{and} \quad |b^0, q^0| < |u^0| \leqslant |v|.$$

By Lemma 3.2,

$$|c^0, \mathbf{m}^0| \leqslant |b^1, \mathbf{m}^1| \quad \text{and} \quad |b^0, q^0| \leqslant |b^1, \mathbf{m}^1|,$$

and thus

$$\beta_v(\langle c^0, \mathbf{m}^0\rangle, \langle b^1, \mathbf{m}^1\rangle) = 0 = \beta_v(\langle b^0, q^0\rangle, \langle b^1, \mathbf{m}^1\rangle).$$

Furthermore, since $|c^0, \mathbf{m}^0| < |v|$, $\gamma_v(\langle c^0, \mathbf{m}^0\rangle) = q^0$ and we have $\mathsf{Q}_{1,2}(u^0, u^1, \beta_v, \gamma_v)$ and hence $\mathsf{Q}(u^0, u^1, \beta_v, \gamma_v)$. If, on the other hand, $|u^1| < |u^0|$, then $|b^1, \mathbf{m}^1| < |u^1| \leqslant |v|$. If $|b^1, \mathbf{m}^1| < |c^0, \mathbf{m}^0|$, then $\beta_v(\langle c^0, \mathbf{m}^0\rangle, \langle b^1, \mathbf{m}^1\rangle) = 1$ which implies $\sim\mathsf{Q}(u^0, u^1, \beta_v, \gamma_v)$. Otherwise, $|c^0, \mathbf{m}^0| \leqslant |b^1, \mathbf{m}^1|$, and thus $\beta_v(\langle c^0, \mathbf{m}^0\rangle, \langle b^1, \mathbf{m}^1\rangle) = 0$. But then $|c^0, \mathbf{m}^0| < |v|$, so $\{c^0\}^1(\mathbf{m}^0) \simeq \gamma_v(\langle c^0, \mathbf{m}^0\rangle) \simeq q^0$ (say) and necessarily $|b^1, \mathbf{m}^1| < |b^0, q^0|$ (by the assumption $|u^1| < |u^0|$), so $\beta_v(\langle b^0, q^0\rangle, \langle b^1, \mathbf{m}^1\rangle) = 1$ and again we have $\sim\mathsf{Q}(u^0, u^1, \beta_v, \gamma_v)$.

Now to derive (iv) from (vi), suppose first that $\beta_v(u^0, u^1) = 0$ so that $|u^0| \leqslant |u^1|$ and $|u^0| < |v|$. Since $\min\{|u^0|,|u^1|\} = \min\{|u^0|,|v|\} = |u^0| < |v|$, (vi) yields immediately the right-hand side of (iv). Conversely, since $\min\{|v|,|u^0|\} \leqslant |v|$, if the right-hand side of (iv) holds, then from (vi) and $\sim\mathsf{Q}(v, u^0, \beta_v, \gamma_v)$ we conclude that $|u^0| < |v|$. Then also $\min\{|u^0|,|u^1|\} \leqslant |u^0| < |v|$, so from (vi) and $\mathsf{Q}(u^0, u^1, \beta_v, \gamma_v)$ we conclude that $|u^0| \leqslant |u^1|$ and thus that $\beta_v(u^0, u^1) = 0$. The proof of (v) from (vi) is similar. This concludes the proof of (i).

For (ii), suppose $v \notin U^1$ and that β and γ are any functions such that $\mathsf{P}(v, \beta, \gamma)$ holds. We propose to prove that for all u^0, u^1, a, and $\mathbf{m}$,

(vii) $\quad u^0 \in U^1 \wedge |u^0| \leqslant |u^1| \rightarrow \beta(u^0, u^1) = 0;$

(viii) $\quad u^1 \in U^1 \wedge |u^1| < |u^0| \rightarrow \beta(u^0, u^1) = 1;$

(ix) $\quad \langle a, \mathbf{m}\rangle \in U^1 \rightarrow \gamma(\langle a, \mathbf{m}\rangle) = \{a\}^1(\mathbf{m}).$

As in the proof of Theorem 3.3, the proof of (vii)–(ix) is again by induction on $\min\{|u^0|,|u^1|\} = |\langle a, \mathbf{m}\rangle| = \sigma < \aleph_1$. If $|u^0| = 0$, then $\mathsf{Q}_{0,s}(u^0, u^1, \beta, \gamma)$ holds for

u^1 falling under any case (s) $(0 \leqslant s \leqslant 4)$ and $\sim\mathsf{Q}_{r,0}(v, u^0, \beta, \gamma)$ holds for r such that v falls under case (r) $(1 \leqslant r \leqslant 4$, because $v \notin U^1$ and thus cannot fall under case (0)). Hence by the definition of P, $\beta(u^0, u^1) = 0$ in accord with (vii). If, on the other hand, $|u^1| = 0 < |u^0|$, then $\sim\mathsf{Q}_{r,0}(u^0, u^1, \beta, \gamma)$ and $\sim\mathsf{Q}_{r',0}(v, u^1, \beta, \gamma)$ for some r and r' $(1 \leqslant r, r' \leqslant 4)$ representing the cases applicable to u^0 and v, respectively. Hence $\beta(u^0, u^1) = 1$ in accord with (viii). Finally, if $|\langle a, \mathbf{m}\rangle| = 0$, then it follows from (vii) that $\beta(\langle a, \mathbf{m}\rangle, \langle a, \mathbf{m}\rangle) = 0$ and thus from $\mathsf{S}(v, \beta, \gamma)$ we have $\gamma(\langle a, \mathbf{m}\rangle) = \{a\}^{\mathrm{I}}(\mathbf{m})$.

Now suppose $\sigma > 0$ and as induction hypothesis that (vii)–(ix) hold for all u^0, u^1, a, and $\mathbf{m}$ with $\min\{|u^0|, |u^1|\} < \sigma$ and $|\langle a, \mathbf{m}\rangle| < \sigma$. We first claim that for all u^0 and u^1,

$$\text{(x)} \qquad \min\{|u^0|, |u^1|\} \leqslant \sigma \rightarrow [|u^0| \leqslant |u^1| \leftrightarrow \mathsf{Q}(u^0, u^1, \beta, \gamma)].$$

As in the case of (vi), the proof of (x) by cases is virtually a copy of the corresponding argument for Theorem 3.3. We omit details and only mention that at those points in the argument where for (vi) we appealed to the fact that some ordinal is less than $|v|$ to justify an assertion concerning β_v or γ_v, we here appeal to the fact that the corresponding ordinal is less than σ and apply the relevant clause (vii)–(ix) of the induction hypothesis.

To complete the induction step in the proof of (vii)–(ix), let $\min\{|u^0|, |u^1|\} = \sigma$ and suppose first that $|u^0| \leqslant |u^1|$. Then by (x), $\mathsf{Q}(u^0, u^1, \beta, \gamma)$ holds. Since $v \notin U^1$, $\min\{|v|, |u^0|\} = |u^0| = \sigma$ and thus as $|v| \nleqslant |u^0|$, by (x) again $\sim\mathsf{Q}(v, u^0, \beta, \gamma)$. Hence $\beta(u^0, u^1) \simeq 0$ as required by (vii). If $|u^1| < |u^0|$, then

$$\min\{|v|, |u^1|\} = |u^1| = \sigma < |v|,$$

and two more applications of (x) yield $\sim\mathsf{Q}(u^0, u^1, \beta, \gamma)$ and $\sim\mathsf{Q}(v, u^1, \beta, \gamma)$, so $\beta(u^0, u^1) = 1$ as required by (viii). Finally, suppose $|a, \mathbf{m}| = \sigma$. If, for example, $a = \langle 1, k, 0, b, c\rangle$, then $|c, \mathbf{m}| < \sigma$ so by (ix) of the induction hypothesis $\gamma(\langle c, \mathbf{m}\rangle) = \{c\}^{\mathrm{I}}(\mathbf{m})$. Also $|b, \{c\}^{\mathrm{I}}(\mathbf{m})| < \sigma$ so $\gamma(\langle b, \{c\}^{\mathrm{I}}(\mathbf{m})\rangle) = \{b\}^{\mathrm{I}}(\{c\}^{\mathrm{I}}(\mathbf{m}))$. By (vii), $\beta(\langle a, \mathbf{m}\rangle, \langle a, \mathbf{m}\rangle) = 0$ so since $\mathsf{S}(\beta, \gamma)$ holds, we have

$$\gamma(\langle a, \mathbf{m}\rangle) = \gamma(b, \gamma(\langle a, \mathbf{m}\rangle)) = \{b\}^{\mathrm{I}}(\{c\}^{\mathrm{I}}(\mathbf{m})) = \{a\}^{\mathrm{I}}(\mathbf{m}).$$

The other cases are similar. □

4.19–4.31 Exercises

4.19. Can $\mathrm{Sel}^{\mathrm{I}}$ be defined to have the additional property present in other cases that

$$\exists p . \{a\}^{\mathrm{I}}(p, \mathbf{m}, \boldsymbol{\alpha}) \downarrow \leftrightarrow \mathrm{Sel}^{\mathrm{I}}(\mathbf{m}, \boldsymbol{\alpha}) \downarrow ?$$

4.20. Modify the proof of the Selection Theorem 4.1 to obtain a functional $\mathrm{Sel}_0^{\mathsf{I}}$ with the additional property that for all p,

$$|a, \mathrm{Sel}_0^{\mathsf{I}}(a, \langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle), \mathbf{m}|_{\boldsymbol{\alpha}}^{\mathsf{I}} \leq |a, p, \mathbf{m}|_{\boldsymbol{\alpha}}^{\mathsf{I}}.$$

4.21. Show that if we do not assume that E is recursive in I, the results of §§ 3 and 4 may fail. In particular, show that it may happen that both R and $\sim\mathsf{R}$ are semi-recursive in I but R is not recursive in I.

4.22. Show that for any I such that E is recursive in I, there exist functions f and g recursive in I such that for all u and m,
(i) if $u \in U^{\mathsf{I}}$ and $|u|^{\mathsf{I}} < \omega$, then $f(u) = |u|^{\mathsf{I}}$
(ii) $g(m) \in U^{\mathsf{I}}$ and $|g(m)|^{\mathsf{I}} = m$.

4.23. Suppose E is recursive in I and F is a function partial recursive in I such that $F : U^{\mathsf{I}} \to U^{\mathsf{I}}$. Show that there exists a function $G : U^{\mathsf{I}} \to U^{\mathsf{I}}$ partial recursive in I such that for all $u \in U^{\mathsf{I}}$

$$|G(u)|^{\mathsf{I}} = \sup{}^{+}\{|F(v)|^{\mathsf{I}} : |v|^{\mathsf{I}} < |u|^{\mathsf{I}}\}.$$

4.24. Prove the following *Effective Boundedness Principle*: there exists a primitive recursive function h such that for any d, if $\forall p. \{d\}^{\mathsf{I}}(p) \in U^{\mathsf{I}}$, then $h(d) \in U^{\mathsf{I}}$ and $\forall p. |\{d\}^{\mathsf{I}}(p)|^{\mathsf{I}} < |h(d)|^{\mathsf{I}}$.

4.25. For any I such that E is recursive in I, there exists a well-ordering of ω of order-type $\omega_1[\mathsf{I}]$ which is semi-recursive in I.

4.26. Show that if E is recursive in I, then $\mathsf{W}_{\omega_1[\mathsf{I}]}$ is semi-recursive in I and if E_1 is recursive in I, then $\mathsf{W}_{\omega_1[\mathsf{I}]}$ is recursive in I. (Define a functional G partial recursive in I such that for all $\gamma \in \mathsf{W}$ and all u,

$$\mathsf{G}(u, \gamma) \simeq \begin{cases} 0, & \text{if } |u|^{\mathsf{I}} < \|\gamma\|; \\ 1, & \text{otherwise.)} \end{cases}$$

4.27. Show that $|a, \langle \mathbf{m}\rangle|_0^{\mathsf{I}} = |a, \langle \mathbf{m}\rangle, \langle\ \ \rangle|^{\mathsf{I}}$.

4.28. For any I such that E is recursive in I, if A and B are sets of numbers both semi-recursive in I but not recursive in I, then A is recursive in I and B and B is recursive in I and A. (Cf. Corollary IV.2.13).

4.29 (cf. Exercise IV.2.24). The hierarchy of Theorem 4.11 is deficient in that it may happen that for some $\rho < \sigma < \kappa^{\mathsf{I}}$ no new relations are reducible to some U_τ^{I}

$(\tau < \sigma)$ that are not already reducible to some U^{I}_{τ} $(\tau < \rho)$. This can be remedied by omitting superfluous levels in the hierarchy. Let

$$X = \{\sigma : \sigma < \kappa^{\mathsf{I}} \wedge U^{\mathsf{I}}_{\sigma} \not\leqslant U^{\mathsf{I}}_{\rho} \text{ for any } \rho < \sigma\}$$

and

$$\bar{\kappa}^{\mathsf{I}} = \text{order-type of } X.$$

Show that if E is recursive in I, then $\bar{\kappa}^{\mathsf{I}} = \kappa^{\mathsf{I}}$.

Hint. Let

$$\bar{U}^{\mathsf{I}} = \{u : u \in U^{\mathsf{I}} \wedge |u|^{\mathsf{I}} \in X \wedge \forall v\, (|v|^{\mathsf{I}} = |u|^{\mathsf{I}} \rightarrow u \leqslant v)\}$$

and suppose that for some $t \in U^{\mathsf{I}}$, $\bar{\kappa}^{\mathsf{I}} = |t|^{\mathsf{I}} < \kappa^{\mathsf{I}}$. Set

$$\begin{aligned} P(u, v) \leftrightarrow (|u|^{\mathsf{I}} \not< |t|^{\mathsf{I}} \wedge v = 0) \vee (|u|^{\mathsf{I}} < |t|^{\mathsf{I}} \wedge v \in U^{\mathsf{I}} \wedge \\ \forall \alpha\, [\text{if } \alpha \text{ is an ordinal-preserving map of} \\ \{r : |r|^{\mathsf{I}} \leqslant |u|^{\mathsf{I}}\} \text{ into } \{s : s \in \bar{U}^{\mathsf{I}} \wedge |s|^{\mathsf{I}} \leqslant |v|^{\mathsf{I}}\}, \\ \text{then } \alpha(u) = v]). \end{aligned}$$

Show that P is semi-recursive in I, there is a function β recursive in I such that $\forall u P(u, \beta(u))$ and

$$v \in \bar{U}^{\mathsf{I}} \leftrightarrow \exists u\, (|u|^{\mathsf{I}} < |t|^{\mathsf{I}} \wedge \beta(u) = v).$$

4.30. Show that for any $v \notin U^{\mathsf{I}}$ and any β and γ, if $\mathsf{P}(v, \beta, \gamma)$ holds (P from the proof of Theorem 4.18), then β is not recursive in I.

4.31 (Gandy). Show that $\{\alpha : \alpha$ is recursive in $\mathsf{E}_1\}$ is a model of the Δ^1_2 Comprehension schema.

4.32 Notes. To anyone who has reached this note legitimately — that is, by following the proof of Theorem 4.18 — we offer our congratulations and suggest that some strong refreshment is in order. Try combining some hard-frozen strawberries, raspberries, or peaches in a blender with enough dark rum so that the result is a stiff mush (add powdered sugar if the fruit was not sweetened). Pour into a stemmed cocktail glass and relax! For an alternative, see the Notes to Barwise [1975, § II.6].

A simpler proof of a weaker version of Theorem 4.18 is sketched in Exercise 6.26.

5. Hierarchies of Relations Recursive in a Type-2 Functional

In § V.5 we constructed hierarchies of relations obtained by repeated application of positive analytic operations Φ and jump operators J. Here we shall study the relationship of these hierarchies with recursion in a functional. With any jump operator J we associate the functional I_{J} defined by

$$\mathsf{I}_{\mathsf{J}}((p) * \alpha) = \mathsf{J}(\alpha)(p)$$

and say that a relation or functional is *recursive in* J iff it is recursive in I_{J}. We write $\{a\}^{\mathsf{J}}$ instead of $\{a\}^{\mathsf{I}_{\mathsf{J}}}$. Recall that $\nabla(\mathsf{J})$ denotes the class of relations on numbers which are recursive in some set D_u^{J} ($u \in O^{\mathsf{J}}$) of Definition V.5.5. The main result of this section is that for any jump operator J,

$$\nabla(\mathsf{J}) = \{R : R \text{ is recursive in } \mathsf{J}\}.$$

Note that for oJ this equation follows from Theorem IV.4.21, Lemma 1.4 and Theorem 1.8; both sides are Δ_1^1.

With any positive analytic operation Φ we may associate the functional I_Φ defined by

$$\mathsf{I}_\Phi(\alpha) = \begin{cases} 0, & \text{if } \quad (\exists A \in \mathsf{B}(\Phi))(\forall p \in A) \,.\, \alpha(p) = 0; \\ 1, & \text{otherwise;} \end{cases}$$

and say that a relation or functional is *recursive in* Φ iff it is recursive in I_Φ. Then, for example, $\mathsf{I}_\cup = \mathsf{E}$ and $\mathsf{I}_{\mathscr{A}} = \mathsf{E}_1$. Similar methods lead to the result that for normal Φ,

$$\nabla(\Phi) = \{R : R \text{ is recursive in } \Phi\},$$

but we shall not carry out the details. A crucial point is that if Φ is normal, then E is recursive in Φ (Exercise 5.10).

5.1 Lemma. *For any jump operator* J, $\nabla(\mathsf{J}) \subseteq \{R : R$ is recursive in $\mathsf{J}\}$.

Proof. We shall define a primitive recursive function f such that for all $u \in O^{\mathsf{J}}$, $\{f(u)\}^{\mathsf{J}}$ is the characteristic function of D_u^{J}. *Let* h be a primitive recursive function which satisfies the following conditions for any J:

$$\{h(e,1)\}^{\mathsf{J}} = \mathrm{sg}^{+};$$
$$\{h(e,2^{u})\}^{\mathsf{J}} = \mathsf{J}(\{\{e\}(u)\}^{\mathsf{J}});$$
$$\{h(e,3^{a}5^{u})\}^{\mathsf{J}}(\langle m,p\rangle) = \{\{e\}(\{a\}(p,\{\{e\}(u)\}^{\mathsf{J}}))\}^{\mathsf{J}}(m).$$

By the Primitive Recursion Theorem choose $\bar{e}$ such that for all u, $\{\bar{e}\}(u) = h(\bar{e},u)$ and set $f = \{\bar{e}\}$. It is straightforward to prove by induction on the well-founded relation $<^{\mathsf{J}}$ that f is as required. $\square$

The proof of the converse will follow the pattern of IV.4.19–21.

5.2 Lemma. *For any jump operator* J, *there exists a primitive recursive function* g *such that for all* $u, v \in O^{\mathsf{J}}$, $|u|^{\mathsf{J}} \leq |v|^{\mathsf{J}}$, *then* D^{J}_{u} *is recursive in* D^{J}_{v} *with index* $g(u,v)$.

Proof. We shall define g by the Recursion Theorem simultaneously with an auxiliary function f such that for $u, v \in O^{\mathsf{J}}$,

$$|u|^{\mathsf{J}} < |v|^{\mathsf{J}} \leftrightarrow \{f(v)\}(u, D^{\mathsf{J}}_{v^{+}}) \simeq 0$$

and

$$|u|^{\mathsf{J}} \geq |v|^{\mathsf{J}} \leftrightarrow \{f(v)\}(u, D^{\mathsf{J}}_{v^{+}}) \simeq 1.$$

In the proof we shall use the following abbreviations:

$$A_{v,u} \quad \text{for} \quad \{p : \{g(u,v)\}(p,A) \simeq 0\},$$
$$A^{(p)} \quad \text{for} \quad \{m : \langle m,p\rangle \in A\},$$
$$\gamma_a(p) \quad \text{for} \quad \{a\}(p, A_{v,u}),$$

and

$$\delta_b(p) \quad \text{for} \quad \{b\}(p, A^{(0)}).$$

Thus if g satisfies the conclusion of the Lemma, we have:

$$(*) \qquad \begin{array}{l} \text{if} \quad u, v \in O^{\mathsf{J}}, |u|^{\mathsf{J}} \leq |v|^{\mathsf{J}}, \text{ and } A = D^{\mathsf{J}}_{v}, \text{ then} \\ A_{v,u} = D^{\mathsf{J}}_{u} \text{ and for all } p,\ \gamma_a(p) = \{a\}(p, D^{\mathsf{J}}_{u}); \end{array}$$

and without any assumption,

$$(**) \qquad \begin{array}{l} \text{if} \quad 3^{b}5^{v} \in O^{\mathsf{J}} \text{ and } A = D^{\mathsf{J}}_{3^{b}5^{v}}, \text{ then } A^{(0)} = D^{\mathsf{J}}_{v}, \text{ and} \\ \text{for all } p,\ \delta_b(p) = \{b\}(p, D^{\mathsf{J}}_{v}) \text{ and } A^{(p)} = D^{\mathsf{J}}_{\delta_b(p)}. \end{array}$$

We require that f and g satisfy the following conditions: for all u, v, a, b, m, and A,

(1) $\{f(1)\}(u, A) \simeq 1;$ if $v \neq 1, \{f(v)\}(1, A) \simeq 0;$

(2) (a) $\{f(v^+)\}(u^+, \mathsf{J}(A)) \simeq \{f(v)\}(u, A);$

(b) $$\{f(v^+)\}(3^a 5^u, \mathsf{J}(\mathsf{J}(A))) = \begin{cases} \mathsf{F}(u, v, a, \mathsf{J}(\mathsf{J}(A))), & \text{if } \{f(v)\}(u, \mathsf{J}(A)) \simeq 0; \\ 1, & \text{otherwise;} \end{cases}$$

where

$$\mathsf{F}(u, v, a, \mathsf{J}(\mathsf{J}(A))) \simeq \begin{cases} 1, & \text{if } \exists p . \{f(v)\}(\gamma_a(p), \mathsf{J}(A)) \simeq 1; \\ 0, & \text{otherwise.} \end{cases}$$

(3) $$\{f(3^b 5^v)\}(u, \mathsf{J}(A)) \simeq \begin{cases} 0, & \text{if } \exists p . \mathsf{G}(p, u, b, A) \simeq 0; \\ 1, & \text{otherwise;} \end{cases}$$

where

$$\mathsf{G}(p, u, b, A) \simeq \{f(\delta_b(p))\}(u, (A^{(p+1)})_{\delta_b(p+1), \delta_b(p)^+});$$

(4) $\{g(1, v)\}(m, A) \simeq \mathrm{sg}^+(m);$

(5) (a) $\{g(u^+, v^+)\}(m, A) \simeq \{h(g(u, v))\}(m, A),$

where h is the function of Definition V.5.4;

(b) $\{g(3^a 5^u, v^+)\}(m, \mathsf{J}(A)) \simeq \{g(3^a 5^u, v)\}(m, A);$

(6) (a) $\{g(u^+, 3^b 5^v)\}(m, A) \simeq \{g(u^+, \delta_b(\bar{p}))\}(m, A^{(\bar{p})}),$

where $\bar{p} \simeq$ "least" $p . \mathsf{G}(p, u^+, b, A) \simeq 0$ (G as in (3));

(b) $\{g(3^a 5^u, 3^b 5^v)\}(\langle m, p \rangle, A) \simeq \{g(\alpha(p), 3^b 5^v)\}(m, A),$

where $\alpha(p) \simeq \{a\}(p, A_{3^b 5^v, u}).$

The proof that there exist primitive recursive f and g which satisfy these conditions is essentially as for Lemma IV.4.20. The fact that $\mathsf{oJ}(A)$ is recursive in $\mathsf{J}(A)$ is needed to account for the quantifiers. We prove by induction on $|v|$ $(= |v|^{\mathsf{J}})$ that f and g have the required properties. If $|v| = 0$ this is obvious, so suppose it is true with v replaced by any w such that $|w| < |v^+|$. Then $(*)$ holds for this v and any u and

(1) $|1| < |v^+|$ is true and $\{f(v^+)\}(1, D^{\mathsf{J}}_{v^{++}}) \simeq 0.$

(2) (a) If $u^+ \in O^{\mathsf{J}}$, then $u \in O^{\mathsf{J}}$, so

$$|u^+| < |v^+| \leftrightarrow |u| < |v| \leftrightarrow \{f(v)\}(u, D^{\mathsf{J}}_{v^+}) \simeq 0$$
$$\leftrightarrow \{f(v^+)\}(u^+, D^{\mathsf{J}}_{v^{++}}) \simeq 0.$$

(b) If $3^a 5^u \in O^{\mathsf{J}}$, then $u \in O^{\mathsf{J}}$ and for all p, $\{a\}(p, D^{\mathsf{J}}_u) \in O^{\mathsf{J}}$. Thus

$$|3^a 5^u| < |v^+| \leftrightarrow |u| < |v| \quad \text{and} \quad \forall p . |\{a\}(p, D^{\mathsf{J}}_u)| < |v|$$
$$\leftrightarrow \{f(v)\}(u, D^{\mathsf{J}}_{v^+}) \simeq 0 \text{ and } \forall p . \{f(v)\}(\{a\}(p, D^{\mathsf{J}}_u), D^{\mathsf{J}}_{v^+}) \simeq 0$$
$$\leftrightarrow \{f(v^+)\}(3^a 5^u, D^{\mathsf{J}}_{v^{++}}) \simeq 0.$$

The last equivalence follows from $(*)$.

(5) (a) If $|u^+| \leqslant |v^+|$, then $|u| \leqslant |v|$, so

$$m \in D^{\mathsf{J}}_{u^+} \leftrightarrow m \in \mathsf{J}(D^{\mathsf{J}}_u)$$
$$\leftrightarrow \mathsf{J}(\lambda n . \{g(u, v)\}(n, D^{\mathsf{J}}_v))(m) \simeq 0$$
$$\leftrightarrow \{h(g(u, v))\}(m, D^{\mathsf{J}}_{v^+}) \simeq 0$$
$$\leftrightarrow \{g(u^+, v^+)\}(m, D^{\mathsf{J}}_{v^+}) \simeq 0.$$

(b) If $|3^a 5^u| \leqslant |v^+|$, then $|3^a 5^u| \leqslant |v|$, so

$$m \in D^{\mathsf{J}}_{3^a 5^u} \leftrightarrow \{g(3^a 5^u, v)\}(m, D^{\mathsf{J}}_v) \simeq 0$$
$$\leftrightarrow \{g(3^a 5^u, v^+)\}(m, D^{\mathsf{J}}_{v^+}) \simeq 0.$$

Now suppose that the result holds with v replaced by any w such that $|w| < |3^b 5^v|$.

(3) First note that by $(**)$ and the induction hypothesis,

$$((D^{\mathsf{J}}_{3^b 5^v})^{(p+1)})_{\delta_b(p+1), \delta_b(p)^+} = D^{\mathsf{J}}_{\delta_b(p)^+}.$$

Then as in the corresponding part of the proof of IV.4.20, for any $u \in O^{\mathsf{J}}$ and any p,

$$|u| < |\delta_b(p)| \leftrightarrow \mathsf{G}(p, u, b, D^{\mathsf{J}}_{3^b 5^v}) \simeq 0.$$

Hence

$$|u| < |3^b 5^v| \leftrightarrow \exists p . |u| < |\delta_b(p)|$$
$$\leftrightarrow \exists p . \mathsf{G}(p, u, b, D^{\mathsf{J}}_{3^b 5^v}) \simeq 0$$
$$\leftrightarrow \{f(3^b 5^v)\}(u, D^{\mathsf{J}}_{(3^b 5^v)^+}) \simeq 0.$$

(6) We proceed by induction on $|u|$.

(a) If $|u^+| \leqslant |3^b 5^v|$, then $|u^+| < |3^b 5^v|$ so (by (**)) for some (least) $\bar{p}$, $|u^+| < |\delta_b(\bar{p})|$. Then $\mathsf{G}(\bar{p}, u^+, b, D^{\mathsf{J}}_{3^b 5^v}) \simeq 0$ and

$$\begin{aligned} m \in D^{\mathsf{J}}_{u^+} &\leftrightarrow \{g(u^+, \delta_b(\bar{p}))\}(m, D^{\mathsf{J}}_{\delta_b(\bar{p})}) \simeq 0 \\ &\leftrightarrow \{g(u^+, \delta_b(\bar{p}))\}(m, (D^{\mathsf{J}}_{3^b 5^v})^{(\bar{p})}) \simeq 0 \\ &\leftrightarrow \{g(u^+, 3^b 5^v)\}(m, D^{\mathsf{J}}_{3^b 5^v}) \simeq 0. \end{aligned}$$

(b) If $|3^a 5^u| \leqslant |3^b 5^v|$, then $|u| < |3^b 5^v|$ and for all p, $|\{a\}(p, D^{\mathsf{J}}_u)| < |3^b 5^v|$. By the induction hypothesis on u,

$$(D^{\mathsf{J}}_{3^b 5^v})_{3^b 5^v, u} = D^{\mathsf{J}}_u$$

so that $\alpha(p) = \{a\}(p, D^{\mathsf{J}}_u)$. Then

$$\begin{aligned} \langle m, p\rangle \in D^{\mathsf{J}}_{3^a 5^u} \leftrightarrow m &\in D^{\mathsf{J}}_{\alpha(p)} \\ &\leftrightarrow \{g(\alpha(p), 3^b 5^v)\}(m, D^{\mathsf{J}}_{3^b 5^v}) \simeq 0 \\ &\leftrightarrow \{g(3^a 5^u, 3^b 5^v)\}(\langle m, p\rangle, D^{\mathsf{J}}_{3^b 5^v}) \simeq 0. \quad \square \end{aligned}$$

5.3 Lemma. *For any jump operator* J, *there exists a partial recursive function* $+^{\mathsf{J}}$ *such that for all* $u, v \in O^{\mathsf{J}}$, $u +^{\mathsf{J}} v \in O^{\mathsf{J}}$, $|u +^{\mathsf{J}} v|^{\mathsf{J}} = |u|^{\mathsf{J}} + |v|^{\mathsf{J}}$, *and if* $v \neq 1$, *then* $u <^{\mathsf{J}} (u +^{\mathsf{J}} v)$.

Proof. We define $+^{\mathsf{J}}$ by the Recursion Theorem to satisfy the following conditions:

(1) $u +^{\mathsf{J}} 1 \simeq u$;

(2) $u +^{\mathsf{J}} v^+ \simeq (u +^{\mathsf{J}} v)^+$;

(3) $u +^{\mathsf{J}} 3^b 5^v \simeq 3^c 5^{u +^{\mathsf{J}} v}$, where c is an index such that

$$\{c\}(p, D^{\mathsf{J}}_{u +^{\mathsf{J}} v}) \simeq u +^{\mathsf{J}} \{b\}(p, D^{\mathsf{J}}_v).$$

That is, with g as in the preceding lemma,

$$\{c\}(p, A^{\boldsymbol{\cdot}}) \simeq u +^{\mathsf{J}} \{b\}(p, \lambda n . \{g(v, u +^{\mathsf{J}} v)\}(n, A)).$$

If follows easily by induction over O^{J} that $+^{\mathsf{J}}$ has the required properties. $\square$

5.4 Corollary (Effective Boundedness). *For any jump operator* J, *there exists a primitive recursive function* h *such that for any* $u \in O^{\mathsf{J}}$ *and any* d, *if for all* p,

$\{d\}(p, D^J_u) \in O^J$, *then* $h(d, u) \in O^J$, $|u|^J < |h(d, u)|^J$, *and for all* p, $|\{d\}(p, D^J_u)|^J < |h(d, u)|^J$.

Proof. We set $h(d, u) = 3^c 5^u$, where c is an index such that

$$\{c\}(0, A) \simeq u, \quad \text{and}$$
$$\{c\}(p+1, A) \simeq \{c\}(p, A) +^J \{d\}(p, A)^+.$$

The required properties of h follow easily from the properties of $+^J$. □

5.5 Theorem. *For any jump operator* J, *there exist primitive recursive functions* f *and* g *such that for any* a *and* $\mathbf{m}$, *if* $\{a\}^J(\mathbf{m})$ *is defined, then* $f(a, \langle \mathbf{m} \rangle) \in O^J$ *and*

$$\{a\}^J(\mathbf{m}) \simeq \{g(a)\}(\mathbf{m}, D^J_{f(a, \langle \mathbf{m} \rangle)}).$$

Proof. We shall use the notation $A_{v,u}$ as in the proof of Lemma 5.2 and the function h of the preceding Corollary. In particular if $u \in O^J$ and for all p, $\{d\}(p, D^J_u) \in O^J$,

$$(D^J_{h(d,u)})_{h(d,u), \{d\}(p, D^J_u)} = D^J_{\{d\}(p, D^J_u)}.$$

We require that f and g satisfy the following conditions for all a and $\mathbf{m}$,
(0) If $a = \langle 0, k, 0, \ldots \rangle$, then $f(a, \langle \mathbf{m} \rangle) = 1$ and

$$\{g(a)\}(\mathbf{m}, D^J_1) \simeq \{a\}^J(\mathbf{m});$$

(1) if $a = \langle 1, k, 0, b, c_1, c_2 \rangle$ (say), then $f(a, \langle \mathbf{m} \rangle) = h(d_1, h(d_0, 1))$, where

$$\{d_0\}(p, D^J_1) \simeq \begin{cases} f(c_1, \langle \mathbf{m} \rangle), & \text{if} \quad p = 0; \\ f(c_2, \langle \mathbf{m} \rangle), & \text{if} \quad p > 0; \end{cases}$$
$$\{d_1\}(p, A) \simeq f(b, \langle \{g(c_1)\}(\mathbf{m}, A_{h(d_0,1), f(c_1, \langle \mathbf{m} \rangle)}), \{g(c_2)\}(\mathbf{m}, A_{h(d_0,1), f(c_2, \langle \mathbf{m} \rangle)}) \rangle);$$
$$\{g(a)\}(\mathbf{m}, A) \simeq \{g(b)\}(q_1, q_2, A_{f(a, \langle \mathbf{m} \rangle), f(b, \langle q_1, q_2 \rangle)}),$$

where

$$q_i \simeq \{g(c_i)\}(\mathbf{m}, A_{f(a, \langle \mathbf{m} \rangle), f(c_i, \langle \mathbf{m} \rangle)});$$

(2) if $a = \langle 2, k+1, 0 \rangle$, then $f(a, \langle b, \mathbf{m} \rangle) = h(d_2, 1)$, where

$$\{d_2\}(p, D^J_1) \simeq f(b, \langle \mathbf{m} \rangle);$$
$$\{g(a)\}(b, \mathbf{m}, A) \simeq \{g(b)\}(\mathbf{m}, A_{f(a, \langle b, \mathbf{m} \rangle), f(b, \langle \mathbf{m} \rangle)});$$

(3) if $a = \langle 3, k, l, b\rangle$, then $f(a, \langle\mathbf{m}\rangle) = h(d_3, 1)^+$, where $\{d_3\}(p, D^{\mathsf{J}}_1) \simeq f(b, \langle p, \mathbf{m}\rangle)$; $g(a)$ is an index such that

$$\{g(a)\}(\mathbf{m}, \mathsf{J}(A)) \simeq \mathsf{J}(\lambda p.\{d_4\}(p+1, \mathbf{m}, A))(\{d_4\}(0, A)),$$

where

$$\{d_4\}(p, \mathbf{m}, A) \simeq \{g(b)\}(p, \mathbf{m}, A_{h(d_3,1), f(b, \langle p, \mathbf{m}\rangle)}).$$

The proof that there exist such f and g is again similar to the corresponding proof in Lemma IV.4.20. We prove that f and g have the required properties by induction on $\Omega[\mathsf{J}]$ $(= \Omega[\mathsf{I}_{\mathsf{J}}])$.

(0) If $a = \langle 0, k, 0, \dots\rangle$, then the result is clear.

(1) If $a = \langle 1, k, 0, b, c_1, c_2\rangle$, $\{c_i\}^{\mathsf{J}}(\mathbf{m}) \simeq q_i$ $(i = 1, 2)$, and $\{a\}^{\mathsf{J}}(\mathbf{m}) \simeq \{b\}^{\mathsf{J}}(q_1, q_2) \simeq n$, then by the induction hypothesis, $f(c_i, \langle\mathbf{m}\rangle) \in O^{\mathsf{J}}$. Hence for all p, $\{d_0\}(p, D^{\mathsf{J}}_1) \in O^{\mathsf{J}}$, $h(d_0, 1) \in O^{\mathsf{J}}$, $|f(c_i, \langle\mathbf{m}\rangle)|^{\mathsf{J}} < |h(d_0, 1)|^{\mathsf{J}}$ $(i = 1, 2)$, and

$$(D^{\mathsf{J}}_{h(d_0,1)})_{h(d_0,1), f(c_i, \langle\mathbf{m}\rangle)} = D^{\mathsf{J}}_{f(c_i, \langle\mathbf{m}\rangle)}.$$

By the induction hypothesis for g, $\{g(c_i)\}(\mathbf{m}, D^{\mathsf{J}}_{f(c_i,\langle\mathbf{m}\rangle)}) \simeq q_i$, so for all p, $\{d_1\}(p, D^{\mathsf{J}}_{h(d_0,1)}) \simeq f(b, \langle q_1, q_2\rangle) \in O^{\mathsf{J}}$, and thus $f(a, \langle\mathbf{m}\rangle) = h(d_1, h(d_0, 1)) \in O^{\mathsf{J}}$. Furthermore, by the induction hypothesis,

$$\{g(a)\}(\mathbf{m}, D^{\mathsf{J}}_{f(a,\langle\mathbf{m}\rangle)}) \simeq \{g(b)\}(q_1, q_2, D^{\mathsf{J}}_{f(b,\langle q_1, q_2\rangle)}) \simeq \{b\}^{\mathsf{J}}(q_1, q_2) \simeq n.$$

(2) If $a = \langle 2, k+1, 0\rangle$ and $\{a\}^{\mathsf{J}}(b, \mathbf{m}) \simeq \{b\}^{\mathsf{J}}(\mathbf{m}) \simeq n$, then for all p, $\{d_2\}(p, D^{\mathsf{J}}_1) \simeq f(b, \langle\mathbf{m}\rangle) \in O^{\mathsf{J}}$, so $f(a, \langle b, \mathbf{m}\rangle) \simeq h(d_2, 1) \in O^{\mathsf{J}}$. Furthermore,

$$\{g(a)\}(b, \mathbf{m}, D^{\mathsf{J}}_{f(a,\langle b, \mathbf{m}\rangle)}) \simeq \{g(b)\}(\mathbf{m}, D^{\mathsf{J}}_{f(b,\langle\mathbf{m}\rangle)}) \simeq \{b\}^{\mathsf{J}}(\mathbf{m}) \simeq n.$$

(3) If $a = \langle 3, k, 0, b\rangle$ and $\{a\}^{\mathsf{J}}(\mathbf{m}) \simeq \mathsf{I}_{\mathsf{J}}(\lambda p.\{b\}^{\mathsf{J}}(p, \mathbf{m})) \simeq n$, then for all p, $\{d_2\}(p, D^{\mathsf{J}}_1) \simeq f(b, \langle p, \mathbf{m}\rangle) \in O^{\mathsf{J}}$, so $f(a, \langle\mathbf{m}\rangle) \simeq h(d_3, 1)^+ \in O^{\mathsf{J}}$. Then also

$$\begin{aligned}\{g(a)\}(\mathbf{m}, D^{\mathsf{J}}_{f(a,\langle\mathbf{m}\rangle)}) &\simeq \mathsf{J}(\lambda p.\{d_4\}(p+1, \mathbf{m}, D^{\mathsf{J}}_{h(d_3,1)}))(\{d_4\}(0, D^{\mathsf{J}}_{h(d_3,1)}))\\ &\simeq \mathsf{J}(\lambda p.\{g(b)\}(p+1, \mathbf{m}, D^{\mathsf{J}}_{f(b,\langle p+1, \mathbf{m}\rangle)}))(\{g(b)\}(0, \mathbf{m}, D^{\mathsf{J}}_{f(b,\langle 0, \mathbf{m}\rangle)}))\\ &\simeq \mathsf{J}(\lambda p.\{b\}^{\mathsf{J}}(p+1, \mathbf{m}))(\{b\}^{\mathsf{J}}(0, \mathbf{m}))\\ &\simeq \mathsf{I}_{\mathsf{J}}(\lambda p.\{b\}^{\mathsf{J}}(p, \mathbf{m})) \simeq n. \quad \square\end{aligned}$$

5.6 Corollary. *For any jump operator* J *and any* $R \subseteq {}^k\omega$, *R is recursive in* J *iff for some* $u \in O^{\mathsf{J}}$, *R is recursive in* D^{J}_u.

Proof. Immediate from Lemma 5.1, Corollary 5.4, and Theorem 5.5. $\square$

Since for any jump operator J and any A, $o\mathsf{J}(A)$ is recursive in $\mathsf{J}(A)$, it follows that $\mathsf{J}(A)$ is not recursive in A. In particular, for all $u \in O^{\mathsf{J}}$, $D^{\mathsf{J}}_{u^+}$ is not recursive in D^{J}_u, whence by Lemma 5.2, if $|u|^{\mathsf{J}} < |v|^{\mathsf{J}}$, then D^{J}_v is not recursive in D^{J}_u. Thus if we set

$$\Delta^{\mathsf{J}}_\sigma = \{R : R \text{ is recursive in some } D^{\mathsf{J}}_u \text{ such that } |u|^{\mathsf{J}} \leqslant \sigma\},$$

the classes $\Delta^{\mathsf{J}}_\sigma$ form a properly increasing hierarchy of length

$$\lambda^{\mathsf{J}} = \sup{}^+\{|u|^{\mathsf{J}} : u \in O^{\mathsf{J}}\},$$

which includes exactly the relations on numbers which are recursive in J. The evaluation of λ^{J} provides no surprise:

5.7 Theorem. *For any jump operator* J, $\lambda^{\mathsf{J}} = \omega_1[\mathsf{J}]$.

Sketch of proof. An easy induction over $\Omega[\mathsf{J}]$ shows that for all a and $\mathbf{m}$, if $\{a\}^{\mathsf{J}}(\mathbf{m})$ is defined, then $|\langle a, \mathbf{m}\rangle|^{\mathsf{J}} \leqslant |f(a, \langle \mathbf{m}\rangle)|^{\mathsf{J}}$, where the first ordinal is that assigned in § 3, the second is the ordinal assigned to members of O^{J}, and f is the function of Theorem 5.5. It then follows from Theorem 4.16 that $\lambda^{\mathsf{J}} \geqslant \omega_1[\mathsf{J}]$. Conversely, it is possible to define a primitive recursive function h such that for all $u \in O^{\mathsf{J}}$, $h(u)$ is an index from D^{J}_u of $\{(v, w) : v <^{\mathsf{J}} w <^{\mathsf{J}} u\}$ which is a well-ordering of type $|u|^{\mathsf{J}}$ and is recursive in J. It follows that $\omega_1[\mathsf{J}] \geqslant \lambda^{\mathsf{J}}$ (Exercise 5.11). □

These results provide a hierarchy for the relations on numbers recursive in any jump operator. Happily, for any functional I in which E is recursive there is a jump operator J_I of the same degree (each of I and J_I is recursive in the other) so that we obtain a hierarchy for the relations on numbers recursive in any such I.

Let

$$\mathsf{J}_\mathsf{I}(\alpha)(\langle a, n\rangle) = \begin{cases} 0, & \text{if } \mathsf{I}(\lambda p . \{a\}(p, \alpha)) \simeq n; \\ 1, & \text{otherwise.} \end{cases}$$

5.8 Lemma. *For any* I, J_I *is a jump operator.*

Proof. Let $n_0 = \mathsf{I}(\lambda p . 0)$ and f be a primitive recursive function such that for all a, p, $\mathbf{m}$, and α,

$$\{f(a, \langle \mathbf{m}\rangle)\}(p, \alpha) \simeq 0 \cdot \{a\}(\mathbf{m}, \alpha).$$

Then

$$\begin{aligned}\alpha^{\mathsf{oJ}}(\langle a, \mathbf{m}\rangle) = 0 &\leftrightarrow \{a\}(\mathbf{m}, \alpha) \text{ is defined}\\ &\leftrightarrow \forall p\,.\,\{f(a, \langle \mathbf{m}\rangle)\}(p, \alpha) \simeq 0\\ &\leftrightarrow \mathsf{J}_{\mathsf{I}}(\alpha)(\langle f(a, \langle \mathbf{m}\rangle), n_0\rangle) = 0\\ &\leftrightarrow \{d\}(\langle a, \mathbf{m}\rangle, \mathsf{J}_{\mathsf{I}}(\alpha)) = 0\end{aligned}$$

for an appropriate index d. Condition (ii) of Definition V.5.4 follows easily from Theorem 2.10. □

5.9 Theorem. *For any* I, *if* E *is recursive in* I, *then* I *and* J_{I} *are each recursive in the other.*

Proof. Suppose that E is recursive in I. Then one can decide recursively in I whether or not $\lambda p\,.\,\{a\}(p, \alpha)$ is total and, if it is, whether or not $\mathsf{I}(\lambda p\,.\,\{a\}(p, \alpha)) = n$. Hence J_{I} is recursive in I. For the converse, let a_0 be an index such that $\{a_0\}(m, \alpha) = \alpha(m)$. Then

$$\mathsf{I}(\alpha) = n \leftrightarrow \mathsf{J}_{\mathsf{I}}(\alpha)(\langle a_0, n\rangle) = 0,$$

so Gr_{I} is recursive in J_{I}. Since oJ is recursive in J_{I}, so is E by Lemma 1.4. Hence by Corollary 4.6, I is recursive in J_{I}. □

5.10–5.11 Exercises

5.10. Show that if Φ is a normal positive analytic operation, then E is recursive in Φ.

5.11. Complete the proof of Theorem 5.7. Give an alternative proof of the inequality $\lambda^{\mathsf{J}} \geqslant \omega_1[\mathsf{J}]$ by showing that for any $\gamma \in \mathsf{W}$ recursive in J, there exists a function f recursive in J such that for all $p \in \mathrm{Fld}(\gamma)$, $|p|_\gamma \leqslant |f(p)|^{\mathsf{J}}$.

5.12 Notes. The construction of a hierarchy for the relations on numbers recursive in a type-2 functional is due independently to Hinman [1966] and Shoenfield [1968]. The method here is Shoenfield's. A proof of the result mentioned just before Lemma 5.1 may be found in Hinman [1969].

6. Extended Functionals

The notion of extended functionals arises from the following question. Recall that a positive analytic operation Φ has a natural interpretation as a quantifier:

$$(\Phi \mathsf{R})(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow (\exists A \in \mathsf{B}(\Phi))(\forall p \in A)\, \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}).$$

These would be natural objects of study even in the absence of their classical origins. In § 5 we defined the notion of recursion relative to Φ. It is immediate from the definition that for any positive analytic operation Φ,

(1) $\{\mathsf{R} : \mathsf{R}$ is recursive in $\Phi\}$ is closed under the quantifier Φ.

Since "(semi-) recursive in $\cup$" coincides with "(semi-) recursive in E", it follows from either Corollary 1.10 or Corollary 4.3 that

(2) $\{\mathsf{R} : \mathsf{R}$ is semi-recursive in $\cup\}$ $(= \Pi^1_1)$ is closed under the quantifier $\cup\,(= \exists^0)$.

We ask, therefore, if also (2) holds with "$\cup$" replaced by an arbitrary "Φ". The answer is no (Corollary 6.3). To restore the analogy we define a "functional" $\Phi^{\#}$ with the property that $\{\mathsf{R} : \mathsf{R}$ is semi-recursive in $\Phi^{\#}\}$ is closed under the quantifier Φ. $\Phi^{\#}$ is not a functional in the sense that we have used the term as its domain includes some partial unary functions from ω into ω as well as all total unary functions. Although the previous definitions and results do not apply directly, it turns out that the theory of recursion relative to $\Phi^{\#}$ is very much like the theory of recursion relative to an ordinary functional. In the latter part of the section we investigate the (close) connections among $\Phi^{\#}$, Φ^*, and Φ-positive inductive definitions.

6.1 Lemma. *For any functional* I, *there exists a relation* P^{I} *semi-recursive in* I *such that for all* u, v, *and* $\boldsymbol{\alpha}$,

(i) $u \in U^{\mathsf{I}}_{\boldsymbol{\alpha}} \wedge \mathsf{P}^{\mathsf{I}}(u, v, \langle\boldsymbol{\alpha}\rangle) \rightarrow |v|^{\mathsf{I}}_{\boldsymbol{\alpha}} < |u|^{\mathsf{I}}_{\boldsymbol{\alpha}}$;

(ii) $u \notin U^{\mathsf{I}}_{\boldsymbol{\alpha}} \rightarrow \exists v\, [v \notin U^{\mathsf{I}}_{\boldsymbol{\alpha}} \wedge \mathsf{P}^{\mathsf{I}}(u, v, \langle\boldsymbol{\alpha}\rangle)]$.

Proof. Intuitively, $\mathsf{P}^{\mathsf{I}}(u, v, \langle\boldsymbol{\alpha}\rangle)$ means that v represents a computation relative to $\boldsymbol{\alpha}$ which is an immediate subcomputation of that represented by u. We define $\mathsf{P}^{\mathsf{I}}(u, v, \langle\boldsymbol{\alpha}\rangle)$ to hold just in case one of the following holds for some $k, \mathbf{m} \in {}^k\omega$, and $l = \lg(\boldsymbol{\alpha})$:

(1) for some $b, c_0, \ldots, c_{k'-1}$, $u = \langle\langle 1, k, l, b, c_0, \ldots, c_{k'-1}\rangle, \mathbf{m}\rangle$, and either $v = \langle c_i, \mathbf{m}\rangle$ for some $i < k'$ or there exist $q_0, \ldots, q_{k'-1}$ such that for all $i < k'$, $\{c_i\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq q_i$ and $v = \langle b, \mathbf{q}\rangle$;

(2) for some b, $u = \langle\langle 2, k+1, l\rangle, b, \mathbf{m}\rangle$ and $v = \langle b, \mathbf{m}\rangle$;

(3) for some b, $u = \langle\langle 3, k, l, b\rangle, \mathbf{m}\rangle$ and for some p,

$$v = \langle b, p, \mathbf{m}\rangle;$$

(4) u is of none of these forms, $|u|^{\mathsf{I}}_{\boldsymbol{\alpha}} \neq 0$, and $v = 0$.

Suppose first that $u \in U^{\mathsf{I}}_{\alpha}$ and $\mathsf{P}^{\mathsf{I}}(u, v, \langle \alpha \rangle)$ holds. Then u falls under one of clauses (1)–(3) and it is clear from Lemma 3.2 that in each case $|v|^{\mathsf{I}}_{\alpha} < |u|^{\mathsf{I}}_{\alpha}$. For (ii), suppose that $u \notin U^{\mathsf{I}}_{\alpha}$. If u is not of the proper form for α, then by clause (4), $\mathsf{P}^{\mathsf{I}}(u, 0, \langle \alpha \rangle)$ holds and (ii) is satisfied becasuse $0 \notin U^{\mathsf{I}}_{\alpha}$. Otherwise u satisfies the hypotheses of one of (1)–(3). Suppose $u = \langle\langle 1, k, l, b, c_0, \ldots, c_{k'-1}\rangle, \mathbf{m}\rangle$. If for some $i < k'$, $\langle c_i, \mathbf{m}\rangle \notin U^{\mathsf{I}}_{\alpha}$, let v be such a $\langle c_i, \mathbf{m}\rangle$. Otherwise, there exist $q_0, \ldots, q_{k'-1}$ such that $\{c_i\}^{\mathsf{I}}(\mathbf{m}, \alpha) \simeq q_i$ and $\langle b, \mathbf{q}\rangle \notin U^{\mathsf{I}}_{\alpha}$, as otherwise we would have $u \in U^{\mathsf{I}}_{\alpha}$, contrary to assumption. Thus $v = \langle b, \mathbf{q}\rangle$ satisfies the conclusion of (ii). The other cases are similar. □

6.2 Theorem. *For any functional* I *and all* a, $\mathbf{m}$, *and* α,

$$\{a\}^{\mathsf{I}}(\mathbf{m}, \alpha) \textit{ is defined} \leftrightarrow \neg \exists \beta \, \forall p \, [\mathsf{P}^{\mathsf{I}}(\beta(p), \beta(p+1), \langle \alpha \rangle) \wedge \beta(0) = \langle a, \mathbf{m}\rangle].$$

Proof. Suppose first that $\{a\}^{\mathsf{I}}(\mathbf{m}, \alpha)$ is defined but for some β, $\beta(0) = \langle a, \mathbf{m}\rangle$ and for all p, $\mathsf{P}^{\mathsf{I}}(\beta(p), \beta(p+1), \langle \alpha \rangle)$. Since $\langle a, \mathbf{m}\rangle \in U^{\mathsf{I}}_{\alpha}$ by assumption, it follows by induction from Lemma 6.1(i) that for all p, $\beta(p) \in U^{\mathsf{I}}_{\alpha}$ and thus for all p, $|\beta(p+1)|^{\mathsf{I}}_{\alpha} < |\beta(p)|^{\mathsf{I}}_{\alpha}$, a contradiction.

If, on the other hand, $\{a\}^{\mathsf{I}}(\mathbf{m}, \alpha)$ is not defined, clause (ii) of the preceding lemma guarantees that there is a unique function β such that $\beta(0) = \langle a, \mathbf{m}\rangle$ and for all p,

$$\beta(p+1) = \text{least } v[v \notin U^{\mathsf{I}}_{\alpha} \text{ and } \mathsf{P}^{\mathsf{I}}(\beta(p), v, \langle \alpha \rangle)]. \quad \square$$

6.3 Corollary. *For any functional* I *such that* E *is recursive in* I, *there exists a relation* R *semi-recursive in* I *such that* $\mathscr{A}\mathsf{R}(= \{(\mathbf{m}, \alpha) : \exists \beta \, \forall p \, \mathsf{R}(\bar{\beta}(p), \mathbf{m}, \alpha)\})$ *is not semi-recursive in* I. *In particular, the class of relations semi-recursive in* $\mathscr{A}$ ($=$ *semi-recursive in* E_1) *is not closed under* $\mathscr{A}$.

Proof. Let

$$\mathsf{R}(s, \langle \mathbf{m}\rangle, \langle \alpha \rangle) \leftrightarrow [(\forall p < \lg(s) \dot{-} 1)\, \mathsf{P}^{\mathsf{I}}((s)_p, (s)_{p+1}, \langle \alpha \rangle) \wedge (s)_0 = \langle a, \mathbf{m}\rangle].$$

By the preceding theorem $\mathscr{A}\mathsf{R} = {\sim}\mathsf{U}^{\mathsf{I}}$. If $\mathscr{A}\mathsf{R}$ were semi-recursive in I, then by Corollary 4.5, U^{I} would be recursive in I, a contradiction. □

For any positive analytic operation Φ, we define the *extended functional* $\Phi^{\#}$ as follows: for any *partial* function f from ω into ω,

$$\Phi^{\#}(f) \simeq \begin{cases} 0, & \text{if } (\exists A \in \mathsf{B}(\Phi))(\forall p \in A) . f(p) \simeq 0; \\ 1, & \text{if } (\forall A \in \mathsf{B}(\Phi))(\exists p \in A)(\exists n > 0) . f(p) \simeq n; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Note that $\Phi^{\#}$ is an extension of the functional I_Φ defined in § 5. In fact, for any partial function f,

$$\Phi^{\#}(f) \simeq n \leftrightarrow \forall \alpha\, [f \subseteq \alpha \rightarrow \mathsf{I}_\Phi(\alpha) = n].$$

Note also that $\Phi^{\#}$ is *consistent* — that is, if $\Phi^{\#}(f) \simeq n$ and $f \subseteq g$, then also $\Phi^{\#}(g) \simeq n$. We shall write $\mathsf{E}^{\#}$ for $\mathsf{U}^{\#}$ and $\mathsf{E}_1^{\#}$ for $\mathscr{A}^{\#}$. Thus, for any partial f

$$\mathsf{E}_1^{\#}(f) \simeq \begin{cases} 0, & \text{if} \quad \exists \beta\, \forall p \,.\, f(\bar{\beta}(p)) \simeq 0; \\ 1, & \text{if} \quad \forall \beta\, \exists p\, (\exists n > 0) \,.\, f(\bar{\beta}(p)) \simeq n; \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

Recursion relative to extended functionals may be defined in nearly the same way as for ordinary functionals:

6.4 Definition. For any extended functional $\Phi^{\#}$, $\Omega[\Phi^{\#}]$ is the smallest set such that for all k, l, n, p, q, r, and s, all $i < k$, and $j < l$, and all $(\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$,

(0), (1), (2) identical to the corresponding clauses of Definition II.2.1;

(3)$^{\#}$ for any b and any f such that $\Phi^{\#}(f) \simeq n$, if for all p and q,

$$f(p) \simeq q \rightarrow (b, p, \mathbf{m}, \boldsymbol{\alpha}, q) \in \Omega[\Phi^{\#}],$$

then $(\langle 3, k, l, b\rangle, \mathbf{m}, \boldsymbol{\alpha}, n) \in \Omega[\Phi^{\#}]$.

The proof that for all a, $\mathbf{m}$, and $\boldsymbol{\alpha}$ there is at most one n such that $(a, \mathbf{m}, \boldsymbol{\alpha}, n) \in \Omega[\Phi^{\#}]$ depends on the consistency of $\Phi^{\#}$ but otherwise proceeds as in the previous cases and we set

$$\{a\}^{\Phi^{\#}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n \leftrightarrow (a, \mathbf{m}, \boldsymbol{\alpha}, n) \in \Omega[\Phi^{\#}]$$

and define the other relevant notions as in § 1. The remainder of the theory of §§ 1 and 2 may now be carried over to recursion in an extended functional with exactly the same proofs.

We show first that replacing Φ by $\Phi^{\#}$ has the intended effect.

6.5 Theorem. *For any extended functional* $\Phi^{\#}$, $\{\mathsf{R} : \mathsf{R}$ *is semi-recursive in* $\Phi^{\#}\}$ *is closed under the quantifier* Φ.

Proof. Let R be semi-recursive in $\Phi^{\#}$, say with semi-index a from $\Phi^{\#}$, and suppose $\mathsf{S} = \Phi\mathsf{R}$. Then $\mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \Phi^{\#}(\lambda p \,.\, 0 \cdot \{a\}^{\Phi^{\#}}(p, \mathbf{m}, \boldsymbol{\alpha}))$ is defined, so S is also semi-recursive in $\Phi^{\#}$. $\square$

Since $\Phi^{\#}$ is an extension of Φ, it is to be expected that functionals (partial) recursive in Φ are also (partial) recursive in $\Phi^{\#}$. For total functionals this is evident as clause $(3)^{\#}$ implies clause (3) for I_{Φ} and thus $\Omega[\Phi] \subseteq \Omega[\Phi^{\#}]$. Hence if $\{a\}^{\Phi}$ is total, $\{a\}^{\Phi} = \{a\}^{\Phi^{\#}}$. Clearly, however, more indices define computations relative to $\Phi^{\#}$ than to Φ.

Rather than investigate further the subject of extended functionals in full generality we shall concentrate on the particular example $\mathsf{E}_1^{\#}$. The properties of recursion relative to $\mathsf{E}_1^{\#}$ are illustrative of those of all extended functionals and the proofs which establish these properties will benefit greatly from the removal of one layer of technical complexity (see the remarks following Corollary 6.16). $\mathsf{E}_1^{\#}$ is also of particular interest because of its close connection with Σ_1^1 inductive definitions (Theorem 6.14) and the superjump (Theorem 6.11 and Theorem VII.1.8).

6.6 Theorem. *There exists a primitive recursive function g such that for all a,* $\mathbf{m}$, *and* $\boldsymbol{\alpha}$,

$$\{a\}^{\mathsf{E}_1}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \{g(a)\}^{\mathsf{E}_1^{\#}}(\mathbf{m}, \boldsymbol{\alpha}).$$

Proof. The proof is similar to that of Theorem 2.14. We define a primitive recursive function h just as there except that in case (3) we take $h(e, a)$ to be the natural index such that

$$\{h(e, a)\}^{\mathsf{E}_1^{\#}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{E}_1^{\#}(\lambda s\,[0 \cdot \{h(e, b)\}^{\mathsf{E}_1^{\#}}(\mathrm{lg}(s), \mathbf{m}, \boldsymbol{\alpha})]) \\ + \mathsf{E}_1^{\#}(\lambda p.\{h(e, b)\}^{\mathsf{E}_1^{\#}}(p, \mathbf{m}, \boldsymbol{\alpha})).$$

Applying the Primitive Recursion Theorem we obtain an index $\bar{e}$ such that $h(\bar{e}, a) = \{\bar{e}\}(a)$ and set $g = \{\bar{e}\}$.

We need to prove that for all a, $\mathbf{m}$, $\boldsymbol{\alpha}$, and n,

$$\{a\}^{\mathsf{E}_1}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n \leftrightarrow \{g(a)\}^{\mathsf{E}_1^{\#}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n.$$

The implication $(\rightarrow)$ is proved by induction over $\Omega[\mathsf{E}_1]$. Cases (0)–(2) are treated exactly as in Theorem 2.14. Suppose that $a = \langle 3, k, l, b\rangle$ and $\{a\}^{\mathsf{E}_1}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$. Then $\lambda p.\{b\}^{\mathsf{E}_1}(p, \mathbf{m}, \boldsymbol{\alpha})$ is a total function β and $\mathsf{E}_1(\beta) = n$. By the induction hypothesis, for all p, $\{g(b)\}^{\mathsf{E}_1^{\#}}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \beta(p)$. Hence

$$\mathsf{E}_1^{\#}(\lambda p.\{g(b)\}^{\mathsf{E}_1^{\#}}(p, \mathbf{m}, \boldsymbol{\alpha})) \simeq \mathsf{E}_1(\beta) = n.$$

Furthermore, $\mathrm{lg}(\bar{\beta}(p)) = p$, and

$$\forall p\,[0 \cdot \{g(b)\}^{\mathsf{E}_1^{\#}}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq 0],$$

so

$$\exists\gamma\,\forall p\,[0\cdot\{g(b)\}^{\mathsf{E}_1^{\#}}(\lg(\bar{\gamma}(p)),\mathbf{m},\boldsymbol{\alpha})\simeq 0],$$

and thus $\{g(a)\}^{\mathsf{E}_1^{\#}}(\mathbf{m},\boldsymbol{\alpha})\simeq n$.

The converse implication is established by induction on subcomputations for $\mathsf{E}_1^{\#}$. We leave the details to the reader. □

It now follows immediately that every functional partial recursive in E_1 is also partial recursive in $\mathsf{E}_1^{\#}$ (in particular, E is recursive in $\mathsf{E}_1^{\#}$) and every relation semi-recursive in E_1 is also semi-recursive in $\mathsf{E}_1^{\#}$. In particular, $\mathsf{U}^{\mathsf{E}_1}$ is semi-recursive in $\mathsf{E}_1^{\#}$ and from Theorem 6.5 and the proof of Corollary 6.3 we see that also $\sim\mathsf{U}^{\mathsf{E}_1}$ is semi-recursive in $\mathsf{E}_1^{\#}$. We should like to conclude from this that $\mathsf{U}^{\mathsf{E}_1}$ is in fact *recursive* in $\mathsf{E}_1^{\#}$. For this we need the analogue of Corollary 4.5 and thus, in turn, an ordinal comparison theorem.

We could proceed much as in § 3 to compare directly the ordinals of computations relative to $\mathsf{E}_1^{\#}$. The main additional complication is that in contrast with Lemma 3.2(3) we have if $a=\langle 3,k,l,b\rangle$ and $\{a\}^{\mathsf{E}_1^{\#}}(\mathbf{m},\boldsymbol{\alpha})\simeq 0$, then

$$|\,a,\mathbf{m}|_{\boldsymbol{\alpha}}^{\mathsf{E}_1^{\#}}=\inf_{\beta}\sup_p^{+}|\,b,\bar{\beta}(p),\mathbf{m}|_{\boldsymbol{\alpha}}^{\mathsf{E}_1^{\#}}$$

whereas if $\{a\}^{\mathsf{E}_1^{\#}}(\mathbf{m},\boldsymbol{\alpha})\simeq 1$, then

$$|\,a,\mathbf{m}|_{\boldsymbol{\alpha}}^{\mathsf{E}_1^{\#}}=\sup_{\beta}^{+}\inf_p|\,b,\bar{\beta}(p),\mathbf{m}|_{\boldsymbol{\alpha}}^{\mathsf{E}_1^{\#}}.$$

Fortunately we have available here a somewhat simpler method akin to that used in Lemma 1.7. We define inductively a relation M to which all relations semi-recursive in $\mathsf{E}_1^{\#}$ can be reduced. It then suffices to compare the ordinals associated with elements of M.

We define M to be the smallest relation such that for all $d\in\mathrm{Pri}$ and all $\boldsymbol{\alpha}$,

(i) $(0,\langle\boldsymbol{\alpha}\rangle)\in\mathsf{M}$;

(ii) if $\exists\beta\,\forall p.([d](\bar{\beta}(p),\boldsymbol{\alpha}),\langle\boldsymbol{\alpha}\rangle)\in\mathsf{M}$, then $(\langle 0,d\rangle,\langle\boldsymbol{\alpha}\rangle)\in\mathsf{M}$;

(iii) if $\forall\beta\,\exists p.([d](\bar{\beta}(p),\boldsymbol{\alpha}),\langle\boldsymbol{\alpha}\rangle)\in\mathsf{M}$, then $(\langle 1,d\rangle,\langle\boldsymbol{\alpha}\rangle)\in\mathsf{M}$.

Then for each $\boldsymbol{\alpha}$ we set

$$M_{\boldsymbol{\alpha}}=\{u:(u,\langle\boldsymbol{\alpha}\rangle)\in\mathsf{M}\}\quad\text{and}\quad M=M_{\varnothing}.$$

Of course, if $\mathsf{M}=\bar{\Gamma}$, then $M_{\boldsymbol{\alpha}}=\bar{\Gamma}_{\langle\boldsymbol{\alpha}\rangle}$, where Γ_β are the components of the decomposable operator Γ. In view of the (relative) simplicity of the definition of M, the following result is somewhat surprising:

6.7 Theorem. *There exists a primitive recursive function g such that for all a,* $\mathbf{m}$, $\boldsymbol{\alpha}$, *and n,*

$$\{a\}^{\mathsf{E}_1^\#}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n \leftrightarrow g(a, \langle \mathbf{m} \rangle, n) \in M_{\boldsymbol{\alpha}}.$$

Proof. We shall prove the version without parameters ($\boldsymbol{\alpha} = \varnothing$). Note first that there exist primitive recursive functions $h_\forall$ and $h_\exists$ such that for any $d \in \mathrm{Pri}$,

$$h_\forall(d, \langle \mathbf{m} \rangle) \in M \leftrightarrow \forall p . [d](p, \mathbf{m}) \in M,$$

and

$$h_\exists(d, \langle \mathbf{m} \rangle) \in M \leftrightarrow \exists p . [d](p, \mathbf{m}) \in M.$$

Namely, $h_\forall(d, \langle \mathbf{m} \rangle) = \langle 0, f(d, \langle \mathbf{m} \rangle) \rangle$ and $h_\exists(d, \langle \mathbf{m} \rangle) = \langle 1, f(d, \langle \mathbf{m} \rangle) \rangle$, where f is a primitive recursive function such that

$$[f(d, \langle \mathbf{m} \rangle)](s) = [d](\mathrm{lg}(s), \mathbf{m}).$$

Now we shall specify the recursion conditions that g should satisfy and leave it to the reader to provide the details of application of the Recursion Theorem. The conditions are as usual by cases corresponding to those in the definition of $\Omega[\mathsf{E}_1^\#]$.

(0) If $a = \langle 0, k, 0, \ldots \rangle$, then $g(a, \langle \mathbf{m} \rangle, n) = h_\exists(d_0, \langle a, \mathbf{m}, n \rangle)$, where

$$[d_0](u, a, \mathbf{m}, n) = \begin{cases} 0, & \text{if } \;\; \mathsf{T}(a, \langle \mathbf{m} \rangle; u, \langle \;\; \rangle) \wedge (u)_0 = n; \\ 1, & \text{otherwise.} \end{cases}$$

(1) If $a = \langle 1, k, 0, b, c_0, c_1 \rangle$, then $g(a, \langle \mathbf{m} \rangle, n) = h_\exists(d_1, \langle a, \mathbf{m}, n \rangle)$, where

$$[d_1](q_1, a, \mathbf{m}, n) = h_\exists(d_2, \langle q_1, a, \mathbf{m}, n \rangle),$$

$$[d_2](q_0, q_1, a, \mathbf{m}, n) = h_\forall(d_3, \langle q_0, q_1, a, \mathbf{m}, n \rangle), \quad \text{and}$$

$$[d_3](p, q_0, q_1, a, \mathbf{m}, n) = \begin{cases} g(b, \langle q_0, q_1 \rangle, n), & \text{if } \;\; p = 0; \\ g(c_0, \langle \mathbf{m} \rangle, q_0), & \text{if } \;\; p = 1; \\ g(c_1, \langle \mathbf{m} \rangle, q_1), & \text{if } \;\; p \geqslant 2. \end{cases}$$

(2) If $a = \langle 2, k+1, 0 \rangle$, then $g(a, \langle b, \mathbf{m} \rangle, n) = h_\exists(d_4, \langle b, \mathbf{m}, n \rangle)$, where

$$[d_4](p, b, \mathbf{m}, n) = g(b, \langle \mathbf{m} \rangle, n).$$

(3)$^\#$ If $a = \langle 3, k, 0, b \rangle$, then $g(a, \langle \mathbf{m} \rangle, 0) = \langle 0, f_1(a, \langle \mathbf{m} \rangle, n) \rangle$, and $g(a, \langle \mathbf{m} \rangle, 1) = \langle 1, h_\exists(d_5, \langle a, \mathbf{m}, n \rangle) \rangle$, where

$$[f_1(a, \langle \mathbf{m} \rangle, n)](s) = g(b, \langle s, \mathbf{m} \rangle, 0),$$

and

$$[d_5](q,a,\mathbf{m},n)=\begin{cases}1, & \text{if} \quad q=0;\\ g(b,\langle s,\mathbf{m}\rangle,q), & \text{if} \quad q>0.\end{cases}$$

The proof that if $\{a\}^{\mathsf{E}_1^{\#}}(\mathbf{m})\simeq n$, then $g(a,\langle\mathbf{m}\rangle,n)\in M$ is by induction on $\Omega[\mathsf{E}_1^{\#}]$ with cases as follows.

(0) If $\{a\}^{\mathsf{E}_1^{\#}}(\mathbf{m})\simeq n$ and $a=\langle 0,k,0,\ldots\rangle$, then also $\{a\}(\mathbf{m})\simeq n$ and for some u, $\mathsf{T}(a,\langle\mathbf{m}\rangle,u,\langle\ \rangle)$ and $(u)_0=n$. Hence for some u, $[d_0](u,a,\mathbf{m},n)=0\in M$ and thus $g(a,\langle\mathbf{m}\rangle,n)\in M$.

(1) If $\{a\}^{\mathsf{E}_1^{\#}}(\mathbf{m})\simeq\{b\}^{\mathsf{E}_1^{\#}}(q_0,q_1)\simeq n$, where $\{c_i\}^{\mathsf{E}_1^{\#}}(\mathbf{m})\simeq q_i$, then successively

$$\exists q_0\,\exists q_1\,\forall p\,.\,[d_3](p,q_0,q_1,a,\mathbf{m},n)\in M,$$
$$\exists q_0\,\exists q_1\,.\,[d_2](q_0,q_1,a,\mathbf{m},n)\in M,$$
$$\exists q_1\,.\,[d_1](q_1,a,\mathbf{m},n)\in M, \quad \text{and}$$
$$g(a,\langle\mathbf{m}\rangle,n)\in M.$$

The cases (2) and $(3)^{\#}$ are similar. The proof of the converse implication is by induction over M and is left to the reader. □

6.8 Corollary. *There exists a primitive recursive function f such that for all a, $\mathbf{m}$, $\boldsymbol{\alpha}$, and n,*

$$\{a\}^{\mathsf{E}_1^{\#}}(\mathbf{m},\boldsymbol{\alpha}) \text{ is defined} \leftrightarrow f(a,\langle\mathbf{m}\rangle)\in M_{\boldsymbol{\alpha}}.$$

Proof. Let $f(a,\langle\mathbf{m}\rangle)=h_{\exists}(d,\langle a,\langle\mathbf{m}\rangle\rangle)$, where d is an index such that

$$[d](n,a,\langle\mathbf{m}\rangle)=g(a,\langle\mathbf{m}\rangle,n),$$

with g and $h_{\exists}$ as in the preceding theorem. □

For $u\in M_{\boldsymbol{\alpha}}$, $|u|_{\boldsymbol{\alpha}}^{\#}$ denotes the least σ such that $u\in M_{\boldsymbol{\alpha}}^{\sigma}$; otherwise, $|u|_{\boldsymbol{\alpha}}^{\#}=\aleph_1$. Then it is immediate that

$$|0|_{\boldsymbol{\alpha}}^{\#}=0, \quad \text{and if} \quad d\in \mathrm{Pri}^{1,l}, \quad \text{then}$$
$$|\langle 0,d\rangle|_{\boldsymbol{\alpha}}^{\#}=\inf\{\sup{}^{+}\{|[d](\bar{\beta}(p),\boldsymbol{\alpha})|_{\boldsymbol{\alpha}}^{\#}:p\in\omega\}:\beta\in{}^{\omega}\omega\};$$
$$|\langle 1,d\rangle|_{\boldsymbol{\alpha}}^{\#}=\sup{}^{+}\{\inf\{|[d](\bar{\beta}(p),\boldsymbol{\alpha})|_{\boldsymbol{\alpha}}^{\#}:p\in\omega\}:\beta\in{}^{\omega}\omega\}.$$

In what follows we shall abbreviate these expressions by

$$|\langle 0, d\rangle|^{\#}_{\boldsymbol{\alpha}} = \inf_\beta \sup_p^+ |d, \beta, p|^{\#}_{\boldsymbol{\alpha}},$$

etc.

6.9 Ordinal Comparison Theorem. *There exists a functional* H *partial recursive in* $\mathsf{E}_1^{\#}$ *such that for all* u, v, *and* $\boldsymbol{\alpha}$,

(i) *if* $u \in M_{\boldsymbol{\alpha}}$ *and* $|u|^{\#}_{\boldsymbol{\alpha}} \leqslant |v|^{\#}_{\boldsymbol{\alpha}}$, *then* $\mathsf{H}(u, v, \langle\boldsymbol{\alpha}\rangle) \simeq 0$;

(ii) *if* $v \in M_{\boldsymbol{\alpha}}$ *and* $|v|^{\#}_{\boldsymbol{\alpha}} < |u|^{\#}_{\boldsymbol{\alpha}}$, *then* $\mathsf{H}(u, v, \langle\boldsymbol{\alpha}\rangle) \simeq 1$.

Proof. We shall again prove the version without parameters. For this proof only we set

$$\mathsf{A}_1^{\#}(f) \simeq 1 \dot{-} \mathsf{E}_1^{\#}(\lambda m\,.\,1 \dot{-} f(m))$$

so that

$$\mathsf{A}_1^{\#}(f) \simeq \begin{cases} 0, & \text{if } \forall\beta\,\exists p\,.\,f(\bar{\beta}(p)) \simeq 0; \\ 1, & \text{if } \exists\beta\,\forall p\,(\exists n > 0)\,.\,f(\bar{\beta}(p)) \simeq n; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

By the $\mathsf{E}_1^{\#}$-Recursion Theorem there exists a functional H partial recursive in $\mathsf{E}_1^{\#}$ which satisfies the following conditions for all u and v, and all d and $e \in \mathrm{Pri}^{1,0}$:

(1) $\mathsf{H}(0, v) \simeq 0$;
(2) $\mathsf{H}(u+1, 0) \simeq 1$;
(3) $\mathsf{H}(u+1, v+1) \simeq 0$, if $v+1$ is not of the form $\langle 0, c\rangle$ or $\langle 1, c\rangle$ with $c \in \mathrm{Pri}^{1,0}$;
(4) $\mathsf{H}(u+1, \langle 0, e\rangle) \simeq \mathsf{H}(u+1, \langle 1, e\rangle) \simeq 1$, if $u+1$ is not of the form $\langle 0, c\rangle$ or $\langle 1, c\rangle$ with $c \in \mathrm{Pri}^{1,0}$;
(5) $\mathsf{H}(\langle 0, d\rangle, \langle 0, e\rangle) \simeq \mathsf{E}_1^{\#}(\lambda s\,.\,\mathsf{A}_1^{\#}(\lambda t\,.\,\mathsf{H}([d](s), [e](t))))$;
(6) $\mathsf{H}(\langle 1, d\rangle, \langle 0, e\rangle) \simeq \mathsf{A}_1^{\#}(\lambda s\,.\,\mathsf{A}_1^{\#}(\lambda t\,.\,\mathsf{H}([d](s), [e](t))))$;
(7) $\mathsf{H}(\langle 0, d\rangle, \langle 1, e\rangle) \simeq \mathsf{E}_1^{\#}(\lambda t\,.\,\mathsf{E}_1^{\#}(\lambda s\,.\,\mathsf{H}([d](s), [e](t))))$;
(8) $\mathsf{H}(\langle 1, d\rangle, \langle 1, e\rangle) \simeq \mathsf{A}_1^{\#}(\lambda s\,.\,\mathsf{E}_1^{\#}(\lambda t\,.\,\mathsf{H}([d](s), [e](t))))$.

The proof that H satisfies (i) and (ii) is by induction on $\sigma = \min\{|u|^{\#}, |v|^{\#}\}$. Clauses (1) and (2) of the definition of H correspond to the case $\sigma = 0$; (3) and (4) correspond to cases in which either u or v does not belong to M by virtue of being of the wrong form. Both of these are easily seen to be in accord with (i) and (ii), so we consider the four cases $u = \langle i, d\rangle$ and $v = \langle j, e\rangle$, with $d, e \in \mathrm{Pri}^{1,0}$ and at least one of u and v a member of M. For the case $i = j = 0$, we have

$$\begin{aligned} |\langle 0, d\rangle|^{\#} \leqslant |\langle 0, e\rangle|^{\#} &\leftrightarrow \inf_\beta \sup_p^+ |d, \beta, p|^{\#} \leqslant \inf_\gamma \sup_q^+ |e, \gamma, q|^{\#} \\ &\leftrightarrow \exists\beta\,\forall\gamma\,.\,\sup_p^+ |d, \beta, p|^{\#} \leqslant \sup_q^+ |e, \gamma, q|^{\#} \\ &\leftrightarrow \exists\beta\,\forall\gamma\,\forall p\,\exists q\,.\,|d, \beta, p|^{\#} \leqslant |e, \gamma, q|^{\#} \\ &\leftrightarrow \exists\beta\,\forall p\,\forall\gamma\,\exists q\,.\,|d, \beta, p|^{\#} \leqslant |e, \gamma, q|^{\#}. \end{aligned}$$

If these hold, then $\exists\beta\,\forall p\,.\,|d,\beta,p|^{\#} < \sigma$, so by the induction hypothesis,

$$\exists\beta\,\forall p\,\forall\gamma\,\exists q\,.\,\mathsf{H}([d](\bar\beta(p)),[e](\bar\gamma(q))) \simeq 0$$

and thus by clause (5), $\mathsf{H}(\langle 0,d\rangle,\langle 0,e\rangle) \simeq 0$ as required by (i). On the other hand, if these are false, then

$$\forall\beta\,\exists p\,\exists\gamma\,\forall q\,.\,\mathsf{H}([d](\bar\beta(p)),[e](\bar\gamma(p))) \simeq 1$$

so $\mathsf{H}(\langle 0,d\rangle,\langle 0,e\rangle) \simeq 1$.

Consider next the case $i=0$ and $j=1$. We have

$$\begin{aligned}|\langle 0,d\rangle|^{\#} \leq |\langle 1,e\rangle|^{\#} &\leftrightarrow \inf_\beta \sup_p^+ |d,\beta,p|^{\#} \leq \sup_\gamma^+ \inf_q |e,\gamma,q|^{\#}\\ &\leftrightarrow \exists\beta\,\exists\gamma\,.\,\sup_p^+ |d,\beta,q|^{\#} \leq \inf_q |e,\gamma,q|^{\#}+1\\ &\leftrightarrow \exists\beta\,\exists\gamma\,\forall p\,\forall q\,.\,|d,\beta,p|^{\#} \leq |e,\gamma,q|^{\#}\\ &\leftrightarrow \exists\gamma\,\forall q\,\exists\beta\,\forall p\,.\,|d,\beta,p|^{\#} \leq |e,\gamma,q|^{\#}.\end{aligned}$$

The last implication $(\rightarrow)$ is by first-order logic. For $(\leftarrow)$, for any γ_0 choose q_0 to minimize $|e,\gamma_0,q_0|^{\#}$ and β_0 such that for all p, $|d,\beta_0,p|^{\#} \leq |e,\gamma_0,q_0|^{\#}$. Then for all p and q, $|d,\beta_0,p|^{\#} \leq |e,\gamma_0,q|^{\#}$. (Note that this sort of argument would not suffice to obtain the prefix $\exists\beta\,\forall p\,\exists\gamma\,\forall q$, which accounts for the interchange of s and t in clause (7) of the definition of H.) Now just as above we have, under the assumption that one of $\langle 0,d\rangle$ and $\langle 1,e\rangle$ belongs to M,

$$\begin{aligned}|\langle 0,d\rangle|^{\#} \leq |\langle 1,e\rangle|^{\#} &\rightarrow \exists\gamma\,\forall q\,\exists\beta\,\forall p\,.\,\mathsf{H}([d](\bar\beta(p)),[e](\bar\gamma(q))) \simeq 0\\ &\rightarrow \mathsf{H}(\langle 0,d\rangle,\langle 1,e\rangle) \simeq 0,\end{aligned}$$

and

$$\begin{aligned}|\langle 1,e\rangle|^{\#} < |\langle 0,d\rangle|^{\#} &\rightarrow \forall\gamma\,\exists q\,\forall\beta\,\exists p\,.\,\mathsf{H}([d](\bar\beta(p)),[e](\bar\gamma(q))) \simeq 1\\ &\rightarrow \mathsf{H}(\langle 0,d\rangle,\langle 1,e\rangle) \simeq 1.\end{aligned}$$

The other two cases are based similarly on the equivalences:

$$\begin{aligned}|\langle 1,d\rangle|^{\#} \leq |\langle 0,e\rangle|^{\#} &\leftrightarrow \sup_\beta^+ \inf_p |d,\beta,p|^{\#} \leq \inf_\gamma \sup_q^+ |e,\gamma,q|^{\#}\\ &\leftrightarrow \forall\beta\,\forall\gamma\,.\,\inf_p |d,\beta,p|^{\#} < \sup_q^+ |e,\gamma,q|^{\#}\\ &\leftrightarrow \forall\beta\,\forall\gamma\,\exists p\,\exists q\,.\,|d,\beta,p|^{\#} \leq |e,\gamma,q|^{\#}\\ &\leftrightarrow \forall\beta\,\exists p\,\forall\gamma\,\exists q\,.\,|d,\beta,p|^{\#} \leq |e,\gamma,q|^{\#};\end{aligned}$$

and

$$|\langle 1,d\rangle|^{\#} \leqslant |\langle 1,e\rangle|^{\#} \leftrightarrow \sup^{+}_{\beta} \inf_p |d,\beta,p|^{\#} \leqslant \sup^{+}_{\gamma} \inf_q |e,\gamma,q|^{\#}$$
$$\leftrightarrow \forall\beta\,\exists\gamma\,.\inf_p |d,\beta,p|^{\#} \leqslant \inf_q |e,\gamma,q|^{\#}$$
$$\leftrightarrow \forall\beta\,\exists\gamma\,\exists p\,\forall q\,.|d,\beta,p|^{\#} \leqslant |e,\gamma,q|^{\#}$$
$$\leftrightarrow \forall\beta\,\exists p\,\exists\gamma\,\forall q\,.|d,\beta,p|^{\#} \leqslant |e,\gamma,q|^{\#}. \quad \square$$

From Theorems 6.7 and 6.9 we can derive results corresponding to theose of § 4 for the class of relations semi-recursive in $\mathsf{E}_1^{\#}$:

6.10 Corollary.
(i) *There exists a functional* $\mathrm{Sel}^{\mathsf{E}_1^{\#}}$ *partial recursive in* $\mathsf{E}_1^{\#}$ *such that for all* a, $\mathbf{m}$, *and* $\boldsymbol{\alpha}$,

$$\exists p.\{a\}^{\mathsf{E}_1^{\#}}(p,\mathbf{m},\boldsymbol{\alpha}) \text{ is defined} \leftrightarrow \{a\}^{\mathsf{E}_1^{\#}}(\mathrm{Sel}^{\mathsf{E}_1^{\#}}(a,\langle\mathbf{m}\rangle,\langle\boldsymbol{\alpha}\rangle),\mathbf{m},\boldsymbol{\alpha}) \text{ is defined};$$

(ii) *for any relation* R *semi-recursive in* $\mathsf{E}_1^{\#}$, *there exists a functional* $\mathrm{Sel}_{\mathsf{R}}$ *partial recursive in* $\mathsf{E}_1^{\#}$ *such that for all* $\mathbf{m}$ *and* $\boldsymbol{\alpha}$,

$$\exists p\, \mathsf{R}(p,\mathbf{m},\boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathrm{Sel}_{\mathsf{R}}(\mathbf{m},\boldsymbol{\alpha}),\mathbf{m},\boldsymbol{\alpha});$$

(iii) *the class of relations semi-recursive in* $\mathsf{E}_1^{\#}$ *is closed under finite union and bounded and unbounded existential number quantification*;
(iv) *the class of functionals partial recursive in* $\mathsf{E}_1^{\#}$ *and relations semi-recursive in* $\mathsf{E}_1^{\#}$ *is closed under definition by positive cases*;
(v) *a relation is recursive in* $\mathsf{E}_1^{\#}$ *iff it is both semi-recursive in* $\mathsf{E}_1^{\#}$ *and co-semi-recursive in* $\mathsf{E}_1^{\#}$;
(vi) *for any partial functional* F, F *is partial recursive in* $\mathsf{E}_1^{\#}$ *iff* Gr_{F} *is semi-recursive in* $\mathsf{E}_1^{\#}$ *and* F *is recursive in* $\mathsf{E}_1^{\#}$ *iff* F *is total and* Gr_{F} *is recursive in* $\mathsf{E}_1^{\#}$.

Proof. For (i) we proceed as for Theorem 4.1 except that in place of $\mathsf{H}(\langle a,\mathbf{m}\rangle,\langle b,\mathbf{n}\rangle,\langle\boldsymbol{\alpha}\rangle)$, we write $\mathsf{H}(f(a,\langle\mathbf{m}\rangle),f(b,\langle\mathbf{n}\rangle),\langle\boldsymbol{\alpha}\rangle)$, with f the function of Corollary 6.8. (ii)–(vi) then follow as in § 4. $\square$

In particular, it follows from (v) and the discussion following Theorem 6.6 that $\mathsf{U}^{\mathsf{E}_1}$ is recursive in $\mathsf{E}_1^{\#}$. In fact,

6.11 Theorem. *For any functional* I, *if* I *is recursive in* $\mathsf{E}_1^{\#}$, *then also* I^{sJ} *is recursive in* $\mathsf{E}_1^{\#}$.

Proof. Suppose that I is recursive in $\mathsf{E}_1^{\#}$. An obvious modification of the proof of

Theorem 6.6 establishes that there exists a primitive recursive function g such that for all a, $\mathbf{m}$, and $\boldsymbol{\alpha}$,

$$\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \{g(a)\}^{\mathsf{E}_1^{\#}}(\mathbf{m}, \boldsymbol{\alpha}).$$

Hence U^{I} and $\sim\mathsf{U}^{\mathsf{I}}$ are both semi-recursive in $\mathsf{E}_1^{\#}$, so by 6.10(v), U^{I} is recursive in $\mathsf{E}_1^{\#}$. Since

$$\mathsf{I}^{\mathsf{sJ}}((\langle a, \mathbf{m}\rangle) * \alpha) = \begin{cases} 0, & \text{if} \quad \mathsf{U}^{\mathsf{I}}(a, \langle \mathbf{m}\rangle, \langle \alpha \rangle); \\ 1, & \text{otherwise,} \end{cases}$$

also I^{sJ} is recursive in $\mathsf{E}_1^{\#}$. $\square$

Thus if we define a sequence of functionals E_r by:

$$\mathsf{E}_0 = \mathsf{E} \quad \text{and} \quad \mathsf{E}_{r+1} = (\mathsf{E}_r)^{\mathsf{sJ}},$$

then E_r is recursive in E_s iff $r \leq s$ and all E_r are recursive in $\mathsf{E}_1^{\#}$ (cf. Exercise 1.20). The functionals E_r will play a role in § VIII.4, and in § VII.1. Theorem 6.11 will be extended to show that the type-3 functional corresponding to the superjump sJ is recursive in $\mathsf{E}_1^{\#}$ (Theorem VII.1.8).

We turn now to the relationship of $\mathsf{E}_1^{\#}$ to inductive definability and the $*$-operation.

6.12 Theorem. *For any Σ_1^1 monotone operator Γ over ω, $\bar{\Gamma}$ is semi-recursive in $\mathsf{E}_1^{\#}$.*

Proof. Suppose that Γ is Σ_1^1 and monotone, say

$$\mathsf{P}_\Gamma(m, \alpha) \leftrightarrow \exists \gamma\, \forall p\, R(\bar{\alpha}(p), \bar{\gamma}(p), m)$$

with R recursive. Since Γ is monotone,

$$m \in \Gamma(A) \leftrightarrow \exists B\, [B \subseteq A \wedge m \in \Gamma(B)].$$

Thus,

$$\begin{aligned} \mathsf{P}_\Gamma(m, \alpha) &\leftrightarrow \exists \beta\, \exists \gamma\, [\forall n\, (\beta(n) = 0 \rightarrow \alpha(n) = 0) \wedge \forall p\, R(\bar{\beta}(p), \bar{\gamma}(p), m)] \\ &\leftrightarrow \exists \beta\, \forall n\, [((\beta)_0(n) = 0 \rightarrow \alpha(n) = 0) \wedge R(\overline{(\beta)_0}(n), \overline{(\beta)_1}(n), m)]. \end{aligned}$$

Let G be the function partial recursive in $\mathsf{E}_1^{\#}$ which computes according to the following flow diagram:

$$\begin{tikzcd}
G(e,\bar{\beta}(n),m) \arrow[r] & \boxed{(\beta)_0(n)} \arrow[rr, ">0"] \arrow[dr, "0"'] & & \boxed{\mathsf{K}_R\,(\overline{(\beta)_0(n)},\overline{(\beta)_1(n)},m)} \arrow[r] & {} \\
& & \boxed{\{e\}^{\mathsf{E}_1^\#}(n)} \arrow[ur, "0"'] \arrow[r, ">0"] & {} &
\end{tikzcd}$$

Set $H(e,m)\simeq \mathsf{E}_1^\#(\lambda s\,.\,G(e,s,m))$ and let $\bar{e}$ be the natural index (provided by the proof of the $\mathsf{E}_1^\#$-Recursion Theorem) such that $H(\bar{e},m)\simeq\{\bar{e}\}^{\mathsf{E}_1^\#}(m)$. We claim that

$$m\in\bar{\Gamma}\leftrightarrow\{\bar{e}\}^{\mathsf{E}_1^\#}(m)\simeq 0.$$

The implication ($\rightarrow$) is proved by a straightforward induction over $\bar{\Gamma}$. The implication ($\leftarrow$) is proved by an induction on subcomputations using the fact that if $\{\bar{e}\}^{\mathsf{E}_1^\#}(m)\simeq 0$, then for some β and all n, the computation of $G(\bar{e},\bar{\beta}(n),m)$ is a subcomputation and thus so is that of $\{\bar{e}\}^{\mathsf{E}_1^\#}(n)\simeq 0$ for all n such that $(\beta)_0(n)=0$. Hence all such n belong to $\bar{\Gamma}$ by the induction hypothesis. □

The inductive definition of M (and of M) has both Σ_1^1 and Π_1^1 clauses — that is, there are monotone operators $\Gamma_0\in\Sigma_1^1$ and $\Gamma_1\in\Pi_1^1$ such that if for all A, $\Gamma(A)=\Gamma_0(A)\cup\Gamma_1(A)$, then $\bar{\Gamma}=M$. Because of the special form of Γ_1 and the fact that Π_1^1 relations are all Π_1^0 inductively definable, we can in fact replace Γ by a Σ_1^1 operator:

6.13 Lemma. *Let Γ_0 and Γ_1 be monotone operators defined by*

$$m\in\Gamma_0(A)\leftrightarrow\exists\gamma\,\forall p\;\mathsf{R}_0(\bar{\gamma}(p),m,A),$$

and

$$m\in\Gamma_1(A)\leftrightarrow\forall\delta\,\exists q\;\mathsf{R}_1(\bar{\delta}(q),m,A),$$

where R_0 and R_1 are recursive relations and R_1 satisfies:

$$(*)\qquad \mathsf{R}_1(s,m,A)\wedge A\subseteq B\rightarrow\mathsf{R}_1(s,m,B),$$

and set $\Gamma(A)=\Gamma_0(A)\cup\Gamma_1(A)$. Then there exists a Σ_1^1 monotone operator Λ such that for all m,

$$m\in\bar{\Gamma}\leftrightarrow\langle 0,m\rangle\in\bar{\Lambda}.$$

Proof. For any set A, let $A_{(0)} = \{m : \langle 0, m\rangle \in A\}$ and let Λ be the operator defined by:

(1) $\langle 0, m\rangle \in \Lambda(A) \leftrightarrow m \in \Gamma_0(A_{(0)}) \vee \langle 1, m, \langle\ \rangle\rangle \in A$;

(2) $\langle 1, m, s\rangle \in \Lambda(A) \leftrightarrow \mathsf{R}_1(s, m, A_{(0)}) \vee \forall n . \langle 1, m, s * \langle n\rangle\rangle \in A$.

Clearly Λ is Σ_1^1 and monotone. We first establish that

(3) $\forall \delta\, \exists q\, \mathsf{R}_1(s * \bar{\delta}(q), m, \bar{\Lambda}_{(0)}) \to \langle 1, m, s\rangle \in \bar{\Lambda}$;

(4) $\langle 1, m, s\rangle \in \Lambda^{\sigma} \to \forall \delta\, \exists q\, \mathsf{R}_1(s * \bar{\delta}(q), m, (\Lambda^{\sigma})_{(0)})$.

For (3), suppose that $\langle 1, m, s\rangle \notin \bar{\Lambda}$. Then as in the latter part of the proof of Theorem III.3.2 there exists a unique function δ such that

$$\delta(q) = \text{least } n . \langle 1, m, s * \bar{\delta}(q) * \langle n\rangle\rangle \notin \bar{\Lambda}.$$

In particular, $\forall q \sim \mathsf{R}_1(s * \bar{\delta}(q), m, \bar{\Lambda}_{(0)})$.

For (4), we proceed by induction on σ and assume as induction hypothesis that (4) holds for all $\tau < \sigma$ in place of σ. Suppose that $\langle 1, m, s\rangle \in \Lambda^{\sigma}$. Then either $\mathsf{R}_1(s, m, (\Lambda^{(\sigma)})_{(0)})$ or $\forall n . \langle 1, m, s * \langle n\rangle\rangle \in \Lambda^{(\sigma)}$. In the first case, by the property $(*)$ of R_1, $\mathsf{R}_1(s, m, (\Lambda^{\sigma})_{(0)})$ and thus $\forall \delta\, \mathsf{R}_1(s * \bar{\delta}(0), m, (\Lambda^{\sigma})_{(0)})$. In the second case, for each n there exists a $\tau_n < \sigma$ such that $\langle 1, m, s * \langle n\rangle\rangle \in \Lambda^{\tau_n}$ and hence, by the induction hypothesis, $\forall \delta\, \exists q\, \mathsf{R}_1(s * \langle n\rangle * \bar{\delta}(q), m, (\Lambda^{\tau_n})_{(0)})$. Then, again by $(*)$, $\forall n\, \forall \delta\, \exists q\, \mathsf{R}_1(s * \langle n\rangle * \bar{\delta}(q), m, (\Lambda^{\sigma})_{(0)})$ and thus $\forall \delta\, \exists q\, \mathsf{R}_1(s * \bar{\delta}(q), m, (\Lambda^{\sigma})_{(0)})$.

To prove that $\bar{\Gamma} \subseteq \bar{\Lambda}_{(0)}$ we show that $\Gamma(\bar{\Lambda}_{(0)}) \subseteq \bar{\Lambda}_{(0)}$. First, if $m \in \Gamma_0(\bar{\Lambda}_{(0)})$, then by (1), $\langle 0, m\rangle \in \Lambda(\bar{\Lambda}) = \bar{\Lambda}$, so $m \in \bar{\Lambda}_{(0)}$. Suppose that $m \in \Gamma_1(\bar{\Lambda}_{(0)})$, so $\forall \delta\, \exists q\, \mathsf{R}_1(\bar{\delta}(q), m, \bar{\Lambda}_{(0)})$. Then by (3), $\langle 1, m, \langle\ \rangle\rangle \in \bar{\Lambda}$ and thus by (1), $\langle 0, m\rangle \in \bar{\Lambda}$ so $m \in \bar{\Lambda}_{(0)}$.

For the converse, we prove by induction on σ that $(\Lambda^{\sigma})_{(0)} \subseteq \bar{\Gamma}$. As induction hypothesis we assume that $(\Lambda^{(\sigma)})_{(0)} \subseteq \bar{\Gamma}$. Then if $m \in (\Lambda^{\sigma})_{(0)}$, either $m \in \Gamma_0((\Lambda^{(\sigma)})_{(0)})$ or $\langle 1, m, \langle\ \rangle\rangle \in \Lambda^{(\sigma)}$. In the first case we have by the induction hypothesis and the monotonicity of Γ_0 that $m \in \Gamma_0(\bar{\Gamma}) \subseteq \bar{\Gamma}$. In the second case, there exists an ordinal $\tau < \sigma$ such that $\langle 1, m, \langle\ \rangle\rangle \in \Lambda^{\tau}$. By (4), $\forall \delta\, \exists q\, \mathsf{R}_1(\bar{\delta}(q), m, (\Lambda^{\tau})_{(0)})$, that is $m \in \Gamma_1((\Lambda^{\tau})_{(0)})$. By the induction hypothesis and the monotonicity of Γ_1, $m \in \Gamma_1(\bar{\Gamma}) \subseteq \bar{\Gamma}$. $\square$

6.14 Theorem. *For all $R \subseteq {}^k\omega$, the following are equivalent*:

(i) *R is semi-recursive in $\mathsf{E}_1^{\#}$*;

(ii) *R is reducible to the closure of a monotone Σ_1^1 operator*;

(iii) $R \in \Pi_1^{\mathscr{A}^*}$.

Proof. (i)$\to$(ii): The Π^1_1 clause in the inductive defintion of M clearly satisfies the condition $(*)$ of the preceding lemma. Hence M is reducible to the closure of a monotone Σ^1_1 operator, and by Corollary 6.8, so is every R semi-recursive in $\mathsf{E}_1^{\#}$.

(ii)$\to$(i): Immediate from Theorem 6.12.

(iii)$\to$(ii): From Definition V.5.2 we see easily that a relation R is $\Pi_1^{\mathscr{A}^*}$ just in case there is a recursive relation P such that

$$(**)\qquad R(\mathbf{m}) \leftrightarrow (\forall A \in \mathsf{B}(\mathscr{A}^*))(\exists t \in A)\exists q\, P(t, q, \mathbf{m}).$$

Let Γ be the monotone operator defined by:

$$\langle \mathbf{m}, s\rangle \in \Gamma(D) \leftrightarrow \exists q\, P(s, q, \mathbf{m}) \vee \forall\beta\, \exists q\, \exists\gamma\, \forall q\,.\langle \mathbf{m}, s * \langle\bar\beta(p), \bar\gamma(q)\rangle\rangle \in D.$$

The proof of Theorem V.4.10 (with obvious simple modifications) shows that

$$R(\mathbf{m}) \leftrightarrow \langle \mathbf{m}, \langle\ \ \rangle\rangle \in \bar\Gamma.$$

As it stands, Γ is nowhere near being a Σ^1_1 operator. However, it is not hard to see that this equivalence holds also for the operator Γ defined by:

$$\begin{aligned}\langle \mathbf{m}, s\rangle \in \Gamma(D) \leftrightarrow \exists q\, P(s, q, \mathbf{m}) \vee &\\ &\vee [\lg(s) \text{ is even} \wedge \forall\beta\, \exists p\,.\langle \mathbf{m}, s * \langle\bar\beta(p)\rangle\rangle \in D]\\ &\vee [\lg(s) \text{ is odd} \wedge \exists\gamma\, \forall q\,.\langle \mathbf{m}, s * \langle\bar\gamma(q)\rangle\rangle \in D].\end{aligned}$$

This Γ is in the form specified in the hypothesis of Lemma 6.13 and thus may be in turn replaced by a Σ^1_1 monotone operator.

(ii)$\to$(iii): Let Γ be a monotone Σ^1_1 operator, say

$$m \in \Gamma(A) \leftrightarrow \exists\gamma\, \forall q\, \mathsf{R}(\bar\gamma(q), m, A)$$

for a recursive relation R. Because Γ is monotone, we have

$$\begin{aligned} m \in \Gamma(A) &\leftrightarrow \exists B\, [B \subseteq A \wedge m \in \Gamma(B)]\\ &\leftrightarrow \exists B\, \exists\gamma\, \forall q\, \forall r\, [(r \notin B \vee r \in A) \wedge \mathsf{R}(\bar\gamma(q), m, B)].\end{aligned}$$

For any $A \neq \omega$ we have then

$$m \in \Gamma(A) \leftrightarrow \exists B\, \exists\gamma\, \forall q\, \forall r\, [(r \notin B \wedge \mathsf{R}(\bar\gamma(q), m, B)) \vee r \in A].$$

Hence for a suitable recursive relation S,

$$(\dagger)\qquad m \in \Gamma(A) \leftrightarrow \exists\beta\, \forall p\, [S(\bar\beta(p), m) \vee (p)_0 \in A].$$

Let P and Q be recursive relations such that

$$Q(\langle s_0, \ldots, s_n\rangle, m) \leftrightarrow S(s_0, m) \vee (\exists i < n)\, S(s_{i+1}, (\lg(s_i))_0),$$

$P(\langle s_0\rangle, m)$ is always false, and for all n, $s_0, \ldots, s_{2n+2}$,

$$P(\langle s_0, \ldots, s_{2n+2}\rangle, m) \leftrightarrow P(\langle s_0, \ldots, s_{2n+1}\rangle, m) \leftrightarrow Q(\langle s_1, s_3, s_5, \ldots, s_{2n+1}\rangle, m).$$

We shall show that

$$m \in \bar{\Gamma} \leftrightarrow (\forall A \in \mathrm{B}(\mathscr{A}^*))(\exists t \in A)\, P(t, m)$$

which by (**) above implies that $\bar{\Gamma} \in \Pi_1^{\mathscr{A}^*}$.

First observe that by formula (8) of § V.4, the right side of this equivalence is equivalent to

$$(\forall \beta_0 \exists p_0)(\exists \beta_1 \forall p_1)(\forall \beta_2 \exists p_2) \cdots \exists n\, P(\langle \bar{\beta}_0(p_0), \ldots, \bar{\beta}_n(p_n)\rangle, m)$$

and hence to

$$(\exists \beta_0 \forall p_0)(\exists \beta_1 \forall p_1)(\exists \beta_2 \forall p_2) \cdots \exists n\, Q(\langle \bar{\beta}_0(p_0), \ldots, \bar{\beta}_n(p_n)\rangle, m),$$

which we abbreviate by: $m \in Q^*$. We recall from § V.4 that $m \in Q^*$ just in case player I has a winning strategy in the game $\mathscr{G}_m$ played as follows: the players play alternately, at his n-th turn player I chooses β_n and player II chooses p_n, and player I wins just in case $\exists n\, Q(\langle \bar{\beta}_0(p_0), \ldots, \bar{\beta}_n(p_n)\rangle, m)$. Thus we need to prove:

$$m \in \bar{\Gamma} \leftrightarrow \text{player I has a winning strategy for } \mathscr{G}_m.$$

Suppose first that $m \notin \bar{\Gamma}$; we shall describe a winning strategy for player II in $\mathscr{G}_m$. Since $m \notin \Gamma(\bar{\Gamma})$, we have by (†),

$$\forall \beta_0 \exists p_0 [\sim S(\bar{\beta}_0(p_0), m) \wedge (p_0)_0 \notin \bar{\Gamma}]$$

so let player II at his first turn respond to I's choice of β_0 with such a p_0. Then, since $(p_0)_0 \notin \Gamma(\bar{\Gamma})$, again by (†),

$$\forall \beta_1 \exists p_1 [\sim S(\bar{\beta}_1(p_1), (p_0)_0) \wedge (p_1)_0 \notin \bar{\Gamma}],$$

and player II at his second turn responds to player I's choice of β_1 with such a p_1. It is clear that by following this strategy, player II ensures that for all n,

$$\sim S(\bar{\beta}_0(p_0), m) \wedge (\forall i < n) \sim S(\bar{\beta}_{i+1}(p_{i+1}), (p_i)_0)$$

and thus that for all n, $\sim Q(\langle\bar{\beta}_0(p_0),\ldots,\bar{\beta}_n(p_n)\rangle, m)$. Hence this is a winning strategy for player II.

For the converse implication we prove that $\Gamma(Q^*) \subseteq Q^*$. Suppose that $m \in \Gamma(Q^*)$ so by (†),

$$\exists\gamma\,\forall q\,[S(\bar{\gamma}(q), m) \vee (q)_0 \in Q^*].$$

Then by the definition of Q^*,

$$\exists\gamma\,\forall q\,[S(\bar{\gamma}(q), m) \vee (\exists\beta_0\,\forall p_0)\cdots\exists n\, Q(\langle\bar{\beta}_0(p_0),\ldots,\bar{\beta}_n(p_n)\rangle, (q)_0)]$$

or equivalently,

$$\exists\gamma\,\forall q\,[S(\bar{\gamma}(q), m) \vee (\exists\beta_0\,\forall p_0)\cdots\exists n \\ (S(\bar{\beta}_0(p_0), (q)_0) \vee (\exists i < n)\, S(\bar{\beta}_{i+1}(p_{i+1}), (p_i)_0))].$$

By a rule of first-order logic, which is easily seen to apply also to the current situation involving infinite strings of quantifiers,

$$\exists\gamma\,\forall q\,\exists\beta_0\,\forall p_0\cdots\exists n\,[S(\bar{\gamma}(q), m) \\ \vee S(\bar{\beta}_0(p_0), (q_0)) \vee (\exists i < n)\, S(\bar{\beta}_{i+1}(p_{i+1}), (p_i)_0)].$$

From this follows immediately that $m \in Q^*$, as required. □

6.15 Corollary. *For all $R \subseteq {}^k\omega$, the following are equivalent:*

(i) *R is recursive in $\mathsf{E}_1^{\#}$;*

(ii) *R and $\sim R$ are each reducible to the closure of a Σ_1^1 monotone operator;*

(iii) *$R \in \Delta_1^{\mathscr{A}^*}$.*

Proof. Immediate from Theorem 6.14 and Corollary 6.10(v). □

6.16 Corollary. *$\nabla(\mathscr{A})$ is a proper subclass of $\Delta_1^{\mathscr{A}^*}$.*

Proof. U^{E_1} is recursive in $\mathsf{E}_1^{\#}$, hence belongs to $\Delta_1^{\mathscr{A}^*}$, but U^{E_1} is not recursive in E_1 and thus by the results stated at the beginning of § 5, U^{E_1} does not belong to $\nabla(\mathscr{A})$. □

We note that the proof of Theorem 6.14 shows that a relation R is reducible to the closure of a Σ_1^1 monotone operator iff it is in the form

$$\exists\beta_0\,\forall p_0\,\exists\beta_1\,\forall p_1\cdots\exists n\, P(\langle\bar{\beta}_0(p_0),\ldots,\bar{\beta}_n(p_n)\rangle, \mathbf{m}),$$

with P recursive. The reader should contrast this with Exercise III.3.23.

We conclude this section with a brief discussion of what is needed to extend 6.6–6.16 to extended functionals $\Phi^{\#}$ other than $\mathsf{E}_1^{\#}$. For 6.6 it will suffice that Φ be a normal operation — the crucial point is to have the power of $\mathsf{E}^{\#}$ available during a computation to check that certain functions are total, and if Φ is normal, then $\mathsf{E}^{\#}$ is recursive in $\Phi^{\#}$. The same is true for 6.7–6.10. Theorem 6.11 depends on the special property of the quantifier $\mathscr{A}$ expressed in Corollary 6.3. It holds for any extended functional $\Phi^{\#}$ which arises from an operation Φ which is strongly normal: Φ is *strongly normal* iff Φ is normal and there exists a primitive recursive function h such that for any family $\langle \mathsf{P}_p : p \in \omega \rangle$ of relations,

$$\mathscr{A}\langle \mathsf{P}_p : p \in \omega \rangle = \Phi\langle \mathsf{P}_{h(p)} : p \in \omega \rangle.$$

When Φ is strongly normal, $\mathsf{E}_1^{\#}$ is computable in terms of $\Phi^{\#}$ and thus if I is recursive in $\Phi^{\#}$, so is I^{sJ}. The remaining results are summarized in

6.17 Theorem. *For any normal positive analytic operation Φ and all $R \subseteq {}^k\omega$, the following are equivalent*:

(i) *R is semi-recursive in $\Phi^{\#}$*;

(ii) *R is reducible to the closure of an effective Φ-positive inductive operator*;

(iii) $R \in \Pi_1^{\Phi^*}$.

Sketch of proof. That (i)$\rightarrow$(ii) is immediate from the observation that the inductive operator which defines $M[\Phi^{\#}]$ is Φ-positive. The proof that (ii)$\rightarrow$(i) is nearly the same as that of Theorem 6.12, where the main trick was to observe that every Σ_1^1 monotone operator is in fact a Σ_1^1 *positive* operator. The implication (iii)$\rightarrow$(ii) is just the effective version of Theorem V.4.10. For the implication (ii)$\rightarrow$(iii), the main difficulty lies in showing that every Φ-positive operator has a normal form similar to (†) in the proof of Theorem 6.14. This can be done and the proof completed much as before. □

6.18 Corollary. *For any normal positive analytic operation Φ and all $R \subseteq {}^k\omega$, the following are equivalent*:

(i) *R is recursive in $\Phi^{\#}$*;

(ii) *R and $\sim R$ are each reducible to the closure of an effective Φ-positive inductive operator*;

(iii) $R \in \Delta_1^{\Phi^*}$. □

6.19 Corollary. *For any strongly normal positive analytic operation Φ, $\nabla(\Phi)$ is a proper subclass of $\Delta_1^{\Phi^*}$.* □

6.20–6.26 Exercises

6.20. Show that the set of relations semi-recursive in $\mathsf{E}_1^{\#}$ is a proper subset of Δ_2^1.

6.21. Show that for any relation R on numbers semi-recursive in $\mathsf{E}_1^{\#}$, $\exists\beta\,\forall p\,R(\bar\beta(p), m) \leftrightarrow (\exists\beta$ recursive in $\mathsf{E}_1^{\#})\forall p\,R(\bar\beta(p), m)$. (cf. Exercise 1.18)

6.22. Show that M is semi-recursive in $\mathsf{E}_1^{\#}$ without using 6.12–6.14. (Show that if A is semi-recursive in $\mathsf{E}_1^{\#}$ but not recursive in $\mathsf{E}_1^{\#}$, and $A \subseteq M$, then $\{|u|^{\#} : u \in A\}$ is unbounded in $\{|u|^{\#} : u \in M\}$. Then

$$v \in M \leftrightarrow \exists u \in A\,.\,|v|^{\#} \leq |u|^{\#}.)$$

6.23. Give another proof that for any Σ_1^1 monotone operator Γ, $\bar\Gamma$ is semi-recursive in $\mathsf{E}_1^{\#}$ (6.12) by the method of Exercises 2.25–26.

6.24. Show that in Lemma 6.13 the hypothesis $(*)$ is superfluous; every monotone operator $\Gamma_1 \in \Pi_1^1$ may be defined in the given form by a recursive relation R_1 which satisfies $(*)$.

6.25. Use the method of Exercises VI.2.26 to show that in contrast with Corollary 6.3, if E is recursive in I, R is semi-recursive in I, and

$$\mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall\beta\,\exists p\,\mathsf{R}(\bar\beta(p), \mathbf{m}, \boldsymbol{\alpha}),$$

then also S is semi-recursive in I.

6.26. Fill in the following sketch of an alternative proof for Theorem 4.18 for any functional I such that E_I is recursive in I. Let P be a relation such that $\mathsf{P}(u, v, \langle\boldsymbol{\alpha}\rangle, \gamma)$ holds under exactly the same conditions as does $\mathsf{P}^{\mathsf{I}}(u, v, \langle\boldsymbol{\alpha}\rangle)$ in the proof of Lemma 6.1 except that in clause (1) we replace the condition

$$\{c_i\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq q_i \quad \text{by} \quad \gamma(\langle c_i, \mathbf{m}\rangle) = q_i.$$

Clearly P is recursive. Let $<_{\boldsymbol{\alpha},\gamma}$ denote the transitive closure of the relation $<'_{\boldsymbol{\alpha},\gamma}$ defined by:

$$v <'_{\boldsymbol{\alpha},\gamma} u \leftrightarrow \mathsf{P}(u, v, \langle\boldsymbol{\alpha}\rangle, \gamma).$$

Say that a function δ is *closed for* $\boldsymbol{\alpha}, \gamma$ iff for all u and v,

$$\delta(u) = 0 \wedge v <_{\boldsymbol{\alpha},\gamma} u \rightarrow \delta(v) = 0.$$

δ is *well-founded for* $\boldsymbol{\alpha}, \gamma$ iff

$$\neg\exists\beta\,\forall p\,[\delta(\beta(p)) = 0 \wedge \beta(p+1) <_{\boldsymbol{\alpha},\gamma} \beta(p)].$$

Finally, say that γ is *locally correct for* $\boldsymbol{\alpha}, \delta$ iff for all $k, \mathbf{m} \in {}^k\omega$, and $l = \lg(\boldsymbol{\alpha})$, and all u such that $\delta(u) = 0$:

(0) if $u = \langle\langle 0, k, l, \ldots\rangle, \mathbf{m}\rangle$ and $\langle 0, k, l, \ldots\rangle$ is an index of the proper form for $(\mathbf{m}, \boldsymbol{\alpha})$, then $\gamma(u) = \{\langle 0, k, l, \ldots\rangle\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha})$;

(1) if for some $b, c_0, \ldots, c_{k'-1}$, $u = \langle\langle 1, k, l, b, c_0, \ldots, c_{k'-1}\rangle, \mathbf{m}\rangle$ then $\gamma(n) = \gamma(\langle b, \gamma(\langle c_0, \mathbf{m}\rangle), \ldots, \gamma(\langle c_{k'-1}, \mathbf{m}\rangle)\rangle)$;

(2) if for some b, $u = \langle\langle 2, k+1, l\rangle, b, \mathbf{m}\rangle$, then $\gamma(n) = \gamma(\langle b, \mathbf{m}\rangle)$;

(3) if for some b, $u = \langle\langle 3, k, l, b\rangle, \mathbf{m}\rangle$, then $\gamma(n) = \mathsf{I}(\lambda p \,.\, \gamma(\langle b, p, \mathbf{m}\rangle))$.

Now it suffices to show that for all u and $\boldsymbol{\alpha}$,

$$u \in U^{\mathsf{I}}_{\boldsymbol{\alpha}} \leftrightarrow (\exists \gamma, \delta \text{ recursive in } \mathsf{I}, \boldsymbol{\alpha})\,[\gamma \text{ is locally correct for } \boldsymbol{\alpha}, \delta \wedge \delta(u) = 0 \wedge \delta \text{ is closed for } \boldsymbol{\alpha}, \gamma \wedge \delta \text{ is well-founded for } \boldsymbol{\alpha}, \gamma].$$

For the implication $(\rightarrow)$, consider the functions

$$\gamma_u(\langle a, \mathbf{m}\rangle) = \begin{cases} \{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}), & \text{if } \quad |a, \mathbf{m}|^{\mathsf{I}}_{\boldsymbol{\alpha}} \leqslant |u|^{\mathsf{I}}_{\boldsymbol{\alpha}}; \\ 0, & \text{otherwise}; \end{cases}$$

and

$$\delta_u(v) = \begin{cases} 0, & \text{if } \quad |v|^{\mathsf{I}}_{\boldsymbol{\alpha}} \leqslant |u|^{\mathsf{I}}_{\boldsymbol{\alpha}}; \\ 1, & \text{otherwise}. \end{cases}$$

For the implication $(\leftarrow)$, suppose that γ and δ are recursive in $\mathsf{I}, \boldsymbol{\alpha}$ and satisfy the condition in brackets. Prove by induction on $<_{\boldsymbol{\alpha}, \gamma}$ that for all a and $\mathbf{m}$, if $\delta(\langle a, \mathbf{m}\rangle) = 0$ and $\langle a, \mathbf{m}\rangle \leqslant_{\boldsymbol{\alpha}, \gamma} u$, then $\langle a, \mathbf{m}\rangle \in U^{\mathsf{I}}_{\boldsymbol{\alpha}}$ and

$$\gamma(\langle a, \mathbf{m}\rangle) = \{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}).$$

6.27 Notes. The functionals $\Phi^{\#}$ were introduced in Hinman [1969] to prove Corollary 6.19. Many of the results of this section appear there. The key Theorem 6.2 is inspired by a similar result of Moschovakis [1967] (Theorem VII.2.10 below). The connections between $\mathsf{E}^{\#}_1$ and Σ^1_1 inductive definability are due to Aczel [1970]. The implication (ii)$\rightarrow$(iii) of Theorem 6.14 was originally (or Moschovakis [1974, Chapter 4]). Exercise 6.26 is due to Gandy and Moschovakis.

7. Recursive Type-3 Functionals and Relations

Recall that a functional $\mathbb{F}$ is of *type* 3 iff it has arguments $(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I})$ of types 0, 1, and 2 — that is $\mathbb{F}$ is a function from ${}^{k,l,l'}\omega = ({}^{k}\omega) \times {}^{l}({}^{\omega}\omega) \times {}^{l'}({}^{({}^{\omega}\omega)}\omega)$ into ω. Similarly a relation $\mathbb{R}$ is of type 3 iff it is a subset of some ${}^{k,l,l'}\omega$. In this section we study the notion of such functionals and relations being recursive or definable from recursive relations by quantification over ω, ${}^{\omega}\omega$, and ${}^{({}^{\omega}\omega)}\omega$. In Chapter VII we shall study the properties of recursion relative to a fixed type-3 functional.

The definition of recursiveness for type-3 functionals is essentially the same as that of recursiveness relative to a fixed functional I. The main difference is that we now explicitly allow for a finite sequence of type-2 arguments (rather than one) and think of them as genuine arguments (rather than parameters). In keeping with the discussions in the Notes to § II.5 and the Introduction to this chapter, it will be obvious that for a fixed I, a functional F is partial recursive in I iff for some partial recursive type-3 functional $\mathbb{G}$,

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathbb{G}(\mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}).$$

7.1 Definition. Ω^3 is the smallest set such that for all k, l, l', p, q, r, and s, all $i < k$, $j < l$, and $j' < l'$, and all $(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \in {}^{k,l,l'}\omega$.

(0) $(\langle 0, k, l, l', 0, n\rangle, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, n) \in \Omega^3$; the remaining parts of (0) and clauses (1) and (2) are similar modifications of the corresponding parts of Definition II.2.1 to accomodate the sequence $\mathbf{I}$ of l' type-2 arguments;

(3) for any b and any β, if for all p, $(b, p, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, \beta(p)) \in \Omega^3$, then $(\langle 3, k, l, l', j', b\rangle, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, \mathsf{I}_{j'}(\beta)) \in \Omega^3$.

Just as in previous cases we can prove easily that for all a, $\mathbf{m}$, $\boldsymbol{\alpha}$, and $\mathbf{I}$, there is at most one n such that $(a, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, n) \in \Omega^3$ and set

$$\{a\}^3(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq n \quad \text{iff} \quad (a, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, n) \in \Omega^3.$$

In particular, it follows from clause (3) that

$$\{\langle 3, k, l, l', j', b\rangle\}^3(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq \mathsf{I}_{j'}(\lambda p\,.\,\{b\}^3(p, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I})).$$

A type-3 functional $\mathbb{F}$ is *partial recursive* iff $\mathbb{F} = \{a\}^3$ for some a and *recursive* iff it is partial recursive and total. A type-3 relation $\mathbb{R}$ is *recursive* iff its characteristic functional $\mathbb{K}_{\mathbb{R}}$ is recursive and *semi-recursive* iff $\mathbb{R} = \mathrm{Dm}\,\mathbb{G}$ for some partial recursive $\mathbb{G}$.

The Recursion Theorem and the consequent closure properties may be established just as in § 1. In particular,

7.2 Lemma. *For any partial recursive functional* F, *if*

$$\mathbb{F}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq \mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}),$$

then $\mathbb{F}$ *is also partial recursive. Conversely, if* $\mathbb{F}$ *is partial recursive of rank* $(k, l, 0)$ *then* $\mathbb{F}$ *is partial recursive in the previous sense.* □

The techniques of § 2 yield the following substitution results:

7.3 Theorem. *For any partial recursive functionals* $\mathbb{G}$ *and* $\mathbb{H}$,

(i) *if* $\mathbb{F}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq \mathbb{G}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p \,.\, \mathbb{H}(p, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}), \mathbf{I})$ *and* $\mathrm{lg}(\mathbf{I}) > 0$, *then* $\mathbb{F}$ *is also partial recursive*;

(ii) *there exists a partial recursive functional* $\mathbb{F}$ *such that for all* $\mathbf{m}$, n, $\boldsymbol{\alpha}$, *and* $\mathbf{I}$,

$$\mathbb{G}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, \lambda\beta \,.\, \mathbb{H}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I})) \simeq n \rightarrow \mathbb{F}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq n,$$

and if $\mathrm{lg}(\mathbf{I}) > 0$ *and* $\lambda\beta \,.\, \mathbb{H}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I})$ *is total, then also*

$$\mathbb{F}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq n \rightarrow \mathbb{G}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, \lambda\beta \,.\, \mathbb{H}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I})) \simeq n.$$

Proof. The proof of (i) may be obtained by a suitable adaptation of that of Theorem 2.10. The hypothesis $\mathrm{lg}(\mathbf{I}) > 0$ is necessary because of Theorem 2.1. (ii) is proved by a similar modification of the proof of Theorem 2.14. (Cf. VII.1.6(ii) below.) □

7.4 Corollary. *For any recursive functionals* $\mathbb{G}$ *and* $\mathbb{H}$, *if*

$$\mathbb{F}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq \mathbb{G}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, \lambda\beta \,.\, \mathbb{H}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I})),$$

then $\mathbb{F}$ *is also recursive.* □

The results of §§ 3 and 4 cannot be generalized to recursion over ${}^{k,l,l'}\omega$ because of the hypothesis in these theorems that E be recursive in I. In fact, we have

7.5 Theorem. *There exists a relation* $\mathbb{R} \subseteq {}^{1,0,1}\omega$ *such that both* $\mathbb{R}$ *and* $\sim\mathbb{R}$ *are semi-recursive, but* $\mathbb{R}$ *is not recursive.*

Proof. Let G be a partial recursive function such that

$$\{m : \lambda p \,.\, G(p, m) \text{ is total}\} \text{ is } \Pi^0_1 \text{ but not recursive}$$

and set

$$\mathbb{R}(m, \mathsf{I}) \leftrightarrow \mathsf{I}(\lambda p \,.\, G(p, m)) \text{ is defined.}$$

Clearly $\mathbb{R}$ is semi-recursive. Furthermore, by the assumption and Lemma 7.2, since

$$\sim\mathbb{R}(m, \mathsf{I}) \leftrightarrow \lambda p \,.\, G(p, m) \text{ is not total,}$$

$\sim\mathbb{R}$ is also semi-recursive. Suppose, however, that $\mathbb{R}$ is recursive and set $F(m) = \mathbb{K}_{\mathbb{R}}(m, \lambda\beta\,.\,0)$. It follows from 7.2 and 7.4 that F is a recursive function, but

$$F(m) = 0 \leftrightarrow \mathbb{R}(m, \lambda\beta\,.\,0) \leftrightarrow \lambda p\,.\, G(p, m) \text{ is total,}$$

which contradicts the assumption on G. $\square$

We consider next the relations over ${}^{k,l,l'}\omega$ obtained from the recursive ones by quantification. The classes of arithmetical and analytical relations and the subclasses Σ_r^0, Σ_r^1, etc. may be defined exactly as in §§ III.1–2. We shall write Σ_r^i etc. ambiguously to refer also to these classes of relations over ${}^{k,l,l'}\omega$. Let $\exists^2\mathbb{P}$ denote the relation $\mathbb{R}$ defined by

$$\mathbb{R}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \leftrightarrow \exists \mathsf{H}\,.\,\mathbb{P}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, \mathsf{H})$$

and similarly for $\forall^2\mathbb{R}$. Then we set, for all r,

$$\Sigma_0^2 = \Pi_0^2 = \text{the class of analytical relations over } {}^{k,l,l'}\omega;$$
$$\Sigma_{r+1}^2 = \{\exists^2\mathbb{P} : \mathbb{P} \in \Pi_r^2\};$$
$$\Pi_{r+1}^2 = \{\forall^2\mathbb{P} : \mathbb{P} \in \Sigma_r^2\};$$
$$\Delta_r^2 = \Sigma_r^2 \cap \Pi_r^2.$$

The properties of the arithmetical hierarchy expressed by III.1.4–7 hold here with appropriate modifications by the same proofs. Similarly, the properties of the analytical hierarchy expressed by III.2.4–7 are easily extended to the analytical hierarchy over ${}^{k,l,l'}\omega$. The situation concerning universal relations and hierarchy theorems is, however, somewhat more complicated. On the one hand, the arithmetical and analytical hierarchy theorems follow immediately from those of §§ III.1–2, since a relation R which is $\Sigma_r^i \sim \Delta_r^i$ in the previous sense is still $\Sigma_r^i \sim \Delta_r^i$ as a relation over ${}^{k,l,0}\omega$. This approach does not, however, suggest any way to prove that Σ_1^2 is properly larger than Δ_1^2. The proofs of these results in §§ III.1–2 all spring from the fact that there is a universal Σ_1^0 relation U_1^0. Not too surprisingly, this is no longer true in the present context:

7.6 Lemma. *There is no relation universal for the class of Σ_1^0 relations over ${}^{k,l,l'}\omega$.*

Proof. Suppose $\mathbb{U}$ were such a universal relation and let U be defined by

$$\mathsf{U}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \leftrightarrow \mathsf{U}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, \langle \mathsf{E} \rangle).$$

Then clearly U would be universal for relations Σ_1^0 in E — that is, of the form $\exists^0\mathsf{R}$ with R recursive in E. But if R is recursive in E, so is $\exists^0\mathsf{R}$, so in fact U would be universal for the class of relations recursive in E. A standard diagonal argument shows that this is impossible. □

In find a universal relation, we must go to Π_1^1.

7.7 Theorem. *There exists a relation $\mathbb{U}_1^1 \subseteq {}^{2,1,1}\omega$ which is universal for the class of Π_1^1 relations over ${}^{k,l,l'}\omega$.*

Proof. Let

$$\mathbb{U}_1^1(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, \langle \mathbf{I} \rangle) \leftrightarrow \forall \beta\, \exists p\, [\{a\}^3(p, \mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I}) \text{ is defined}].$$

It is clear that for any Π_1^1 relation $\mathbb{R}$ there is a number a such that

$$\mathbb{R}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \leftrightarrow \mathbb{U}_1^1(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, \langle \mathbf{I} \rangle)$$

and it remains to show that $\mathbb{U}_1^1$ is Π_1^1. This is accomplished by defining a Π_1^1 family of monotone operators $\Lambda_{\langle \boldsymbol{\alpha} \rangle, \langle \mathbf{I} \rangle}$ over ω such that

$$\{a\}^3(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq n \leftrightarrow \langle a, \langle \mathbf{m} \rangle, n \rangle \in \bar{\Lambda}_{\langle \boldsymbol{\alpha} \rangle, \langle \mathbf{I} \rangle}.$$

Λ is defined much as in the proof of Theorem 1.5:

$$\Lambda_{\langle \boldsymbol{\alpha} \rangle, \langle \mathbf{I} \rangle}(A) = \Gamma_{\boldsymbol{\alpha}, \mathbf{I}, 0}(A) \cup \cdots \cup \Gamma_{\boldsymbol{\alpha}, \mathbf{I}, 2}(A) \cup \Lambda_{\boldsymbol{\alpha}, \mathbf{I}, 3}(A)$$

where, for example,

$$\begin{aligned} \Lambda_{\boldsymbol{\alpha}, \mathbf{I}, 3}(A) = \{ & \langle \langle 3, k, l, l', j', b \rangle, \langle \mathbf{m} \rangle, n \rangle : \\ & \forall p\, \exists q\, [\langle b, \langle p, \mathbf{m} \rangle, q \rangle \in A] \wedge \\ & \forall \beta\, (\forall p\, \forall q\, [\langle b, \langle p, \mathbf{m} \rangle, q \rangle \in A \to \beta(p) = q] \to \mathsf{I}_{j'}(\beta) = n)\}. \end{aligned}$$

If follows easily that Λ is as required and that $\bar{\Lambda} \in \Pi_1^1$. Thus also $\mathbb{U}_1^1$ is Π_1^1. □

We may now define, for all $r \geq 1$, relations $\mathbb{U}_r^1$ such that $\mathbb{U}_r^1$ is universal for the class of Π_r^1 relations over ${}^{k,l,l'}\omega$:

$$\mathbb{U}_{r+1}^1(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, \langle \mathbf{I} \rangle) \leftrightarrow \exists \beta \sim \mathbb{U}_r^1(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha}, \beta \rangle, \langle \mathbf{I} \rangle).$$

The following analogue of Lemma III.2.9 will allow us to extend this to the classes Σ_r^2:

7.8 Lemma. $\Sigma_1^2 = (\exists^2 \mathbb{P} : \mathbb{P} \in \Pi_1^1\}$.

Proof. The inclusion ($\supseteq$) is immediate from the definitions. For the converse we show that the class of relations of the form $\exists^2\mathbb{P}$ with $\mathbb{P} \in \Pi_1^1$ is closed under $\exists^1$, $\forall^1$, and $\exists^2$. This follows easily from the equivalences:

$$\exists\beta\,\exists\mathsf{H}\,\mathbb{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I}, \mathsf{H}) \leftrightarrow \exists\mathsf{H}\,\mathbb{P}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p\,.\,(\mathsf{H})^{p+1}(\lambda q\,.\,0), \mathbf{I}, (\mathsf{H})^0);$$

$$\exists\mathsf{G}\,\exists\mathsf{H}\,\mathbb{P}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, \mathsf{G}, \mathsf{H}) \leftrightarrow \exists\mathsf{H}\,\mathbb{P}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, (\mathsf{H})^0, (\mathsf{H})^1);$$

$$\forall\beta\,\exists\mathsf{H}\,\mathbb{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I}, \mathsf{H}) \leftrightarrow \exists\mathsf{H}\,\forall\beta\,\mathbb{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I}, (\mathsf{H})^\beta);$$

where $(\mathsf{H})^p(\alpha) = \mathsf{H}((p) * \alpha)$ and $(\mathsf{H})^\beta(\alpha) = \mathsf{H}(\langle\beta, \alpha\rangle)$. $\square$

Thus if we set, for $r \geqslant 1$,

$$\mathbb{U}_1^2(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle, \langle\mathbf{I}\rangle) \leftrightarrow \exists\mathsf{H}\,.\,\mathbb{U}_1^1(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle, \langle\mathbf{I}, \mathsf{H}\rangle)$$

and

$$\mathbb{U}_{r+1}^2(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle, \langle\mathbf{I}\rangle) \leftrightarrow \exists\mathsf{H} \sim \mathbb{U}_r^2(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle, \langle\mathbf{I}, \mathsf{H}\rangle)$$

and

$$\mathbb{U}_{r+1}^2(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle, \langle\mathbf{I}\rangle) \leftrightarrow \exists\mathsf{H} \sim \mathbb{U}_r^2(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle, \langle\mathbf{I}, \mathsf{H}\rangle)$$

we have relations $\mathbb{U}_r^2$ universal for Σ_r^2 and $\sim\mathbb{U}_r^2$ universal for Π_r^2. Then just as in §§ III.1–2,

7.9 Theorem (The Functional Quantifier Hierarchy). *For all* $r \geqslant 1$,

(i) $\Sigma_r^2 \not\subseteq \Delta_r^2$ *and* $\Pi_r^2 \not\subseteq \Delta_r^2$;

(ii) $\Delta_{r+1}^2 \not\subseteq \Sigma_r^2 \cup \Pi_r^2$. $\square$

We turn now to the consideration of inductive definitions of subsets of ${}^{k,l}\omega$ which we were forced to abandon in § III.3. For simplicity we restrict attention to inductively defined subsets of ${}^{\omega}\omega$; extension to the general case is obtained by the usual codings. If Γ is an operator over ${}^{\omega}\omega$, we set

$$\mathbb{P}_\Gamma(\alpha, \mathsf{I}) \leftrightarrow \alpha \in \Gamma(\mathsf{Z}_\mathsf{I}),$$

where $\mathsf{Z}_\mathsf{I} = \{\beta : \mathsf{I}(\beta) = 0\}$, and classify Γ as Σ_r^i, etc. according to the classification of $\mathbb{P}_\Gamma$. Parallel to Theorem III.3.1, we have:

7.10 Theorem. *For any* $r > 0$ *and any monotone operator* Γ *over* ${}^{\omega}\omega$,

$$\Gamma \in \Pi^2_r \rightarrow \bar{\Gamma} \in \Pi^2_r.$$

Proof. We have

$$\begin{aligned}\alpha \in \bar{\Gamma} &\leftrightarrow \forall \mathsf{A}\,(\Gamma(\mathsf{A}) \subseteq \mathsf{A} \rightarrow \alpha \in \mathsf{A})\\ &\leftrightarrow \forall \mathsf{I}\,(\forall \beta\,[\mathbb{P}_\Gamma(\beta, \mathsf{I}) \rightarrow \mathsf{I}(\beta) = 0] \rightarrow \mathsf{I}(\alpha) = 0). \quad \square\end{aligned}$$

To obtain further results of this sort, we shall need to examine the closure ordinals of operators over ${}^\omega\omega$. By Lemma I.3.2, $|\Gamma|$ is less than the least cardinal greater than $2^{\aleph_0}$. Hence the ordinals $\leq |\Gamma|$ can be coded as well-orderings of subsets of ${}^\omega\omega$. For any I, let

$$\leqslant_{\mathsf{I}} = \{(\gamma, \delta) : \mathsf{I}(\langle \gamma, \delta \rangle) = 0\},$$

and let

$$\mathsf{W} = \{\mathsf{I} : \leqslant_{\mathsf{I}} \text{ is a well-ordering}\}.$$

In contrast to the fact that $\mathsf{W} \in \Pi^1_1 \sim \Delta^1_1$ (Corollary IV.1.2), we have here

7.11 Lemma. $\mathsf{W} \in \Delta^2_1$; *in fact,* $\mathsf{W} \in \Delta^1_{(\omega)}$.

Proof. To express that $\leqslant_{\mathsf{I}}$ is a linear ordering clearly requires only function quantifiers. That $\leqslant_{\mathsf{I}}$ is well founded is expressed by: $\neg \exists \alpha\, \forall p.\, (\alpha)^{p+1} <_{\mathsf{I}} (\alpha)^p$, which is also analytical. $\square$

For $\mathsf{I} \in \mathsf{W}$, we denote by $\|\mathsf{I}\|$ the order-type of $\leqslant_{\mathsf{I}}$ and define $\mathsf{I} \restriction \gamma$ and $|\gamma|_{\mathsf{I}}$ in the natural way so that the analogues of (8)–(11) of § I.1 hold. Then for any Γ over ${}^\omega\omega$,

$$\begin{aligned}\alpha \in \bar{\Gamma} &\leftrightarrow \exists \mathsf{I}\,(\mathsf{I} \in \mathsf{W} \wedge \alpha \in \Gamma^{\|\mathsf{I}\|})\\ &\leftrightarrow \forall \mathsf{I}\,(\mathsf{I} \in \mathsf{W} \wedge \Gamma^{\|\mathsf{I}\|} \subseteq \Gamma^{(\|\mathsf{I}\|)} \rightarrow \alpha \in \Gamma^{\|\mathsf{I}\|})\end{aligned}$$

and to classify $\bar{\Gamma}$ it remains to evaluate the complexity of $\Gamma^{\|\mathsf{I}\|}$ and $\Gamma^{(\|\mathsf{I}\|)}$.

7.12 Theorem. *For any $r > 0$ and any inductive operator $\Gamma \in \Delta^2_r$ over ${}^\omega\omega$, there exist relations $\mathsf{V}^{()}_\Sigma$ and $\mathsf{V}_\Sigma \in \Sigma^2_r$ and $\mathsf{V}^{()}_\Pi$ and $\mathsf{V}_\Pi \in \Pi^2_r$ such that for all $\mathsf{I} \in \mathsf{W}$ and all α,*

(i) $\alpha \in \Gamma^{(\|\mathsf{I}\|)} \leftrightarrow \mathsf{V}^{()}_\Sigma(\alpha, \mathsf{I}) \leftrightarrow \mathsf{V}^{()}_\Pi(\alpha, \mathsf{I})$;
(ii) $\alpha \in \Gamma^{\|\mathsf{I}\|} \leftrightarrow \mathsf{V}_\Sigma(\alpha, \mathsf{I}) \leftrightarrow \mathsf{V}_\Pi(\alpha, \mathsf{I})$.

Proof. Similar to that of Theorem III.3.9. $\square$

7.13 Corollary. *For any $r > 0$ and any inductive operator Γ over ${}^{\omega}\omega$,*

$$\Gamma \in \Delta_r^2 \rightarrow \bar{\Gamma} \in \Delta_r^2. \quad \square$$

Parallel to Theorem III.3.13 we have

7.14 Theorem. *For any $r > 0$ and any monotone operator $\Gamma \in \Sigma_r^2$ (Π_r^2), there exist relations $\mathbb{V}^{(\,)}$ and $\mathbb{V} \in \Sigma_r^2$ (Π_r^2) such that for all $\mathsf{I} \in \mathbb{W}$ and all α,*

(i) $\alpha \in \Gamma^{(\|\mathsf{I}\|)} \leftrightarrow \mathbb{V}^{(\,)}(\alpha, \mathsf{I})$;

(ii) $\alpha \in \Gamma^{\|\mathsf{I}\|} \leftrightarrow \mathbb{V}(\alpha, \mathsf{I})$. $\square$

7.15 Corollary. *For any $r > 0$ and any monotone operator Γ over ${}^{\omega}\omega$,*

$$\Gamma \in \Sigma_r^2 \rightarrow \bar{\Gamma} \in \Sigma_r^2. \quad \square$$

Very little more is known about the classes Σ_r^2 and Π_r^2. It follows from the Hypothesis of Constructibility (V = L) that there is a Δ_1^2 well-ordering of ${}^{({}^{\omega}\omega)}\omega$ and thus that for all $r \geqslant 1$, Σ_r^2 has the pre-wellordering property. Determinacy of all subsets of ${}^{\omega}\omega$ also implies that Σ_1^2 has the pre-wellordering property and the proposition that Π_1^2 has the pre-wellordering property is known to be consistent with ZFC.

For investigations under the Hypothesis of Determinacy or some other hypothesis which contradicts the Axiom of Choice, it is useful to note that the theory of the functional quantifier hierarchy can be developed without this axiom (cf. discussion preceding Theorem V.3.1). One use of choice is in the last of the three equivalences used in the proof of Lemma 7.8. Unfortunately, choice seems essential for the lemma and to avoid it we must find a new way to define the Σ_r^2 universal relations. The idea is provided by Theorems III.3.6–7. Following the pattern of these theorems we may define a Σ_1^1 inductive operator Γ such that $|\Gamma| = \omega$ and $\bar{\Gamma}$ is a relation $\mathbb{U}_{(\omega)}^1$ which is universal for $\Delta_{(\omega)}^1$. Although $\mathbb{U}_{(\omega)}^1$ cannot be $\Delta_{(\omega)}^1$, it is easily seen to be Δ_1^2. Hence we may take

$$\mathbb{U}_1^2(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, \langle \mathbf{I} \rangle) \leftrightarrow \exists \mathsf{H}\,.\,\mathbb{U}_{(\omega)}^1(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, \langle \mathbf{I}, \mathsf{H} \rangle)$$

and $\mathbb{U}_{r+1}^2$ as before.

The same equivalence is required to show that Σ_r^2 and Π_r^2 are closed under function quantification ($\exists^1$ and $\forall^1$). Here we cannot avoid choice altogether, but can replace it with the weaker *Collection Principle*:

$$\forall \beta\, \exists \mathsf{H}\,.\,\mathbb{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I}, \mathsf{H}) \rightarrow \exists \mathsf{H}\, \forall \beta\, \exists \gamma\,.\,\mathbb{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I}, (\mathsf{H})^{\gamma}).$$

This principle holds in $L[{}^{\omega}\omega]$, the class of sets constructible from ${}^{\omega}\omega$ (cf. Moschovakis [1970]). Many people have conjectured that some strong form of determinacy also holds in this model.

The final use of the Axiom of Choice occurs in the discussion of the closure ordinals of inductive definitions over ${}^{\omega}\omega$. If ${}^{\omega}\omega$ is not well-orderable, then "the least cardinal greater than $2^{\aleph_0}$" is meaningless and we need a new bound for these closure ordinals. Fortunately, this is easy to compute. Any inductive operator Γ over any set X induces a pre-wellordering $\leqslant_\Gamma$ of X of type $|\Gamma|+1$:

$$x \leqslant_\Gamma y \leftrightarrow \forall\sigma\,[y \in \Gamma^\sigma \rightarrow x \in \Gamma^\sigma].$$

Hence if we set

$$o(X) = \sup{}^{+}\{\sigma : \text{there exists a pre-wellordering of } X \text{ of type } \sigma\},$$

then $|\Gamma| < o(X)$. Furthermore, it is no harder to code ordinals with pre-wellorderings. Thus 7.11–12 and 7.14 may be proved with $\mathbb{W}$ replaced by $\mathbf{p}\mathbb{W} = \{\mathsf{I} : \leqslant_{\mathsf{I}} \text{ is a pre-wellordering}\}$, so 7.13 and 7.15 hold also in a set theory without the Axiom of Choice.

7.16–7.22 Exercises

7.16. Sketch a proof of Theorem 7.3.

7.17. Does there exist a subset of ${}^{({}^{\omega}\omega)}\omega$ which is Σ_1^0 but not recursive?

7.18 (Tugué [1960]). Show that the class of Σ_1^1 relations over ${}^{k,l,l'}\omega$ does not have the separation property.

7.19. Formulate and prove substitution theorems for recursive type-3 functionals analogous to those of § 2.

7.20. Formulate and prove substitution theorems for the functional quantifier hierarchy analogous to those for the arithmetical and analytical hierarchies (III.1.11 and III.2.11). Consider substitution for both type-1 and type-2 arguments.

7.21. Suppose that F is defined from G and $\mathbb{H}$ by the following recursion:

$$\mathsf{F}(0, \mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq \mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}, \beta),$$

$$\mathsf{F}(p+1, \mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq \mathbb{H}(p, \mathbf{m}, \boldsymbol{\alpha}, \beta, \lambda\gamma\,.\,\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}, \gamma)).$$

If G and $\mathbb{H}$ are recursive, does it follow that also F is recursive?

7.22. Prove the fact mentioned following Corollary 7.15 that if V = L, there is a Δ_1^2 well-ordering of ${}^{({}^{\omega}\omega)}\omega$ and for all r, Σ_r^2 has the pre-wellordering property.

7.23 Notes. The results of this section are largely from the folk literature.