Chapter 6
Degree Structure

The study of degrees, in particular of r.e. degrees, is a characteristic and important part of recursion theory. And no account of general recursion theory can be claimed to be successful unless at least an introduction to notions of *reducibilities* and the associated *degree theory* is given. This is precisely what we will do in this chapter: to present an *introduction* to this topic within the general framework of infinite theories and to give an example of a non-trivial result in the extended framework.

But we should really like to do something more. In the spirit of an axiomatic analysis we want to determine the "true" domain for degree theory and priority arguments. This is the question we turn to in Section 6.3. Our discussion is fragmentary and we do not claim any complete solution. Indeed there may not be any well-defined "solution". But we hope that this section may give some clue as to how far recursion-theoretic regularities extend.

Our discussion is, in principle, self-contained, but some familiarity with the basic notions of α-recursion theory would be helpful: we recommend the introductory paper of R. A. Shore, *α-recursion theory* [152], in a precise sense we continue his discussion in this chapter.

6.1 Basic Notions

The setting is an infinite computation theory Θ on a prewellordered domain $(\mathfrak{A}, \preccurlyeq)$, see Definition 5.1.5. We shall need a suitable notion of enumeration and of parametrization of the Θ-semicomputable sets. But as usual we must preface our definitions by introducing some necessary notation.

Let f be a mapping which to every $x \in A$ gives us a canonical Θ-index for a Θ-finite set, i.e. $f(x)$ is an index for the function $\mathbf{E}_{W^x}$, where W^x is the Θ-finite set associated with x. It will be convenient to write the mapping as

$$f = \lambda x \cdot W^x,$$

but we should always remember that the value of f at x is a canonical Θ-index for the Θ-finite set W^x.

6.1.1 Definition. A $\preccurlyeq$-*enumeration* of a set W is a Θ-computable mapping $\lambda x \cdot W^x$ whose values are canonical Θ-indices for the Θ-finite sets W^x, such that

(i) $y \preccurlyeq x \Rightarrow W^y \subseteq W^x$,
(ii) $W = \bigcup \{W^x : x \in A\}$.

A $\preccurlyeq$-*parametrization* of the Θ-semicomputable sets is a Θ-computable mapping $\lambda ax.\ W_a{}^x$ such that

(iii) $y \preccurlyeq x \Rightarrow W_a^y \subseteq W_a^x$,
(iv) for each Θ-semicomputable set W there is an a such that $W = \bigcup \{W_a^x ; x \in A\}$.

From axioms A and B (see 5.1.1 and 5.1.2) the reader will have no difficulty in constructing a $\preccurlyeq$-enumeration of the sets

$$\Theta_n = \{\langle a, \sigma, z\rangle : \{a\}_\Theta(\sigma) \simeq z \wedge \mathrm{lh}(\sigma) = n\},$$

from which he may derive a $\preccurlyeq$-parametrization of the Θ-semicomputable sets.

A number of arguments in α-recursion theory seem to require the use of the μ-operator. Let $R(\sigma, x)$ be a Θ-computable relation, we would like to introduce a function $\mu x R(\sigma, x)$ by the equivalence

$$\mu x R(\sigma, x) = z \quad \text{iff} \quad R(\sigma, z) \wedge (\forall y \prec z) \neg R(\sigma, y).$$

In α-recursion theory the domain, a segment of the ordinals, is well-ordered, so there is a unique z satisfying the equivalence. When the domain has a prewell-ordering, the μ-operator would in general have to be multiple-valued. But there is a way of getting around this obstacle.

6.1.2 Proposition. *Let $R(\sigma, x)$ be a Θ-semicomputable relation such that*

(i) *$R(\sigma, x) \Rightarrow x$ is a canonical Θ-index for some Θ-finite set K_x.*
(ii) *$R(\sigma, x) \wedge R(\sigma, y) \Rightarrow K_x = K_y$.*

Then there is a Θ-computable mapping $q(\sigma)$ obtained uniformly from an index r of R such that

$$K_x = K_{q(\sigma)},$$

for all x such that $R(\sigma, x)$

We concentrate on the key point of the proof. $q(\sigma)$ will be a canonical Θ-index for the set

$$N_\sigma = \bigcup \{K_x : R(\sigma, x)\}.$$

To see how to define $q(\sigma)$ we calculate:

$$\begin{aligned} \mathbf{E}_{N_\sigma}(\{\hat{f}\}) = 0 \quad &\text{iff} \quad \exists y \in N_\sigma \cdot \{\hat{f}\}(y) = 0 \\ &\text{iff} \quad \exists y \exists w (R(\sigma, w) \wedge y \in K_w \wedge \{\hat{f}\}(y) = 0) \\ &\text{iff} \quad \exists w (R(\sigma, w) \wedge \{w\}(\hat{f}) = 0). \end{aligned}$$

In the same way we get

$$\begin{aligned} \mathbf{E}_{N_\sigma}(\{\hat{f}\}) = 1 \quad &\text{iff} \quad \forall y \in N_\sigma \cdot \{\hat{f}\}(y) = 1 \\ &\text{iff} \quad \forall y \forall w (R(\sigma, w) \wedge y \in K_w \to \{\hat{f}\}(y) = 1) \\ &\text{iff} \quad \forall w (R(\sigma, w) \to \{w\}(\hat{f}) = 1) \\ &\text{iff} \quad \exists w (R(\sigma, w) \wedge \{w\}(\hat{f}) = 1). \end{aligned}$$

The last equivalence follows from assumption (ii). We now choose a code $q'(\sigma)$ such that $\{q'(\sigma)\}(\hat{f}, j) \simeq 0$ iff $\exists w(R(\sigma, w) \wedge \{w\}(\hat{f}) = j)$, and we get our function $q(\sigma)$ by using selection over the integers.

6.1.3 Theorem. *There is a Θ-computable function $q(a, \sigma)$ such that if $B_\sigma = \{x : \{a\}(x, \sigma) \simeq 0\}$ is a non-empty Θ-semicomputable set, then $q(a, \sigma)$ gives a canonical Θ-index for a non-empty Θ-finite subset N of B_σ.*

This is another variation of a familiar theme. We are not in general able to computably select a *unique* element of B_σ, but we can *effectively compute* an index of a finite subset of B_σ. For the proof let $\lambda z \cdot W^z$ be a $\preccurlyeq$-enumeration of the set $\{\langle a, x, \sigma, 0\rangle : \{a\}(x, \sigma) \simeq 0\}$. For each z we introduce the Θ-finite set $N_{\sigma,z} = \{y : y \prec z \wedge \langle a, y, \sigma, 0\rangle \in W^z\}$. Consider the following Θ-semicomputable relation

$$\begin{aligned} H(\sigma, z) \quad \text{iff} \quad &\exists y \prec z \cdot \langle a, y, \sigma, 0\rangle \in W^z \\ &\wedge (\forall w \prec z) \neg (\exists y \prec w) \cdot \langle a, y, \sigma, 0\rangle \in W^w. \end{aligned}$$

From this we see that if $H(\sigma, z_1)$ and $H(\sigma, z_2)$, then $z_1 \sim z_2$ and further $N_{\sigma,z_1} = N_{\sigma,z_2}$. Let

$$N = \bigcup \{N_{\sigma,z} : H(\sigma, z)\}.$$

Since the canonical Θ-indices involved are effectively computable from the given data, Proposition 6.1.2 allows us to compute $q(a, \sigma)$ as canonical Θ-index of N, and obviously $\varnothing \neq N \subseteq B_\sigma$.

6.1.4 Definition. A theory Θ is *projectible* into a subset W of its domain A if there is a Θ-computable function p such that $\text{dom}(p) \subseteq W$ and p maps onto all of A.

Here the set $p^{-1}(x)$ is a set of "notations" for $x \in A$, but, lacking a well-ordering, we have in general no unique notation for each $x \in A$. For some purposes it is important to know that $p^{-1}(x)$ is Θ-finite uniformly in x. We can always so arrange things by using 6.1.3. Usually we shall study projections into sets of the form

$$L^\beta = \{x \in A : x \prec \beta\}.$$

This is a slight abuse of notation; what we mean is that $|x|_\preccurlyeq < \beta$, where $|x|_\preccurlyeq$ is the ordinal of x in the prewellordering $\preccurlyeq$. In the same way $L^x = \{y \in A : y \prec x\}$. Note that we can always assume that $p^{-1}(L^x)$ is Θ-finite. This is used e.g. in the proof of Theorem 6.1.18.

6.1.5 Definition. Let Θ be an infinite theory on a domain $(\mathfrak{A}, \preccurlyeq)$. The *projectum* of Θ, denoted by $|\preccurlyeq|^*$, is the least ordinal β such that Θ is projectible into L^β.

This means that we have a notation for each $x \in A$ below $|\preccurlyeq|^*$. And more importantly it follows that we have a $\preccurlyeq$-parametrization of the Θ-semicomputable sets below $|\preccurlyeq|^*$.

6.1.6 Lemma. (i) *Let $W = \{a : W_a \neq \varnothing\}$ for a given $\preccurlyeq$-parametrization of the Θ-semicomputable sets. Then Θ is projectible into W.*

(ii) *Let p be a projection. Then there is a $\preccurlyeq$-parametrization of the Θ-semicomputable sets such that $\{a : W_a \neq \varnothing\} \subseteq$ dom(p).*

To clarify our notation, if $\lambda za \cdot W_a^z$ is a $\preccurlyeq$-parametrization, then $W_a = \bigcup \{W_a^z; z \in A\}$. We prove (ii): Let $\lambda az \cdot V_a^z$ be any $\preccurlyeq$-parametrization of the Θ-semicomputable sets. Let W be the domain of the projection p and $\lambda z \cdot W^z$ a $\preccurlyeq$-enumeration of W. Define a relation R by

$$R(a, x) \quad \text{iff} \quad p(a) = x.$$

By 6.1.3 let R_a be a Θ-finite subset of $\{x : R(a, x)\}$. Note that if $R(a, x)$, then $R_a = \{x\}$. Introduce sets $R_{a,z}$ by

$$R_{a,z} = \begin{cases} \varnothing & \text{if} \quad a \notin W^z \\ R_a & \text{if} \quad a \in W^z. \end{cases}$$

We have our $\preccurlyeq$-parametrization by setting

$$W_a^z = \{y \in L^z : (\exists x \in R_{a,z})[y \in V_x^z]\}.$$

So far things have extended. But now we come to a difficulty. An important technical lemma of α-recursion theory states that any α-semicomputable subset bounded below the projectum is α-finite. This may not be true for arbitrary infinite theories. It could also happen that the projectum $|\preccurlyeq|^*$ is not a limit ordinal. Both properties seem necessary for a decent degree theory. And since we cannot prove them in general we get around these difficulties by a definition. We shall return to this point in Section 6.3.

6.1.7 Definition. Let Θ be an infinite theory on a domain $(\mathfrak{A}, \preccurlyeq)$. The *r.e.-projectum* of Θ, denoted by $|\preccurlyeq|^+$, is the least ordinal β for which there is a Θ-semicomputable non Θ-finite set $W \subseteq L^\beta$.

It is always true that $|\preccurlyeq|^+ \leqslant |\preccurlyeq|^*$. The converse is a definition.

6.1.8 Definition. Let Θ be an infinite theory on a domain $(\mathfrak{A}, \preccurlyeq)$. Θ is called *adequate* if $|\preccurlyeq|^+ = |\preccurlyeq|^* =$ limit ordinal.

We shall in Section 6.3 discuss the "true" domain of degree theory and priority arguments. Here we just note that there are non-wellorderable adequate theories.

Let Θ be infinite and let $\lambda z \cdot K_z$ be an enumeration of the Θ-finite sets, i.e. the values of the function are canonical Θ-indices for Θ-finite sets. Every Θ-finite set K is K_z for some z. Sometimes we may require of the enumeration that $K_z \subseteq L^z$, i.e. every $x \in K_z$ satisfies $x \prec z$.

Given two subsets B, C of the domain A there is an immediate reducibility notion that comes to mind, viz. B is reducible to C if B is $\Theta[C]$-computable. But aside from fixing the proper version of $\Theta[C]$ there are difficulties. We want a notion of reducibility relative to a given theory Θ, i.e. we want to decide questions about B using Θ-finite information about C and its complement. But the notion of finiteness may change in passing from Θ to $\Theta[C]$. Thus we are led inevitably to the following notion of "Θ-computable in".

6.1.9 Definition. Let $B, C \subseteq A$, f a function, and $\lambda z \cdot K_z$ a fixed enumeration of Θ-finite sets.

(i) f is *weakly Θ-computable in C*, denoted $f \leqslant_w C$, if there is a Θ-semicomputable set W such that for all σ, y

$$f(\sigma) \simeq y \quad \text{iff} \quad \exists z, w(\langle \sigma, y, z, w\rangle \in W \wedge K_z \subseteq C \wedge K_w \cap C = \varnothing).$$

B is *weakly Θ-computable* in C, $B \leqslant_w C$, in case $c_B \leqslant_w C$.

(ii) B is *Θ-computable in C*, denoted $B \leqslant C$, if there is a Θ-semicomputable set W such that for all z_1, z_2

$$K_{z_1} \subseteq B \wedge K_{z_2} \cap B = \varnothing \quad \text{iff} \quad \exists w_1, w_2(\langle z_1, z_2, w_1, w_2\rangle \in W \wedge K_{w_1} \subseteq C \wedge K_{w_2} \cap C = \varnothing).$$

(iii) B is *weakly Θ-semicomputable in C* if there is a Θ-semicomputable set W such that for all x

$$x \in B \quad \text{iff} \quad \exists z, w(\langle x, z, w\rangle \in W \wedge K_z \subseteq C \wedge K_w \cap C = \varnothing).$$

(iv) B is *Θ-semicomputable* in C if there is a Θ-semicomputable set W such that for all z

$$K_z \subseteq B \quad \text{iff} \quad \exists w_1, w_2(\langle z, w_1, w_2\rangle \in W \wedge K_{w_1} \subseteq C \wedge K_{w_2} \cap C = \varnothing).$$

The definitions are independent of the particular enumeration of the Θ-finite sets. As usual we set $B \equiv C$ iff $B \leqslant C$ and $C \leqslant B$.

The reducibility notion $B \leqslant C$ will be a focus of our attention. It is the one among several possible generalizations of Turing reducibility in ORT which has led to the most interesting results in the general framework. However, the relation "B is $\Theta[C]$-computable" (i.e. c_B is $\Theta[C]$-computable) also merits some comment. We shall not pursue any philosophic discussions of notions of reducibilities here, the reader may want to consult Kriesel [90] and also the excellent and annotated bibliography of Shore [152]. We shall return to more general matters in Section 6.3.

Given a set $C \subseteq A$ we construct $\Theta[C]$ along the lines of the construction in Section 5.2, and arrange things such that a tuple (a, σ, z) is added at stage $\Theta^\beta[C]$ only if a, σ, z and $\langle a, \sigma, z\rangle$ are elements of L^β.

6.1.10 Definition. Let B, C be subsets of the domain of Θ and f a function.

(i) $f \leqslant_d C$ if f is $\Theta[C]$-computable.
(ii) $B \leqslant_d C$ if c_B is $\Theta[C]$-computable.

A simple argument shows that $\leqslant_d$ is transitive. The following lemma is also immediate.

6.1.11 Lemma. *Let f be an integer-valued function. Then $f \leqslant_w C$ implies $f \leqslant_d C$.*

$f \leqslant_w C$ means that for some Θ-semicomputable W

$$f(\sigma) \simeq x \quad \text{iff} \quad \exists z, w(\langle \sigma, x, z, w\rangle \in W \ \wedge \ K_z \subseteq C \ \wedge \ K_w \cap C = \varnothing).$$

Thus f has a $\Theta[C]$-semicomputable graph. Using selection over the integers, which is available in $\Theta[C]$, we define f as a $\Theta[C]$-computable function.

The lemma allows us to conclude that

$$B \leqslant C \Rightarrow B \leqslant_w C \Rightarrow B \leqslant_d C.$$

But none of these implications can be reversed. (See Driscoll [21] where an example is given that $\leqslant_w$ need not be transitive even on the Θ-semicomputable sets.)

But there is one case where $B \leqslant_d C$ implies $B \leqslant C$, viz. the *regular* and *hyperregular* case. These notions are due to Sacks [140]. Before introducing the definition let us note that the sets weakly Θ-semicomputable in C are enumerated by setting

$$W_a^C = \{x : \exists z, w(\langle x, z, w\rangle \in W_a \ \wedge \ K_z \subseteq C \ \wedge \ K_w \cap C = \varnothing)\}.$$

As an approximation to W_a^C let

$${}^zW_a^C = \{x : \exists w_1, w_2(\langle x, w_1, w_2\rangle \in W_a^z \ \wedge \ K_{w_1} \subseteq C \ \wedge \ K_{w_2} \cap C = \varnothing)\}.$$

6.1.12 Definition. (i) A set B is *regular* if $B \cap K$ is Θ-finite whenever K is Θ-finite.

(ii) A set B is *hyperregular* if for all Θ-finite sets K and all indices a, $K \subseteq W_a^B$ implies that $K \subseteq {}^zW_a^B$, for some z.

The reader will notice the similarity of (i) to Δ_0-separation and of (ii) to Δ_0-collection. It is perhaps not too surprising that when B is regular and hyperregular, then $\Theta[B]$ will be an infinite theory and the notion of Θ-finite and $\Theta[B]$-finite will coincide. This is the substance of the following proposition.

6.1.13 Proposition. *Let B be regular. Then the following are equivalent:*

(i) *B is hyperregular.*
(ii) *$\Theta[B]$ is an infinite theory.*
(iii) *$f \leqslant_w B$ iff $f \leqslant_d B$, whenever f is integer-valued.*

In this setting the result is due to V. Stoltenberg-Hansen [163]. For the proof we need to be a bit more careful in how we construct $\Theta[B]$. Let B_1 and B_2 be disjoint sets and define a theory $\Theta[B_1, B_2]$ by the following modification of the construction of $\Theta[B]$. Let b be the index in $\Theta[B]$ which introduces the characteristic function of B. Then if $b, x, 0, \langle b, x, 0\rangle \in L^\beta$ and $x \in B_1$ we add $(b, x, 0)$ to $\Theta^\beta[B_1, B_2]$. And if $b, x, 1, \langle b, x, 1\rangle \in L^\beta$ and $x \in B_2$ we add $(b, x, 1)$ to $\Theta^\beta[B_1, B_2]$. Obviously $\Theta[B] = \Theta[B, A - B]$, where A is the domain of Θ. Now introduce

$$^mH_{z,w}^x = \{\langle a, \sigma, y\rangle : (a, \sigma, y) \in \Theta^{|x|}[K_z, K_w], \operatorname{lh}(\sigma) = m\}.$$

An analysis of the definitions will show that $^mH_{z,w}^x$ is Θ-finite uniformly in the parameters m, x, z, w. And we further note that if

$$\langle a, \sigma, y\rangle \in {}^mH_{z,w}^x \ \wedge\ K_z \subseteq B \ \wedge\ K_w \cap B = \varnothing,$$

then $(a, \sigma, y) \in \Theta^{|x|}[B]$.

We now return to the proof of Proposition 6.1.13.

(i) ⇒ (ii). It suffices to show that the inductive definition of $\Theta[B]$ closes off at the ordinal $|\preccurlyeq|$. This reduces to studying the case of bounded universal quantification. So assume that $(a, y, 1) \in \Theta^{<|\preccurlyeq|}[B]$ (i.e. has been added before stage $|\preccurlyeq|$) for each $y \prec x$. We must show that $(a_0, a, z, 1) \in \Theta^{<|\preccurlyeq|}[B]$, where a_0 is a code for $\mathbf{E}^{\preccurlyeq}$ in $\Theta[B]$.

By regularity of B there are for each $y \prec x$ some z, w_1, w_2 such that $\langle a, y, 1\rangle \in {}^1H_{w_1,w_2}^z$ where $K_{w_1} \subseteq B$ and $K_{w_2} \cap B = \varnothing$. But now we can play with our notation. Letting $W^z = \{\langle y, w_1, w_2\rangle \in L^z : \langle a, y, 1\rangle \in {}^1H_{w_1,w_2}^z\}$, we see that $L^x \subseteq W^B$ (where $\lambda z \cdot W^z$ is a $\preccurlyeq$-enumeration of W). By hyperregularity of B there is a z such that $L^x \subseteq {}^zW^B$. But then $(a, y, 1) \in \Theta^{|z|}[B]$ for all $y \prec x$, and hence $(a_0, a, x, 1) \in \Theta^{<|\preccurlyeq|}[B]$.

(ii) ⇒ (iii). Let f be $\Theta[B]$-computable with index a. We then see that

$$f(\sigma) \simeq y \quad \text{iff} \quad \exists\beta < |\preccurlyeq|((a, \sigma, y) \in \Theta^\beta[B])$$
$$\text{iff} \quad \exists z, w_1, w_2(\langle a, \sigma, y\rangle \in {}^mH^z_{w_1,w_2} \ \wedge \ K_{w_1} \subseteq B \ \wedge \ K_{w_2} \cap B = \varnothing).$$

It follows that $f \leqslant_w B$.

(iii) ⇒ (i). Let V be $\Theta[B]$-semicomputable, it follows from (iii) that V is of the form W^B for some Θ-semicomputable set W. Letting

$$V^z = \{x \prec z : \exists w_1 w_2 \prec z(\langle x, w_1, w_2\rangle \in W^z \ \wedge \ K_{w_1} \subseteq B \ \wedge \ K_{w_2} \cap B = \varnothing)\}$$

we see that $\lambda z \cdot V^z$ is a $\preccurlyeq$-enumeration of V in $\Theta[B]$. Since every $\Theta[B]$-semicomputable set has a $\preccurlyeq$-enumeration in $\Theta[B]$ it follows that the domain A is $\Theta[B]$-infinite.

Let now $K \subseteq W^B_a$ where K is Θ-finite. Introduce the relation $F(x, z)$ by defining z to be a minimal element such that $x \in {}^zW^B_a$. Let F_x be non-empty $\Theta[B]$-finite subset of $\{z : F(x, z)\}$ and let $M = \bigcup\{F_x : x \in K\}$. Then M is $\Theta[B]$-finite and hence bounded by some $w \in A$ (since the domain A is $\Theta[B]$-infinite). Then $K \subseteq {}^wW^B_a$, so B is hyperregular.

6.1.14 Remark. We now observe that when B is regular and hyperregular, then for any set C, $C \leqslant B$ iff $C \leqslant_a B$. Just let $f(z, w) \simeq 0$ iff $K_z \subseteq A \ \wedge \ K_w \cap A = \varnothing$ in 6.1.13 (iii).

Hyperregularity and the relation $\leqslant_a$ is a digression from the main line of development of this chapter, whereas regularity is not. The importance of regularity comes from the following observation. Let $\lambda z \cdot W^z$ be a $\preccurlyeq$-enumeration of the Θ-semicomputable set W. Let $V^z = W^z - \bigcup\{W^w : w \prec z\}$. Then $\lambda z \cdot V^z$ is a disjoint $\preccurlyeq$-enumeration of W. And W is regular iff $(\forall\beta < |\preccurlyeq|)(\exists z)(\forall w \succ z)\cdot(V^w \cap L^\beta = \varnothing)$. This means that in enumerating W, given any level β, there is a stage z after which we always enumerate beyond β.

The anomaly of non-regularity can be circumvented by the following theorem when studying Θ-semicomputable degrees for adequate theories.

6.1.15 Theorem. *Let Θ be an adequate theory. Then for every Θ-semicomputable set B there is a regular Θ-semicomputable set D such that $B \equiv D$. D can be taken to consist of levels, i.e. $\forall x(\forall y \sim x)(x \in D \rightarrow y \in D)$.*

This was proved in the context of α-recursion theory by Sacks [140]; his proof was simplified by Simpson [153]. For adequate theories the result is due to Stoltenberg-Hansen [164], who had to go back to the original and more complicated proof of Sacks due to the lack of a well-ordering of the domain.

We shall not prove the general version in this book. For many purposes a simpler result is sufficient. This we now develop.

6.1.16 Lemma. *Let Θ be an infinite theory, and B a Θ-semicomputable non Θ-finite* *set. Let $\lambda z \cdot B^z$ be a disjoint enumeration of B such that each B^z is non-empty and contained in one level of the* pwo $\preccurlyeq$. *Then the deficiency set of B,*

$$D = \{z : (\exists w \succ z)(B^w \prec B^z)\},$$

is a regular Θ-semicomputable set with unbounded complement, and further $\forall x(\forall y \sim x)(x \in D \rightarrow y \in D)$.

It is clear that D is Θ-semicomputable with unbounded complement. To prove regularity it suffices to show that $D \cap L^x$ is Θ-finite for each $x \in A$.

Fix x and let $z_0 = x$. Suppose that we have defined $z_0, z_1, \ldots, z_n$. Choose, if possible, z_{n+1} such that $z_{n+1} \succ z_n$ and $B^{z_{n+1}} \prec B^{z_n}$. By the well-foundedness of $\prec$ the sequence is finite. Let z_n be its last element. It is then easily seen that

$$D \cap L^x = \{z \prec x : (\exists w \preccurlyeq z_n)(w \succ z \;\wedge\; B^w \prec B^z)\},$$

which is Θ-finite.

We shall apply the construction of the lemma in two situations, both important for the theory in Section 6.2.

6.1.17 Corollary. *If the set B of Lemma* 6.1.16 *is regular, then $B \equiv D$.*

To show that $B \leqslant D$ we define a relation $Q(z, w)$ iff w is minimal such that $w \notin D \wedge K_z \subseteq L^{B^w}$. Observe then that $K_z \cap B = \varnothing$ iff $\exists w[Q(z, w) \;\wedge\; K_z \cap B^{\prec w} = \varnothing]$. (Note that when the sets involved are Θ-semicomputable we need only worry about the "negative" requirements $K \cap B = \varnothing$; the "positive" requirements $K \subseteq B$ take care of themselves.)

To prove that $D \leqslant B$ we first introduce a relation $F(z, w)$ iff w is minimal such that $(\forall w_1 \in K_z)(B^{\preccurlyeq w_1} \subseteq L^w)$. Then we define $N(z, w)$ iff w is of minimal level such that $\exists w_1[F(z, w_1) \;\wedge\; L^{w_1} - B^{\prec w} \subseteq \bar{B}]$. We see that $K_u \cap D = \varnothing$ iff $(\forall z \in K_u)(\exists w)[N(z, w) \;\wedge\; (\forall w_1 \preccurlyeq w)(w_1 \preccurlyeq z \;\vee\; B^z \preccurlyeq B^{w_1})$.

Where did we use the regularity of B? Simply to know that given z there is some w such that $Q(z, w)$, and similarly for N.

We shall now state our approximation to 6.1.15. The result in the multiple-valued setting is due to Stoltenberg-Hansen [162].

6.1.18 Theorem. *Let Θ be an adequate theory. Then for every Θ-semicomputable non Θ-computable set B there are regular Θ-semicomputable sets D_1 and D_2 such that D_1 is not Θ-computable and $D_1 \leqslant B \leqslant D_2$.*

From B we shall construct two sets B_1^* and B_2^* and then let D_1 and D_2 be the deficiency sets of B_1^* and B_2^*, respectively.

For the definition of B_2^* assume that B is not regular. Then by adequacy, $|\preccurlyeq|^* < |\preccurlyeq|$. Let p be a projection into $L^{|\preccurlyeq|^*}$ and set $B_2^* = \{z : p(z)\downarrow \;\wedge\; K_{p(z)} \cap B \neq \varnothing\}$. D_2 will be the deficiency set of B_2^*. We leave the proof of $B \leqslant D_2$ to the reader. (It is not entirely trivial, but see [162] for details.)

We now turn to the existence of D_1. Since B is not regular, there is an x such that $B \cap L^x$ is not regular. Let $B_1^* = p^{-1}(B \cap L^x)$. (At this point recall the discussion following Definition 6.1.4.) We observe that $K \subseteq \bar{B}^*$ iff $K \cap p^{-1}(L^x) \subseteq \overline{p^{-1}(B)}$ iff $p(K \cap p^{-1}(L^x)) \subseteq \bar{B}$, from which we conclude that $B_1^* \leqslant B$.

Toward defining D_1 we first note that B_1^* is not Θ-finite since $p(B_1^*) = B \cap L^x$ is not Θ-finite. Let $\lambda w \cdot B^w$ be an enumeration of B_1^* as described in Lemma 6.1.16 and let D_1 be the deficiency set of B_1^* with respect to this enumeration. Then one verifies that

$$K \subseteq \bar{D}_1 \quad \text{iff} \quad \bigcup_{z \in K} (L^{B^z} - B^{\prec z}) \subseteq \bar{B}_1^*.$$

Hence $D_1 \leqslant B_1^* \leqslant B$.

Finally, if D_1 were Θ-computable we see that

$$x \notin B_1^* \quad \text{iff} \quad x \geqslant |{\preccurlyeq}|^* \ \vee \ \exists z (x \prec B^z \ \wedge \ z \notin D \ \wedge \ x \notin B^{\prec z}).$$

This means that $\bar{B}_1^*$ would be Θ-semicomputable, hence B_1^* would be Θ-finite. But we argued above that it is not. (Note that the adequacy of Θ is used to ensure the existence of a suitable z for the last equivalence; $x < |{\preccurlyeq}|^*$, hence by adequacy there must be a z_0 such that $x \prec B^z$ for all $z \succcurlyeq z_0$. Since $\bar{D}$ is unbounded, there must be a z such that $x \prec B^z \ \wedge \ z \notin D$.)

We conclude this section by two definitions.

6.1.19 Definition. A set B is *many-one reducible* to a set C, $B \leqslant_m C$, if there is a Θ-computable mapping $\lambda z \cdot H_z$ where H_z is a non-empty Θ-finite set, such that

(i) $x \in B$ iff $H_x \subseteq C$
(ii) $x \notin B$ iff $H_x \cap C = \varnothing$.

6.1.20 Definition. The jump of a set B is the set

$$B' = \{a : \exists z, w(\langle z, w\rangle \in W_a \ \wedge \ K_z \subseteq B \ \wedge \ K_w \cap B = \varnothing\}.$$

Some basic facts now follow, e.g. a set D is weakly Θ-semicomputable in B iff $D \leqslant_m B'$.

6.2 The Splitting Theorem

We shall give one example of a non-trivial degree-theoretic result.

6.2.1 Theorem. *Let Θ be an adequate theory. Let C be a regular Θ-semicomputable set and let D be a Θ-semicomputable non Θ-computable set. Then there exist Θ-semicomputable sets A and B such that $C = A \cup B$, $A \cap B = \varnothing$, $A \leqslant C$, $B \leqslant C$ and*

(i) $\Theta[A]$ *and* $\Theta[B]$ *are adequate theories (so in particular A and B are hyperregular)*

(ii) $A' \equiv B' \equiv O'$

(iii) $D \not\leqslant_w A$ *and* $D \not\leqslant_w B$.

The splitting theorem in ORT is due to G. E. Sacks [138]. In the context of α-recursion theory it was proved by R. A. Shore [150]. S. Simpson could prove in the context of "thin" admissible sets that there are Θ-semicomputable sets A, B such that $A \not\leqslant_w B$ and $B \not\leqslant_w A$ [154]. The strong version above is due to V. Stoltenberg-Hansen [163]. We must, however, make one reservation; Stoltenberg-Hansen needs to assume for parts (i) and (ii) that the theory Θ has a *reasonable pairing function*. By this we mean that for each $\alpha < |\preccurlyeq|^*$ there is a $\beta < |\preccurlyeq|^*$ such that $L^\alpha \times L^\alpha = \{\langle x, y\rangle; x, y \in L^\alpha\} \subseteq L^\beta$. It is not known whether every adequate theory Θ admits a reasonable pairing function.

We shall in this section prove the following weak version of the splitting theorem.

6.2.2 Theorem. *Let Θ be an adequate theory. Let C be a regular Θ-semicomputable set and let D be a regular Θ-semicomputable non Θ-computable set. Then there are Θ-semicomputable sets A and B such that $C = A \cup B$, $A \cap B = \varnothing$, $A \leqslant C$, $B \leqslant C$, $D \not\leqslant_w A$ and $D \not\leqslant_w B$.*

6.2.3 Remarks. We have the usual corollaries. First note that if A and B are disjoint regular Θ-semicomputable sets, then the join of deg(A) and deg(B), deg(A) $\vee$ deg(B), is deg($A \cup B$). If we let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ range over Θ-semicomputable degrees, we can from 6.2.2 and 6.1.18 conclude that

$$(\forall \mathbf{c} > 0)(\exists \mathbf{a}, \mathbf{b})(\mathbf{a} \vee \mathbf{b} \leqslant \mathbf{c} \wedge \mathbf{a} < \mathbf{c} \wedge \mathbf{b} < \mathbf{c} \wedge \mathbf{a}|\mathbf{b}),$$

where as usual $\mathbf{a}|\mathbf{b}$ means that $\mathbf{a}$ and $\mathbf{b}$ are incomparable. Using the regular set Theorem 6.1.15 we may draw the stronger conclusion that $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$.

Also note that the same results are true for *d*-degrees, i.e. degrees with respect to the relation $\leqslant_d$, see Definition 6.1.10; this is a consequence of 6.1.14 and 6.2.1.

But before we turn to a proof of Theorem 6.2.2 we have to develop a certain "blocking" technique due to Shore [150]. The reason for this is that when the domain of an infinite theory is not computably well-ordered one cannot consider a unique requirement at a given stage of a priority argument. But it will be possible to handle Θ-finite blocks at each stage.

The naive way to do this is to let one level of the pwo of the domain make up one block. And in his thesis [162] Stoltenberg-Hansen was able to obtain a weak positive solution to Post problem in this way.

But stronger results need more refined blocking techniques, even in the context of α-recursion theory. As noted above, this was developed by Shore [150] (see also his survey [152] for further motivation). Simpson [154] observed that this technique also worked for "thin" admissible sets. For adequate theories in general this was developed by Stoltenberg-Hansen [162, 163]. We present here a version for single-valued theories following closely the exposition in [163].

Remark. We have yielded to tradition and used A in the statement of the splitting theorem. For the rest of this section we will use U for the domain of Θ.

As always there are some technical preliminaries. A relation $F(\sigma, z)$ on the domain of Θ induces, in certain circumstances a function on the associated ordinal $|\preccurlyeq|$ of the domain. Let $\sigma \sim \sigma'$, where $\sigma = (x_1, \ldots, x_n)$ and $\sigma' = (x'_1, \ldots, x'_n)$, mean that $x_i \sim x'_i$, $i = 1, \ldots, n$. If F satisfies the requirement that

$$F(\sigma, z) \ \wedge \ F(\sigma', z') \ \wedge \ \sigma \sim \sigma' \Rightarrow z \sim z',$$

then F induces a function f on $|\preccurlyeq|$. We classify f in terms of the associated relation F. Thus f is called Θ-computable if the associated F is Θ-computable. It is called Σ_n if F is Σ_n, where we use the usual Σ_n, Π_n hierarchy starting with $\Sigma_0 = \Pi_0 =$ Θ-computable. For functions on $|\preccurlyeq|$ we use the standard notion of limit

$$\lim_\alpha f'(\alpha, \gamma) = \delta \quad \text{iff} \quad (\exists\beta)(\forall\alpha \geqslant \beta)[f'(\alpha, \gamma) = \delta].$$

With this bit of terminology we have the following standard approximation result.

6.2.4 Lemma. *Let Θ be adequate and f a total Σ_2 function on $|\preccurlyeq|$. Then there is a total Θ-computable f' on $|\preccurlyeq|$ such that* $\lim_\alpha f'(\alpha, \gamma) = f(\gamma)$.

The reader may first establish the following part of Post's theorem: If B is Σ_{n+1} then B is weakly Θ-semicomputable in a Π_n set. From this we may conclude that if B is Σ_2 then B is weakly Θ-semicomputable in a Θ-semicomputable set A, which by 6.1.18 can be taken to be regular.

Let G_f be the graph on the domain of Θ of the function f. By assumption G_f is Σ_2, hence by the remark above G_f is weakly Θ-semicomputable in a regular Θ-semicomputable set A *via* some Θ-semicomputable set W. Let $\lambda z \cdot A^z$ and $\lambda z \cdot W^z$ be $\preccurlyeq$-enumerations of A and W, respectively.

Let N_x^z be the Θ-finite set of all minimal $w \prec z$ such that

$$(\exists y \prec z)(\exists x' \sim x)[\langle x', y, w\rangle \in W^z \ \wedge \ K_w \cap A^z = \varnothing].$$

We define a relation F' by the following requirements: If $N_x^z = \varnothing$, then $\langle z, x, z\rangle \in F'$. If $N_x^z \neq \varnothing$, then

$$\langle z, x, y\rangle \in F' \quad \text{iff} \quad y \text{ is a minimal element such that } (\exists w \in N_x^z)(\exists x' \sim x)[\langle x', y, w\rangle \in W^z].$$

F' is Θ-computable and induces a total function f' on $|\preccurlyeq|$. We must prove that f' converges to f.

Suppose $f(\alpha) = \beta$. Choose elements x, y such that $|x| = \alpha$, $|y| = \beta$ and

$\langle x, y\rangle \in G_f$. Since G_f is weakly Θ-semicomputable in A *via* W, choose some z such that

$$\langle x, y, z\rangle \in W \ \wedge \ K_z \cap A = \varnothing .$$

By the regularity of A we can choose w so large that $y \prec w$, $\langle x, y, z\rangle \in W^z$ and $(U - A) \cap L^z = (U - A^w) \cap L^z$. We want to show that for $\gamma \geqslant |w|, f'(\gamma, \alpha) = \beta$, i.e. $\lim_\gamma f'(\gamma, \alpha) = f(\alpha)$.

Let $w' \succcurlyeq w$. Then $N_x^{w'} \neq \varnothing$ since (possibly except for minimality) z is a candidate for membership. (Note that the enumeration for finite sets is such that $K_z \subseteq L^z$.) Let $u \in N_x^{w'}$. There are elements $x' \sim x$ and y' such that

$$\langle x', y', u\rangle \in W^{w'} \ \wedge \ K_u \cap A^{w'} = \varnothing .$$

Since $u \preccurlyeq z$ and $K_u \subseteq L^u$, $K_u \cap A = \varnothing$. But then $\langle x', y', u\rangle$ is a correct computation of f, i.e. $f(|x'|) = |y'|$, since $|x'| = \alpha$, $|y'| = \beta$. The definition of F' shows that $\langle w', x', y'\rangle \in F'$, i.e. letting $\gamma = |w'|$ we get $f'(\gamma, \alpha) = \beta$, and convergence is proved.

6.2.5 Definition. The Σ_2-cf(α) is the least ordinal β for which there is a Σ_2 function f with domain β and range unbounded in α.

6.2.6 Lemma. *Let Θ be adequate. Then* Σ_2-cf$(|\preccurlyeq|) = \Sigma_2$-cf$(|\preccurlyeq|^*)$.

Let k be a total Θ-computable function on $|\preccurlyeq|$ with range in $|\preccurlyeq|^*$ such that $\{\beta; k(\beta) < \alpha\}$ is bounded for each $\alpha < |\preccurlyeq|^*$. k can be defined from a $\preccurlyeq$-enumeration of a Θ-semicomputable non Θ-computable set $W \subseteq L^{|\preccurlyeq|^*}$.

Let f be Σ_2 with domain β and unbounded in $|\preccurlyeq|$. Then $g(\alpha) = k(f(\alpha))$ is Σ_2 and unbounded in $|\preccurlyeq|^*$. This proves Σ_2-cf$(|\preccurlyeq|^*) \leqslant \Sigma_2$-cf$(|\preccurlyeq|)$.

For the converse, let f be Σ_2 with domain β and unbounded in $|\preccurlyeq|^*$. Let $g(\alpha) = \mu\gamma[(\forall \xi \geqslant \gamma)(f(\alpha) < k(\xi))]$. g is unbounded in $|\preccurlyeq|$. Use 6.2.4 to write f as a limit. Then a simple quantifier analysis shows that g is Σ_2.

By a $\preccurlyeq$-*sequence* of Θ-semicomputable sets we mean a Θ-computable mapping r such that $x \sim y \Rightarrow W_{r(x)} = W_{r(y)}$.

6.2.7 Lemma. *Suppose $\alpha < \Sigma_2$-cf$(|\preccurlyeq|)$ and $\langle I_x : x \prec \alpha\rangle$ is a $\preccurlyeq$-sequence of Θ-semicomputable sets such that I_x is Θ-finite for each $x \prec \alpha$. Then $\bigcup_{x \prec \alpha} I_x$ is Θ-finite.*

This simple lemma is crucial for the later priority construction. The proof is by contradiction. Let α be the least ordinal for which we have a sequence $\langle I_x : x \prec \alpha\rangle$ whose union is not Θ-finite. Let $W = \bigcup_{x \prec \alpha} I_x$ and let $\lambda z \cdot W^z$ be a $\preccurlyeq$-enumeration of W. Define a function g on the associated ordinal $|\preccurlyeq|$ by

$$g(\gamma) = \mu\eta[I_\gamma \subseteq W^\eta].$$

Of course, we should have defined a relation G and let g be the induced function.

In any case, g is Σ_2. g is defined for every $\beta \prec \alpha$, but $g(\alpha)$ is unbounded in $|\preccurlyeq|$ since W is not Θ-finite, hence $\Sigma_2\text{-cf}(|\preccurlyeq|) \leqslant \alpha$.

6.2.8 Blocking Procedure. *Suppose that Θ is an adequate theory. The projectum $L^{|\preccurlyeq|^*}$ can be divided into $\Sigma_2\text{-cf}(|\preccurlyeq|)$ many Θ-finite blocks M_α, each bounded strictly below $|\preccurlyeq|^*$. Each block M_α can be approximated by Θ-finite sets M_α^z. The approximation is uniform in α and z, and is "tame" in the sense that*

$$(\forall \alpha < \Sigma_2\text{-cf}(|\preccurlyeq|))(\exists z)(\forall w \succcurlyeq z)(\forall \beta < \alpha)[M_\beta^w = M_\beta].$$

We know that $\Sigma_2\text{-cf}(|\preccurlyeq|) \leqslant |\preccurlyeq|^*$. If we have equality, we simply set $M_\alpha = M_\alpha^z = \{x : x \sim \alpha\}$. More care is needed when $\Sigma_2\text{-cf}(|\preccurlyeq|) < |\preccurlyeq|^*$.

Let g be a Σ_2 function from the $\Sigma_2\text{-cf}(|\preccurlyeq|)$ to $|\preccurlyeq|^*$ unbounded in $|\preccurlyeq|^*$, and let g' be Θ-computable such that $\lim_\sigma g'(\sigma, \alpha) = g(\alpha)$. These functions exist by 6.2.4 and 6.2.6.

Define an ordinal function

$$h(\sigma, \alpha) = \mu\gamma[(\forall \beta < \alpha)(g'(\sigma, \beta) < \gamma)].$$

Since $\alpha < \Sigma_2\text{-cf}(|\preccurlyeq|)$ there always exists a $\gamma < |\preccurlyeq|^*$ satisfying the requirements inside the brackets $[\dots]$. Now put

$$M_\alpha^z = \{\varepsilon : h(|z|, \alpha) \leqslant |\varepsilon| < h(|z|, \alpha + 1)\}.$$

Each M_α^z is obviously bounded strictly below $|\preccurlyeq|^*$. But we also need to know that a canonical Θ-index can be obtained for M_α^z uniformly in α and z.

To this purpose, let $H(z, a, x)$ be a Θ-computable relation which induces the function h. By the selection principle 6.1.3 we have Θ-finite set $H_{z,a}$ uniformly in z, a such that $x \in H_{z,a}$ implies $H(z, a, x)$. Next observe that given any a in the domain of the pwo $\preccurlyeq$ there is a Θ-finite set S_a (uniform in a) of "successor" notations, i.e. if $a' \in S_a$ then $|a'| = |a| + 1$. Let $H_{z,a}^* = \bigcup \{H_{z,b} : b \in S_a\}$. Then $H_{z,a}^*$ is Θ-finite, and if $y \in H_{z,a}^*$, then $|y| = h(|z|, |a| + 1)$. From $H_{a,z}$ and $H_{a,z}^*$ we now define M_a^z.

To show that the approximation is tame, let

$$I_\beta = \{w : (\exists w' \succ w)(g'(|w'|, \beta) \neq g'(|w|, \beta))\}.$$

Fix $\alpha < \Sigma_2\text{-cf}(|\preccurlyeq|)$. Then $\langle I_\beta : \beta < \alpha + 1\rangle$ is a $\preccurlyeq$-sequence of Θ-semicomputable sets such that each I_β is Θ-finite. By Lemma 6.2.7 the union is Θ-finite. Hence there is some z such that for all $\beta \leqslant \alpha$ and all $w \succcurlyeq z$ we must have $g'(|w|, \beta) = g'(|z|, \beta)$, i.e. tameness.

Let $M_\beta = M_\beta^z$ for sufficiently large z. It remains to verify that

$$L^{|\preccurlyeq|^*} = \bigcup \{M_\beta : \beta < \Sigma_2\text{-cf}(|\preccurlyeq|)\}.$$

Fix $\varepsilon \prec |\preccurlyeq|^*$ and choose the least α for which $\varepsilon \prec h(\sigma, \alpha)$ where σ is fixed and sufficiently large. Such α exists since g is unbounded in $|\preccurlyeq|^*$. By the definition of h there exists a $\beta < \alpha$ such that $\varepsilon \preccurlyeq g'(\sigma, \beta)$. But then $\varepsilon \prec h(\sigma, \beta + 1)$, which by the choice of α means that $\alpha = \beta + 1$. By the minimality of α we also get $h(\sigma, \beta) \preccurlyeq \varepsilon$, hence $\varepsilon \in M_\beta^g$.

We can now return to the proof of Theorem 6.2.2. Let the sets C and D be given. By 6.1.16 and 6.1.17 we may assume that D satisfies in addition the requirement $\forall x(\forall y \sim x)(x \in D \Rightarrow y \in D)$. Let $\lambda z \cdot D^z$ be a $\preccurlyeq$-enumeration of D and $\lambda z \cdot C^z$ a disjoint $\preccurlyeq$-enumeration of C. The sets A and B will be defined via $\preccurlyeq$-enumerations $\lambda z \cdot A^z$ and $\lambda z \cdot B^z$, inductively on the pwo $\preccurlyeq$. If $z \sim w$ then the construction at stages z and w will be identical, but indices may differ. At stage z, C^z will be added to precisely one of $A^{\prec z}$ and $B^{\prec z}$, where $A^{\prec z}$ means $\bigcup \{A^w : w \prec z\}$.

This will ensure that A and B will be Θ-semicomputable, $C = A \cup B$, and $A \cap B = \varnothing$. It is also easy to see that $A \leqslant C$ and $B \leqslant C$. We simply have

$$K_z \cap A = \varnothing \quad \text{iff} \quad \exists w[(K_z - C^{\prec w}) \cap C = \varnothing \ \wedge \ K_z \cap A^w = \varnothing].$$

By regularity of C there is always an element w such that $(K_z - C^{\prec w}) \cap C = \varnothing$.

It remains to ensure that $D \nleqslant_w A$ and $D \nleqslant_w B$. We restrict attention to A, the case for B being similar. It suffices to show that $U - D$ is not weakly Θ-semicomputable in A, i.e. for no index ε is $(U - D) = W_\varepsilon^A$. (For notation see the paragraph immediately preceding 6.1.12.)

To violate the equality $(U - D) = W_\varepsilon^A$ we follow the original procedure of Sacks [138]. The idea is to try to preserve computations $x \in W_\varepsilon^A$ for minimal x not in D. In case $(U - D) = W_\varepsilon^A$ for some ε we would eventually preserve a correct computation for each $x \in W_\varepsilon^A$. But then W_ε^A would be Θ-semicomputable and this would contradict the assumption that D is non Θ-computable. Hence computations $x \in W_\varepsilon^A$ will eventually stop being preserved, and we will eventually violate the equality $U - D = W_\varepsilon^A$.

But there are obstacles to overcome. We need e.g. to have Θ-finite blocks of requirements to settle down by some stage of the computation; to this end we can use the blocking procedure developed in 6.2.8, letting each block play the role of a single requirement in trying to preserve a computation $x \in W_\varepsilon^A$ for $x \notin D$ and some ε in the block considered. And we use the fact that D has the property $\forall x(\forall y \sim x)(x \in D \Rightarrow y \in D)$ to avoid the problem of never finishing creating requirements with arguments from a fixed level of the pwo $\preccurlyeq$; there is a need to create a requirement preserving a computation $x \in W_\varepsilon^A$ only if no other computation $y \in W_\varepsilon^A$ for $y \sim x$ is being preserved.

We turn to the details of the construction. Let M_α^z and M_α for $\alpha < \Sigma_2\text{-cf}(|\preccurlyeq|)$ be the Θ-finite blocks described in 6.2.8. Sets R_A and R_B of requirements will be created, R_A to ensure that $D \nleqslant_w A$. S_A will denote the set of A-requirements, i.e. requirements in R_A, injured during the construction. R_A^z and S_A^z are the Θ-finite parts of R_A and S_A obtained by stage z. Each requirement will be of the form $\langle \varepsilon, x, F \rangle$ where F is (a canonical Θ-index for) a Θ-finite set. Such a requirement in

R_A is called an *ε-A requirement* or an *α-A requirement* (at z) in case $\varepsilon \in M_\alpha$ ($\varepsilon \in M_\alpha^z$): it is said to have argument x. In case $F \cap A^z = \varnothing$ it is said to be *active* at z, else it is *inactive*. $\varepsilon \in M_\alpha^z$ is an *inactive α-A reduction procedure at z* in case there is an active ε-A requirement in R_A^z preserving a computation $x \in W_\varepsilon^A$ for some $x \in D^z$, i.e. there is $\langle \varepsilon, x, F\rangle \in R_A^{\prec z} - S_A^{\prec z}$ such that

$$(\exists w \prec z)[\langle x, w\rangle \in W_\varepsilon^z \ \wedge \ K_w \subseteq F \ \wedge \ x \in D^z].$$

If no such requirement exists, then ε is an *active α-A reduction procedure at z.*

Let $r\colon |\preccurlyeq| \to \Sigma_2\text{-cf}(|\preccurlyeq|)$ be a Θ-computable function such that

$$(\forall \alpha < \Sigma_2\text{-cf}(|\preccurlyeq|))(\forall \beta)(\exists \gamma > \beta)(r(\gamma) = \alpha), \text{ where } \alpha, \beta, \gamma \text{ vary over } |\preccurlyeq|.$$

r indicates which part of the construction to concern ourselves with at a given stage.

The construction at stage z: Let $r(z) = \alpha$. As remarked above, we only treat the A-requirements, the construction of B-requirements being similar.

We recall the motivation above. Every block M_α plays the role of a single A-requirement. This means that if there is an $\varepsilon \in M_\alpha$ which is an active α-A reduction procedure and if there is an active ε-A requirement with argument x, then there is no need to create a new α-A requirement with argument $x' \sim x$. Otherwise we shall contemplate creating new requirements.

To this end let H^z be the Θ-finite set of minimal x such that for each $x' \sim x$, $x' \notin D^z$ and $\neg(\exists\langle \varepsilon, x', F\rangle \in R_A^{\prec z} - S^{\prec z})$ ("ε is an active α-A reduction procedure at z").

Next let

$$N^z = \{\langle \varepsilon, x\rangle \in M_\alpha^z \times H^z : \text{“}\varepsilon \text{ is an active } \alpha\text{-}A \text{ reduction procedure at } z\text{”} \ \wedge \ (\exists w \prec z)[\langle x, w\rangle \in W_\varepsilon^z \ \wedge \ K_w \cap A^{\prec z} = \varnothing]\}.$$

and

$$F_\varepsilon^z = \bigcup \{K_w : (\exists x \in H^z)[\langle x, w\rangle \in W_\varepsilon^z \ \wedge \ K_w \cap A^{\prec z} = \varnothing]\}.$$

Then put

$$R_A^z = R_A^{\prec z} \cup \{\langle \varepsilon, x, F_\varepsilon^z\rangle : \langle \varepsilon, x\rangle \in N^z\}.$$

We must now decide whether to add C^z to $A^{\prec z}$ or to $B^{\prec z}$. Let

$$J_A^z = \{\langle \varepsilon, x, F\rangle \in R_A^z - S_A^{\prec z} : F \cap C^z \neq \varnothing\},$$

i.e. J_A^z is the set of active A requirements which would be injured in case C^z were added to A. Define

$$f_A(z) = \mu\beta[(\exists\langle \varepsilon, x, F\rangle \in J_A^z)(\varepsilon \in M_\beta^z)],$$

if such an ordinal β exists, otherwise let $f_A(z) = |\preccurlyeq|$. In a similar way introduce a function $f_B(z)$. An analysis of the definition tells us that we can Θ-computably decide whether $f_A(z) \leqslant f_B(z)$ or $f_B(z) < f_A(z)$.

If $f_A(z) \leqslant f_B(z)$, let $B^z = B^{\prec z} \cup C^z$ and $A^z = A^{\prec z}$. If $f_B(z) < f_A(z)$, let $A^z = A^{\prec z} \cup C^z$ and $B^z = B^{\prec z}$.

To complete the construction let

$$S_A^z = \{\langle \varepsilon, x, F\rangle \in R_A^z : F \cap A^z \neq \varnothing\}.$$

6.2.9 Lemma. *For each $\alpha < \Sigma_2\text{-cf}(|\preccurlyeq|)$ the set of α-A and α-B requirements is Θ-finite.*

This is proved by induction on $\alpha < \Sigma_2\text{-cf}(|\preccurlyeq|)$. Fix α and assume that the set of β-A and β-B requirements is Θ-finite for each $\beta < \alpha$. By the tameness of the blocking there is a stage z_0 by which all blocks M_β^z for $\beta \leqslant \alpha$ have settled down. Let

$$I_\beta = \{z \succ z_0 : (\exists \langle \varepsilon, x, F\rangle \in R_A^z \cup R_B^z - R_A^{\prec z} \cup R_B^{\prec z})(\varepsilon \in M_\beta^z)\}.$$

Then I_β is Θ-finite for each $\beta < \alpha$ by the induction hypothesis, thus $\bigcup_{\beta<\alpha} I_\beta$ is Θ-finite by Lemma 6.2.7. C is assumed regular so there exists a stage $z_1 \succcurlyeq z_0$ such that all β-requirements for $\beta < \alpha$ have been created by z_1 and no such β-requirement will meet C^w for any $w \succcurlyeq z_1$. It follows that $f_A(w) \geqslant \alpha$ and $f_B(w) \geqslant \alpha$ for $w \succcurlyeq z_1$. Hence, by the way we have arranged the priorities, no α-A requirement will be injured beyond z_1.

This means that an α-A reduction procedure inactive at some $w \succcurlyeq z_1$ will remain inactive forever. We see that the set of α-A reduction procedures which becomes inactive beyond z_1 is Θ-semirecursive and hence Θ-finite. Thus there is a $z_2 \succcurlyeq z_1$ beyond which no α-A reduction procedure is made inactive.

Suppose $z_2 \preccurlyeq z \prec w$ and $r(z) = r(w) = \alpha$. From the choice of z_2 we see that $H^z \preccurlyeq H^w$, i.e. $x \in H^z$ and $y \in H^w$ implies that $x \preccurlyeq y$. Moreover, if an α-A requirement is created at z then $H^z \prec H^w$. From this we may conclude that either the set of α-A requirements it Θ-finite, or for each $x \notin D$ there is a permanent α-A requirement $\langle \varepsilon, x', F\rangle$ where $x' \sim x$ and ε is a reduction procedure active beyond z_2.

But the latter cannot be the case since D then would be Θ-computable, in fact

$$x \notin D \quad \text{iff} \quad (\exists w \succcurlyeq z_2)(\exists x' \sim x)(\exists \langle \varepsilon, x', F\rangle \in R_A^w - S_A^w)$$
("ε is an active α-A reduction procedure at w").

This completes the proof that the set of α-A requirements is Θ-finite. Using the regularity of C choose $z_3 \succcurlyeq z_2$ so large that all α-A requirements have been created and such that no C^w will meet an α-A requirement for $w \succcurlyeq z_3$. No α-B requirement is injured beyond z_3 since $f_A(w) > \alpha$ whenever $w \succcurlyeq z_3$. Now repeat the argument above with B in place of A and starting with z_3 in place of z_1 and conclude that the set of α-B requirements is also Θ-finite.

This ends the proof of Lemma 6.2.9.

It remains to show that $U - D$ is not weakly Θ-semicomputable in A. We argue by contradiction, so suppose that $(U - D) = W_\varepsilon^A$. Choose α and z_0 such that $\varepsilon \in M_\alpha$, all α-A requirements have settled down by stage z_0, and no $\delta \in M_\alpha$ becomes an inactive α-A reduction procedure beyond z_0. Note that ε is an active α-A reduction procedure at z_0, otherwise an erroneous computation would be preserved. Choose a minimal $x \notin D$ such that there is no $x' \sim x$ for which $\langle \delta, x', F \rangle \in R_A^{z_0} - S_A^{z_0}$ where δ is an active α-A reduction procedure at z_0. By the regularity of D there is a stage $z_1 \succcurlyeq z_0$ such that $L^x \cap D = L^x \cap D^{z_1}$. Let $w \succcurlyeq z_1$ be such that $x \in {}^wW_\varepsilon^A$ and $r(w) = \alpha$. Then $H^w = \{x' : x' \sim x\}$ and $\langle \varepsilon, x \rangle \in N^w$. It follows that an ε-requirement with argument x will be created at w, contradicting the fact that $w \succcurlyeq z_0$.

The proof of Theorem 6.2.2 is now complete.

6.3 The Theory Extended

In the last section we gave a non-trivial finite injury argument in a non-wellordered setting: the Sacks' splitting theorem for an arbitrary adequate theory.

But one example is no general proof that the class of adequate theories is the "correct" setting of an axiomatic degree theory. Beyond the finite lies the domain of infinite injuries.

The splitting theorem was our paradigm for the finite injury arguments. The density theorem could be a test case for infinite injury case.

6.3.1 Density Theorem. *Let* **a** *and* **b** *be two Θ-semicomputable degrees such that* **a** $<$ **b**. *There exists a Θ-semicomputable degree* **c** *such that* **a** $<$ **c** $<$ **b**.

In the context of ORT over ω this theorem was proved by G. Sacks [139]. R. A. Shore was able to further refine the techniques he used in [150] (such as the blocking technique 6.2.8) to produce a proof of the density theorem in the setting of α-recursion theory (see [151]). Beyond this the question is open: Give an adequate axiomatic analysis of the density theorem! Which is an injunction to study infinite injury arguments in the abstract.

6.3.2 Remark. There exists at least one example in the context of adequate theories. Stoltenberg-Hansen has proved (unpublished) by an infinite injury argument that if $\Theta[\mathbf{O}']$ is adequate, then there exists a Θ-semicomputable set $A < \mathbf{O}'$ such that $A' \equiv \mathbf{O}''$.

Leaving infinite injuries aside we may still ask whether adequate theories is a "good" category for degree theory. In a certain sense it is *reasonable*. We started with a class of infinite theories corresponding to admissible pwo's, i.e. resolvable admissible structures (possibly with urelements), and we imposed some necessary properties on the projectum: Keep in mind that the Θ-semicomputable sets can be indexed below the projectum; our requirement of adequacy stated that the projectum is a limit ordinal and every Θ-semicomputable set bounded below the projectum is in fact Θ-finite. This is a key combinatorial property, see e.g. Lemma 6.2.9.

The "key combinatorial property" is provable in α-recursion theory, but not in the general setting of infinite theories (in the precise sense of Definition 5.1.5.) We discuss two examples.

6.3.3 Determinacy and Degrees: a Counterexample to Post's Problem. We present a result due to S. Simpson [155]. First a definition: Let M be an admissible set. $B \subseteq M$ is *complete*-$\Sigma(M)$ if B is $\Sigma(M)$ and for each $\Sigma(M)$ set $A \subseteq M$ there is a $\Sigma(M)$ relation C such that:

(a) $\forall x \exists y C(x, y)$

(b) $\forall x \forall y [C(x, y) \rightarrow (x \in A \leftrightarrow y \in B)]$,

i.e. A is $\leqslant_m$-reducible to B.

Simpson proved the following

Theorem. *Assume the Axiom of Determinateness. Let* $M = \mathbb{R}^+$, *the next admissible set after the continuum. Then every* $\Sigma(M)$ *set is either* $\Delta(M)$ *or complete and every regular* $\Sigma(M)$ *set is* $\Delta(M)$.

Here we have a total breakdown of the theory of semicomputable degrees. Note that M is an admissible pwo, hence supports an infinite theory. The only blemish of the counterexample is the use of AD.

Hence the obvious question: Can one get rid of determinacy? There is an unpublished example due to L. Harrington which answers this in the affirmative, but his admissible set is not resolvable.

Thus some restriction, perhaps adequacy, seems necessary. However, there is a rich degree theory in certain non-adequate contexts.

6.3.4 A Non-adequate Theory with a "Good" Degree Structure. This example is due to D. Normann and V. Stoltenberg-Hansen [130]. The setting is Barwise's theory of admissible sets with urelements and the structure is $L(\omega_1)_{\mathfrak{M}}$, where $\mathfrak{M} = V^{\omega}(\mathbb{Q})$ is a countable-dimensional vectorspace over $\mathbb{Q}$.

$L(\omega_1)_{\mathfrak{M}}$ has a degree structure isomorphic to $L(\omega_1)$. This comes from the fact that the structure $\mathfrak{M} = V^{\omega}(\mathbb{Q})$ has a *natural representative* in $L(\omega_1)$. ($\mathfrak{M}' \in L(\omega_1)$ is a natural representative of $\mathfrak{M}$ if (i) $\mathfrak{M}'$ and $\mathfrak{M}$ are isomorphic, and (ii) the set of finite (in the good old sense!) $\tau: \mathfrak{M} \rightarrow \mathfrak{M}'$ which can be extended to an isomorphism from $\mathfrak{M}$ to $\mathfrak{M}'$ is $L(\omega_1)_{\mathfrak{M}}$-computable.) A definability analysis shows that $V^{\omega}(\mathbb{Q})$ has indeed a natural representative in $L(\omega_1)$. And thus the degree theory of $L(\omega_1)_{\mathfrak{M}}$ is reduced to the *adequate* theory $L(\omega_1)$.

But, as shown by Normann and Stoltenberg-Hansen, $L(\omega_1)_{\mathfrak{M}}$ being obviously admissible and resolvable, is *not* adequate. It does, however, satisfy the "regular set" theorem.

We sketch the proof that the structure is not adequate: Let $<$ be a Σ_1 pwo of the structure $L(\omega_1)_{\mathfrak{M}}$, and let $\pi(a)$ be a projection, i.e. a set of "notations" for a below the projectum. An analysis of the construction shows that π and $<$ are definable in a finite set of parameters $p_1, \ldots, p_k$ from the underlying structure. Thus there is a finite-dimensional subspace $\mathfrak{M}_0$ of $\mathfrak{M}$ such that $p_1, \ldots, p_k$ belongs to $\mathfrak{M}_0$. We have the following *basic lemma*: For any two elements $r_1, r_2 \in \mathfrak{M}\backslash\mathfrak{M}_0$ if there is an automorphism τ of $\mathfrak{M}$ leaving $\mathfrak{M}_0$ fixed such that $\tau(r_1) = r_2$, then the sets $\pi(r_1)$ and $\pi(r_2)$ will be at the same level below the projectum of the pwo.

Choose an $r \in \mathfrak{M}\backslash\mathfrak{M}_0$. Let π_1 be a projectum of ω_1 into ω. Define π_2 for $\alpha < \omega_1$ by

$$\pi_2(\alpha) = \pi((1 + \pi_1(\alpha)\cdot r)).$$

It is easily seen that the image of the projection is semirecursive. By the basic lemma the image will be bounded below the projectum; we have a set of non-zero elements $1 + \pi_1(\alpha)\cdot r$ in $\mathfrak{M}\backslash\mathfrak{M}_0$, for any two such elements an automorphism of the required kind exists, hence their projections will lie at the same level.

The projection is not $L(\omega_1)_{\mathfrak{M}}$-finite; for π_2 is essentially a projection of ω_1, the ordinal of the structure.

What is the moral of example 6.3.4? Perhaps this, that adequacy always lurks in the background?

So far *admissibility* has been a dogmatic assumption of the general theory. But irreverent questions began to be asked: Is Σ_1-admissibility a crude global hypothesis which obscures the finer points of recursion theory? (Sacks [144]). And irreverence is always an instrument for progress.

6.3.5 Inadmissible Extensions of the Theory. The basic ideas were announced in S. Friedman and G. Sacks [37] and have been extensively developed in papers of S. Friedman and W. Maass, see [35] and [101] for a preliminary guide to the field.

The Friedman-Sacks' setting for β-recursion theory is the S-hierarchy for L introduced by R. Jensen. An introduction to β-recursion theory from this point of view is given in Friedman [35]. But for our purposes we can equally well, as is done in Maass [101], stick to the more familiar L-hierarchy.

So replace the structure $\langle L_\alpha, \in\rangle$, α admissible, by $\langle L_\beta, \in\rangle$, where β is any limit ordinal. We must choose the "correct" notions of *semicomputable* and *finite* for the structures.

L_β admits a natural hierarchy; this motivates the following definitions: Let $A \subseteq L_\beta$

A is *β-r.e.* iff A is Σ_1-definable over $\langle L_\beta, \in\rangle$
A is *β-recursive* iff $A, \bar{A} = L_\beta - A$ are both β-r.e.

From this we derive, more or less canonically, the notion of a β-recursive function: A partial function f from L_β to L_β is called (partial) *β-recursive* iff its graph is β-r.e.

In the Friedman-Sacks' approach the notion of β-finite is the same as in the admissible case: Let $A \subseteq L_\beta$

A is *β-finite* iff $A \in L_\beta$.

One can now introduce the *reducibilities* of β-recursion theory: $\leqslant_{w\beta}$ and $\leqslant_{\beta}$, exactly as in Definition 6.1.9.

There is a threefold split in the recursion theory on the ordinals. Let $\kappa = \Sigma_1\text{-cf}(\beta)$, i.e. the least ordinal κ such that some Σ_1-function with domain κ has range unbounded in β. Let β^* be the usual (Σ_1-) projectum of β. Given any limit β,

(i) β is *admissible* iff $\kappa = \beta$.
(ii) β is called *weakly inadmissible* if $\beta^* \leqslant \kappa < \beta$.
(iii) β is called *strongly inadmissible* if $\kappa < \beta^*$.

The basic patterns of admissibility theory extend to the weakly inadmissible case. Below κ everything looks very admissible, and since $\beta^* \leqslant \kappa$ we can work below κ. W. Maass developed in [100] a technique which gave a precise meaning to this remark. He associated to each weakly inadmissible β an admissible structure $\mathfrak{A}_\beta$, the *admissible collapse*, such that $\mathfrak{A}_\beta$-r.e. degrees embed into the β-r.e. degrees. Thus one can transfer results from the admissible case to the weakly inadmissible one. A generalization of Maass' construction is given in V. Stoltenberg-Hansen [165].

There are, however, certain complications. "Strange" things start to happen in Friedman-Sacks' β-recursion theory. For example, it is not always possible to effectively enumerate the β-finite subsets of an β-r.e. set. The following definition is not trivial: A set A is said to be *tamely r.e.* (t.r.e.) if the set $\{a \in L_\beta : a \subseteq A\}$ is β-r.e. A β-recursive enumeration $\lambda\sigma \cdot A^\sigma$ of A is called *tame* if $a \subset A$ implies $\exists\sigma(a \subseteq A^\sigma)$. Of course, in the admissible case every enumeration of a β-r.e. set is tame; not so in the inadmissible case.

One can now give a quick summary of the results of Maass and Stoltenberg-Hansen: the structure of the regular t.r.e. β-degrees is non-trivial (and even rich) iff β is admissible or weakly inadmissible.

We will not go into details at this point since we shall outline an alternate approach to β-recursion theory in a moment. But a few results must be mentioned.

For any inadmissible β the following is true: Let W be a universal β-r.e. set. There is a β-recursive set A such that $\mathbf{0} <_\beta A <_\beta W$, and every β-recursive or tamely r.e. set is β-reducible to A, see Friedman [35].

Letting $\mathbf{0}^{1/2}$ denote the degree of A we always have at least three β-r.e. degrees, $\mathbf{0}$, $\mathbf{0}^{1/2}$, and $\mathbf{0}'$. In the weakly inadmissible case there are always more, we have incomparable β-r.e. degrees below $\mathbf{0}'$, even below $\mathbf{0}^{1/2}$.

The situation is at present more "confusing" in the strongly inadmissible case, i.e. when the $\Sigma_1\text{-cf}(\beta)$ is strictly smaller than β^*. Both results and methods become different.

For a large class of ordinals one may answer in the affirmative the following version of Post's problem:

(*) There are β-r.e. sets A, B such that $A \nleqslant_{w\beta} B$, $B \,{}_{w\beta}\!\nleqslant A$?

But there are also ordinals where (*) fails. In [36], S. Friedman shows that (*)

is not true when β is the ω-th primitive-recursively closed ordinal greater than $\aleph^L_{\omega_1}$. But what of the following version?

(**) There are β-r.e. sets A, B such that $A \not\leq_\beta B$, $B \not\leq_\beta A$?

Obviously (**) is true in the admissible or weakly inadmissible case. It is unknown if there is a strongly inadmissible β where (**) fails.

Let us return to basics. The definition of β-finite was lifted verbatim from the admissible case. And one may argue that it is natural from a set-theoretic point of view. But the necessity of the notion of *tameness* indicates that the concept of β-finiteness is awkward for a recursion-theoretic analysis.

W. Maass has recently reanalyzed the foundation of β-recursion theory. He accepts that Σ_1-definable over $\langle L_\beta, \in\rangle$ is the "correct" notion of β-r.e. This gives a unique choice for the notion of a partial β-recursive function.

In his analysis of finiteness Maass was guided by the time-honored principle of seeking a notion invariant under a suitable group of symmetries, in this case the group of β-recursive permutations of the domain.

One cannot start in thin air. Two elementary properties of finiteness seem almost unavoidable: (i) a "finite" set must be β-recursive; (ii) a "finite" set must be bounded. Granted this much Maass [102] proved the following result.

6.3.6 Proposition. *There is a largest class I of subsets of L_β satisfying*

(i) *every $K \in I$ is β-recursive,*
(ii) *every $K \in I$ is bounded, i.e. $K \subseteq L_\gamma$ for some $\gamma < \beta$,*
(iii) *if $K \in I$ and f is a β-recursive permutation of L_β, then $f''K \in I$.*

Moreover, I is explicitly given as

$$I = \{K \subseteq L_\beta : K \in L_\beta \wedge \beta\text{-card}(K) < \Sigma_1\text{-cf}(\beta)\}.$$

Note that if β is admissible we have the usual notion of finiteness of admissibility theory. We also note that the assumption on β is not known to be necessary.

We shall modify Maass' analysis a little, replacing (i) and the invariance property (iii) by

(i′) every $K \in I$ is β-r.e.

(iii′) if K is "finite", f a partial β-recursive function, and $K \subseteq \operatorname{dom} f$, then $f''K$ is "finite".

Our exposition is based on an unpublished note by V. Stoltenberg-Hansen.

One may argue what is more basic. Here we just note that the modified approach expresses the idea that the "finite" sets should be precisely those β-r.e. sets for which *every* enumeration is short compared with those of the universal β-r.e. set.

6.3.7 Lemma. *The class*

$$I = \{K \subseteq L_\beta : K \in L_\beta \wedge \beta\text{-card}(K) < \Sigma_1\text{-cf}(\beta)\}$$

is the largest class satisfying (i′), (ii), *and* (iii′).

It is easy to verify that I satisfies (i′), (ii), and (iii′). Suppose **C** is any class satisfying (i′), (ii), and (iii′) and not contained in I. Let $M \in \mathbf{C} - I$. We argue first that $M \in L_\beta$: Let $\lambda\sigma \cdot M^\sigma$ be a β-recursive enumeration of M which exists by (i′). Define for $x \in M$ a function

$$h(x) = \text{some } \sigma[x \in M^\sigma].$$

h is a partial β-recursive function and by (iii′) $h''M$ belongs to **C**. Thus $h''M$ is bounded using (ii); let σ_0 be a bound. But then $M = M^{\sigma_0} \in L_\beta$.

Since $M \in L_\beta - I$ there is an $f\colon \kappa \xrightarrow{1\text{-}1} M, f \in L_\beta$. But then $f''\kappa \in L_\beta$ since L_β is rudimentarily closed. Define a function $g\colon M \to \kappa$ by

$$g(x) = \begin{cases} f^{-1}(x) & \text{if } \; x \in f''\kappa \\ 0 & \text{otherwise.} \end{cases}$$

Let $h\colon \kappa \to \beta$ be β-recursive and unbounded. By property (iii′) $h \circ g''M \in \mathbf{C}$. But $g''M = \kappa$, therefore $h \circ g''M$ is unbounded, contradicting (ii).

6.3.8 Definition. A set $K \subseteq L_\beta$ is called *invariantly finite*, *i*-finite, if $K \in I$.

This is the notion which will replace the notion of β-finite as used by Friedman-Sacks.

6.3.9 Proposition. *A set $M \subseteq L_\beta$ is i-finite iff for every partial β-recursive function $\varphi(x, y)$ there is a partial β-recursive function $\psi(x)$ such that for all $x \in L_\beta$*

$$\psi(x) \simeq \begin{cases} 0 & \textit{if } \; \exists y \in M \cdot \varphi(x, y) \simeq 0 \\ 1 & \textit{if } \; \forall y \in M \cdot \varphi(x, y) \simeq 1. \end{cases}$$

Thus *i*-finiteness is nothing but the usual axiomatic notion of finiteness, viz. the computability of the functional $\mathbf{E}_M$, $M \in I$. The proof is not difficult. One can argue as in the following 6.3.12 that $\mathbf{E}_M$ is computable if $M \in I$. Conversely, the set $\{M \subseteq L_\beta : \mathbf{E}_M \text{ is computable}\}$ will satisfy (i′), (ii), and (iii′), thus by maximality of I be included in I.

We have the following useful fact:

6.3.10 Proposition. *I is β-recursive.*

Obviously I is β-r.e. As in the proof of 6.3.7 $x \in L_\beta - I$ iff $\exists f \in L_\beta[f\colon \kappa \xrightarrow{1\text{-}1} x]$.

We come to the main construction. Let $h\colon \kappa \to \beta$ be β-recursive and unbounded, and let $\exists y \cdot \varphi(e, x, y)$ be a universal $\Sigma_1(\langle L_\beta, \in\rangle)$ relation where φ is Δ_0. Define the following β-recursive relations:

(i) $U(x)$ iff $x \in L_\beta - I$
(ii) $\psi(e, x, \delta)$ iff $\delta < \kappa \wedge \exists y \in L_{h(\delta)} \cdot \varphi(e, x, y)$.

6.3.11 Definition. The structure

$$\mathfrak{A}_\beta = \langle L_\beta; U, I, \in \upharpoonright L_\beta \times I, \psi\rangle$$

is called the *admissible collapse* of L_β.

Here U is the class of urelements and I the class of sets. When working in $\mathfrak{A}_\beta$ we write $\in$ for the more correct but hopelessly pedantic $\in \upharpoonright L_\beta \times I$.

The construction of $\mathfrak{A}_\beta$ was suggested by Maass [102]. It is the "*i*-finite" version of the original construction [100].

6.3.12 Lemma. *Every* $\Delta_0(\mathfrak{A}_\beta)$ *relation is* $\Delta_1(L_\beta)$.

The following case is representative: Consider $\forall x \in a \cdot \varphi(x, q)$ where $a \in I$ and φ is $\Delta_0(\mathfrak{A}_\beta)$. By the induction hypothesis $\varphi(x, q)$ is $\Delta_1(L_\beta)$. Suppose that

$$\mathfrak{A}_\beta \models \varphi(x, q) \quad \text{iff} \quad L_\beta \models \exists z \theta(z, x, q),$$

where θ is $\Delta_0(L_\beta)$. Suppose further that $\mathfrak{A}_\beta \models \forall x \in a \cdot \varphi(x, q)$. Define f on a by $f(x) =$ some x such that $\theta(z, x, q)$. Since $a \in I$, $f''a$ is *i*-finite. Letting $b = f''a$ it follows that $L_\beta \models \forall x \in a \cdot \exists z \in b \cdot \theta(z, x, q)$. We conclude that

$$\mathfrak{A}_\beta \models \forall x \in a \cdot \varphi(x, q) \quad \text{iff} \quad L_\beta \models \exists b \cdot \forall x \in a \cdot \exists z \in b \cdot \theta(z, x, q),$$

i.e. $\forall x \in a \cdot \varphi(x, q)$ is $\Delta_1(L_\beta)$.

6.3.13 Theorem. $\mathfrak{A}_\beta$ *is an admissible set with urelements. Furthermore, a set* $W \subseteq L_\beta$ *is* β-r.e. *iff it is* $\mathfrak{A}_\beta$-r.e.

The proof that $\mathfrak{A}_\beta$ is admissible is straightforward. For example the union axiom follows since an *i*-finite union of *i*-finite sets is *i*-finite. Δ_0-separation follows since a β-recursive subset of an *i*-finite set is *i*-finite. And Δ_0-collection follows as in Lemma 6.3.12.

Now let $W \subseteq L_\beta$. If W is β-r.e. then for some index e, $x \in W$ iff $L_\beta \models \exists \delta < \kappa \cdot \psi(e, x, \delta)$ iff $\mathfrak{A}_\beta \models \exists \delta \psi(e, x, \delta)$, so W is $\mathfrak{A}_\beta$-r.e. Conversely if W is $\mathfrak{A}_\beta$-r.e. then $x \in W$ iff $\mathfrak{A}_\beta \models \exists z \varphi(x, z)$ where φ is $\Delta_0(\mathfrak{A}_\beta)$. By Lemma 6.3.12 φ is a $\Delta_1(L_\beta)$ relation; we conclude that W is β-r.e.

Over L_β we can now define the notion of an *i-degree* as in Definition 6.1.9. We shall use $\leqslant_{i\beta}$ for the associated reducibility notion. Obviously $A \leqslant_{i\beta} B$ iff

$A \leqslant_{\mathfrak{A}_\beta} B$. Thus the study of i-degrees over L_β is reduced to the study of degrees over the admissible structure $\mathfrak{A}_\beta$.

6.3.14 Theorem. *$\mathfrak{A}_\beta$ is resolvable iff β is admissible or weakly inadmissible.*

The admissible case is trivial, $\mathfrak{A}_\beta$ is then equal to L_β itself. Suppose β is weakly inadmissible. Then there is a β-recursive bijection $q: \kappa \leftrightarrow L_\beta$. For each $\gamma < \kappa$, $q''\gamma$ is i-finite. Thus q induces a β-recursive wellordering on L_β whose initial segments are i-finite, i.e. $\mathfrak{A}_\beta$ is resolvable.

For the converse assume that $\mathfrak{A}_\beta$ is resolvable. Let $\preccurlyeq$ be the induced prewellordering of L_β whose initial segments are $\mathfrak{A}_\beta$-finite. Let $<_\beta$ be the standard β-recursive wellordering of L_β. Define

$$x < y \quad \text{iff} \quad x \prec y \ \vee \ (x \sim y \ \wedge \ x <_\beta y).$$

Then $<$ is an $\mathfrak{A}_\beta$-recursive wellordering of L_β whose initial segments are $\mathfrak{A}_\beta$-finite. $\mathfrak{A}_\beta$ is therefore adequate and hence the deficiency set D (see 6.1.16) of a complete regular $\mathfrak{A}_\beta$-r.e. set is a β-r.e. non-β-recursive subset of κ. In particular, $D \notin L_\beta$; but then $\beta^* \leqslant \kappa$.

We have arrived at the following conclusion: *L_β with the notion of "finite" being i-finite in the sense of Definition* 6.3.8 *is an infinite and adequate computation theory iff β is either admissible or weakly inadmissible.*

Thus there is a natural computation-theoretic analysis of the weakly inadmissible case. But even in the strongly inadmissible case we have the beginnings of a computation-theoretic analysis. Theorem 6.3.13 still holds, but the associated computation theory is no longer resolvable. It is, however, s-normal and p-normal in the sense of the general theory. It is a topic for further research as to how far into the inadmissible one can extend the coherence and unity of concepts and methods that we find in the axiomatic analysis.