Part B

Finite Theories

Chapter 3

Finite Theories on One Type

This chapter develops a general theory abstracting the central features of hyperarithmetic theory or, equivalently, recursion in the functional ${}^2E$. "Higher" recursion theory started in the mid 1950's with S. C. Kleene's work on the analytic hierarchy, constructive ordinals, and the hyperarithmetic sets ([79–81]). A number of important contributions were at that time made by C. Spector, to which we shall return later. The theory of functionals and the connection between hyperarithmetic theory and recursion in ${}^2E$ was developed by S. C. Kleene in 1959 ([83]), and the technique of comparing length of computations, the prewellordering property, was introduced by R. Gandy in 1962 ([38]) and used by him to prove the basic selection principle for recursion in a normal type-2 functional.

The early work did not pay proper attention to the notion of finiteness. It was G. Kreisel in [89] and [91] who stressed the importance of this notion for generalized recursion theory (in particular, in the context of meta-recursion theory, see Chapter 5). Y. Moschovakis [113] gave the definition of finiteness which is now considered to be the appropriate one for the general theory (see Definition 2.5.3).

Prewellordering and *finiteness* are the two key concepts in the general theory. In the first section of this chapter we make some general remarks on notions of finiteness, but concentrate mainly on the effect of the prewellordering property for computations. In the second section we single out the important class of Spector theories, which is the "correct" generalization of hyperarithmetic theory. In a final section we tie up the present study with current work on inductive definability.

3.1 The Prewellordering Property

Let $\langle \Theta, < \rangle$ be a computation theory on a domain $\mathfrak{A} = \langle A, C, N; s, M, K, L\rangle$. We shall not in the sequel distinguish between the set of computations Θ and the coded set $\{\langle a, \sigma, z\rangle : (a, \sigma, z) \in \Theta\}$. For $x \in \Theta$ let $|x|_\Theta$ be the ordinal of the computation x, i.e. the ordinal of the set $\mathrm{S}_x = \{y \in \Theta : y < x\}$.

In Section 2.7 we called a theory Θ s-normal (subcomputation normal) if the sets S_x are uniformly Θ-finite for $x \in \Theta$. In many cases the related notion of p-normality ("p" for prewellordering) turns out to be more useful.

3.1.1 Definition. The theory Θ is called *p-normal* if there is a Θ-computable function p such that $p(x, y)\downarrow$ if $x \in \Theta$ or $y \in \Theta$ and

$$\begin{aligned} x \in \Theta \quad \text{and} \quad & |x|_\Theta \leqslant |y|_\Theta \Rightarrow p(x, y) = 0 \\ & |x|_\Theta > |y|_\Theta \Rightarrow p(x, y) = 1. \end{aligned}$$

This means that if $y \in \Theta$, then the set $\{x \in \Theta : |x|_\Theta \leqslant |y|_\Theta\}$ is Θ-computable. It is easy to organize ordinary recursion theory over ω in such a way that the resulting theory is p-normal but not s-normal.

Conversely, s-normality always implies a "weak" form of the prewellordering property: There is a Θ-computable function $q(x, y)$ such that if $x, y \in \Theta$, then

$$q(x, y) = 0 \quad \text{iff} \quad |x|_\Theta \leqslant |y|_\Theta.$$

Since we can computably quantify over finite sets we have the following recursion equation for q

$$|x|_\Theta \leqslant |y|_\Theta \quad \text{iff} \quad \forall x' \in \mathbf{S}_x \, \exists y' \in \mathbf{S}_y |x'|_\Theta \leqslant |y'|_\Theta.$$

If we assume that Θ has selection operators and that there is a Θ-semicomputable extension of the relation $<$ to all tuples (a, σ, z), s-normality implies p-normality. In this case we have the following recursion equations for the function p:

$$\begin{aligned} p(x, y) &= 0 \quad \text{if} \quad \forall x' \in \mathbf{S}_x \quad \exists y'[y' < y \wedge p(x', y') = 0] \\ p(x, y) &= 1 \quad \text{if} \quad \exists x'[x' < x \wedge \forall y' \in \mathbf{S}_y \cdot p(x', y') = 1]. \end{aligned}$$

These assumptions are satisfied in many cases. But we should note that the usual proofs of the existence of selection operators proceed via the prewellordering property.

To conclude this discussion we remark that if the domain A is Θ-finite then p-normality and s-normality lead to essentially the same class of theories, viz. the Spector theories of the next section.

We return to the general discussion. In Definition 2.5.3 we called a set $S \subseteq A$ Θ-finite if the functional

$$\mathbf{E}_S(f) \simeq \begin{cases} 0 & \text{if} \quad \exists x \in S[f(x) \simeq 0] \\ 1 & \text{if} \quad \forall x \in S[f(x) \simeq 1], \end{cases}$$

is weakly Θ-computable. We now introduce a notion of *weak* Θ-finiteness, and refer to the old notion as *strong* Θ-finiteness.

3.1.2 Definition. A set $S \subseteq A$ is called *weakly Θ-finite* if the functional

$$\mathbf{E}'_S(f) \simeq \begin{cases} 0 & \text{if} \quad \forall x \in S[f(x)\downarrow] \wedge \exists x \in S[f(x) \simeq 0] \\ 1 & \text{if} \quad \forall x \in S \quad \exists y \neq 0[f(x) \simeq y] \end{cases}$$

is weakly Θ-computable.

3.1.3 Examples. We list some familiar examples:

(1) Let Θ = ORT on the integers ω. Then Θ is p-normal and a subset $S \subseteq \omega$ is Θ-finite iff it is finite in the ordinary sense. Hence the domain ω is neither strongly nor weakly Θ-finite.

(2) Let $\Theta = \{(a, \sigma, z) : \{a\}(\sigma, F) \simeq z,\ \sigma$ and z integers$\}$ where F is a normal type-2 functional over ω and $\{a\}(\sigma, F) \simeq z$ is defined by Kleene's schemata S1–S9. (See either Kleene's original treatment in [83] or the more general development of recursion in a normal list in Chapter 4.)

Θ is p-normal and the domain is strongly Θ-finite. The proof of p-normality is based on the following recursion equations: (i) if $x \in \Theta$, then $|x| \leqslant |y|$ iff for all subcomputations x' of x there exists a subcomputation y' of y such that $|x'| \leqslant |y'|$; (ii) $|x| > |y|$ iff there exists a subcomputation x' of x such that $|x'| > |y'|$ for all subcomputations y' of y. Normality of F (i.e. the fact that ${}^2E$ is Kleene recursive in F) allows us to "compute" the quantifiers in (i) and (ii) and hence to define a function $p(x, y)$ with the properties of Definition 3.1.1. (For technical hints see the proof of Proposition 3.1.12.)

(3) Let $A = \bigcup_{i<n} \mathrm{Tp}(i)$, where $\mathrm{Tp}(i)$ is the set of objects of type i. Let F be normal of type $\geqslant n + 2$. Let $\Theta = \{(a, \sigma, z) : \{a\}(\sigma, F) \simeq z,\ z \in \omega$ and σ a list of arguments from $A\}$. $\{a\}(\sigma, F) \simeq z$ is again defined by Kleene's formula S1–S9.

Θ is p-normal and for all $i < n$ $\mathrm{Tp}(i)$ is strongly Θ-finite. If F is of type $n + 2$, then $\mathrm{Tp}(n)$ is weakly but not strongly Θ-finite. If F is of type $> n + 2$, then $\mathrm{Tp}(n)$ is strongly Θ-finite. (Proofs can be extracted from Chapter 4, where the basic references for this example are given.)

(4) As remarked in Section 2.7, computing relative to a *partial* type-2 functional is problematic even over the integers ω. Here is one case where we have p-normality. The example is due to P. Hinman [59], see also Aczel [3].

Let $\mathbf{Q}$ be a monotone quantifier, i.e. $\mathbf{Q} \subseteq 2^\omega$ and $A \in \mathbf{Q}$, $A \subseteq B$ implies $B \in \mathbf{Q}$. The dual of $\mathbf{Q}$ is defined by $\breve{\mathbf{Q}} = \{\omega - A : A \notin \mathbf{Q}\}$. Associated with $\mathbf{Q}$ is a partial type-2 functional $F^\#_{\mathbf{Q}}$ defined by

$$F^\#_{\mathbf{Q}}(f) \simeq \begin{cases} 0 & \text{if } \{x \mid f(x) = 0\} \in \breve{\mathbf{Q}} \\ 1 & \text{if } \{x \mid f(x) > 0\} \in \mathbf{Q}, \end{cases}$$

where f is a partial function. It is easy to see that $\mathbf{E}_\omega$, i.e. the extension of ${}^2E$ to partial objects, is nothing but $F^\#_{\{\omega\}}$, i.e. $\mathbf{Q} = \{\omega\}$.

The basic fact now is that recursion in ${}^2E$, $F^\#_{\mathbf{Q}}$ is p-normal. For an application see Example 3.3.7 below.

The choice of terminology "weak" versus "strong" is justified by the following simple proposition.

3.1.4 Proposition. *Let $\langle \Theta, < \rangle$ be a computation theory on $\mathfrak{A}$ and $S \subseteq A$. If S is strongly Θ-finite, then S is weakly Θ-finite.*

Proof. The domain of definition of $\mathbf{E}_S$ may be larger than the domain of $\mathbf{E}'_S$. We cut it down to the right size by the following simple trick. Let t be the constant function 1. Then $\mathbf{E}_S(\lambda x\cdot t(f(x)))$ is defined iff f is defined on all of S. And if $\mathbf{E}_S(\lambda x\cdot t(f(x)))\downarrow$, let $\mathbf{E}'_S(f) = \mathbf{E}_S(f)$.

In Section 2.4 we discussed various *closure properties* for the class of Θ-semicomputable relations. We note that if the domain A is strongly Θ-finite, then the class of Θ-semicomputable relations is closed under $\exists$-quantifier. From Section 2.4 we further conclude that if N is strongly Θ-finite, then the Θ-semicomputable relations are closed under $\vee$. The following simple observation shows that weak Θ-finiteness suffices to prove the closure of the Θ-semicomputable relations under $\forall$.

3.1.5 Proposition. *Let the domain A be weakly Θ-finite. $\forall x R(x, \sigma)$ is Θ-semicomputable if $R(x, \sigma)$ is, and an index for $\forall x R(x, \sigma)$ can be found uniformly from an index for R.*

Proof. Let e be an index such that $R(x, \sigma)$ iff $\{e\}(x, \sigma) = 1$, and $\{e\}(x, \sigma)$ is undefined whenever $\neg R(x, \sigma)$. Then $\forall x R(x, \sigma)$ iff $\mathbf{E}'_A(\lambda x\cdot\{e\}(x, \sigma)) = 1$.

We noted in Section 2.4 that if Θ has a selection operator over N, then a relation R is Θ-computable iff $R, \neg R$ are Θ-semicomputable. We shall now prove that p-normality of Θ gives us selection operators over ω, see Definition 2.4.3.

3.1.6 Theorem. *A p-normal theory Θ admits selection operators over N.*

The following proof should be compared with the definition of the μ-operator in 1.4.1. (See also the discussion in Section 1.8.) Let e' be an index (computable from e) such that $\{e'\}(\tau) \simeq 0$ iff $\{e\}(\tau)\downarrow$ and such that $|e', \tau, 0|_\Theta > |e, \tau, y|_\Theta$, where $y \simeq \{e\}(\tau)$. Use the fixed-point theorem to define a function $\{e\}$ by

$$\{e\}(n, a, \sigma) \simeq \begin{cases} n, & \text{if} \quad |a, n, \sigma, 0|_\Theta < |e', n+1, a, \sigma, 0|_\Theta \\ \{e\}(n+1, a, \sigma), & \text{otherwise,} \end{cases}$$

p-normality is used in stating the condition on $\{e\}$. Let

$$q(a, \sigma) \simeq \{e\}(0, a, \sigma).$$

Then q is an n-ary selection operator over N, $n = \mathrm{lh}(\sigma)$. For the proof of this we first note:

i If $\{a\}(n, \sigma)\downarrow$, then $p(\langle a, n, \sigma, 0\rangle, \langle e', n+1, a, \sigma, 0\rangle)\downarrow$. Hence $\{e\}(n, a, \sigma)\downarrow$.
ii If $\{e\}(n+1, a, \sigma)\downarrow$, then $\{e\}(n, a, \sigma)\downarrow$.

From **i** and **ii** it follows that if $\exists n\cdot\{a\}(n, \sigma)\downarrow$, then $q(a, \sigma)\downarrow$.

Suppose next that $q(a, \sigma) \simeq k$. We will prove that $\{a\}(k, \sigma) \simeq 0$. First observe that for some n, $|a, n, \sigma, 0|_\Theta \leqslant |e', n+1, a, \sigma, 0|$; otherwise $\{e\}(n, a, \sigma) \simeq \{e\}(n+1, a, \sigma) \simeq k$ for all n, hence we would get an infinite descending chain $|e, 0, a, \sigma, k|_\Theta > |e, 1, a, \sigma, k|_\Theta > \ldots$.

Let $k_0 =$ least n such that $|a, n, \sigma, 0|_\Theta \leqslant |e', n+1, a, \sigma, 0|_\Theta$. Then $\{a\}(k_0, \sigma) \simeq$

0 and $\{e\}(k_0, a, \sigma) \simeq k_0$. Working backwards we get $k_0 = \{e\}(k_0, a, \sigma) = \ldots = \{e\}(0, a, \sigma) = k$. Thus $\{a\}(k, \sigma) \simeq 0$.

It is possible to arrange the construction so that if $\{a\}(n, \sigma) \simeq 0$ and $q(a, \sigma) \simeq n$, then $(a, n, \sigma, 0) < (\hat{q}, a, \sigma, n)$. In particular, $|a, n, \sigma, 0|_\Theta < |\hat{q}, a, \sigma, n|_\Theta$. This concludes the proof.

3.1.7 Corollary. *Let Θ be p-normal. Then the Θ-semicomputable relations are closed under disjunction and $\exists$-quantification over N. A relation R is Θ-computable iff R and $\neg R$ are Θ-semicomputable.*

3.1.8 Remark. We will not always spell out the uniformity involved in the various constructions. To be explicit in one example: There is a Θ-computable mapping r such that if $R_1(\sigma)$, $R_2(\sigma)$ are Θ-semicomputable with indices e_1, e_2, respectively, and $l = \mathrm{lh}(\sigma)$, then $r(e_1, e_2, l)$ is an index for $R_1(\sigma) \vee R_2(\sigma)$. And if $\{r(e_1, e_2, l)\}(\sigma) \simeq 0$, then

$$|r(e_1, e_2, l), \sigma, 0|_\Theta \geqslant \inf\{|e_1, \sigma, 0|_\Theta, |e_2, \sigma, 0|_\Theta\}.$$

Such extra information is often necessary when one is doing iterated constructions. But we will seldom make the details so explicit. Writing about recursion theory one must try to strike a proper balance between completeness in notations and exposition versus an attention to the mathematical core of an argument.

We shall digress for a moment to discuss further the relationship between strong and weak finiteness. The equivalence between the two notions is tied up with the existence of some sort of selection principle. Over ω we have as a consequence of p-normality the existence of selection operators. As we shall see in the next chapter, in higher types we have only the following *selection principle*: Let B be a non-empty Θ-semicomputable subset of the domain. We can effectively compute from the index of B an index of a Θ-computable non-empty subset $B_0 \subseteq B$. It is precisely this principle which allows us to go from weak to strong finiteness.

We start by formulating the principle more carefully. Let $S \subseteq A$:

(*) *There is a Θ-computable mapping r such that for all z, τ: If $B = \{x \in A : \{z\}(x, \tau) \simeq 0\}$, $B \subseteq S$, and $B \neq \varnothing$, then $\lambda x \cdot \{r(z, \mathrm{lh}(\tau))\}(x, \tau)$ is the characteristic function of a non-empty subset $B' \subseteq B$. If $B = \varnothing$, then $\lambda x \cdot \{r(z, \mathrm{lh}(\tau))\}(x, \tau)$ is totally undefined.*

3.1.9 Remark. If S satisfies the condition (*) it is possible to choose r such that, whenever $B \neq \varnothing$,

$$\inf\{|r(z, \mathrm{lh}(\tau)), x, \tau, 0|_\Theta : x \in B'\} \geqslant \inf\{|z, x, \tau, 0|_\Theta : x \in B\}.$$

3.1.10 Proposition. *Let Θ be p-normal and assume that $A = C$. Then for all $S \subseteq A$, S is strongly Θ-finite iff S is weakly Θ-finite and satisfies condition* (*).

Proof. (1) Suppose that S is strongly Θ-finite. From 3.1.4 we know that S is weakly Θ-finite. It remains to verify (*): Let B be a non-empty Θ-semicomputable subset of S, $B = \{x : \{z\}(x, \tau) \simeq 0\}$. $\mathbf{E}_S$ is Θ-computable and by assumption

$$\mathbf{E}_S(\lambda x\cdot\{z\}(x, \tau)) \simeq 0.$$

Let e_S be a Θ-index for $\mathbf{E}_S$; we may define

$$B' = \{x : |z, x, \tau, 0|_\Theta < |e_S, z_\tau, 0|_\Theta\},$$

where z_τ is an index for $\lambda x\cdot\{z\}(x, \tau)$ computable from z and τ. p-normality shows that B' is Θ-computable, and we may easily construct the function $r(z, n)$ as required by (*).

(2) Let S be weakly Θ-finite and satisfy (*). The functional $\mathbf{E}'_S$ is Θ-computable and S, being weakly finite, is Θ-computable. The following instructions give a procedure for computing $\mathbf{E}_S(\lambda x\cdot\{e\}(x, \tau))$:

Choose an index e' such that

$$\{e'\}(x, \tau) \simeq \begin{cases} \{e\}(x, \tau) & \text{if } x \in S \\ 1 & \text{if } x \notin S. \end{cases}$$

Let $B = \{x : \{e'\}(x, \tau) \simeq 0\}$; $B \subseteq S$ and if $B \neq \varnothing$, then $\lambda x\cdot\{r(e', n)\}(x, \tau)$ is the characteristic function of a non-empty subset $B' \subseteq B$. We now consider the following Θ-semicomputable relation

$$\begin{aligned} R(t, e, \tau) \quad \text{iff} \quad & (\mathbf{E}'_S(\lambda x\cdot\{r(e', \mathrm{lh}(\tau))\}(x, \tau)) = 0 \wedge t = 0) \\ & \vee (\mathbf{E}'_S(\lambda x\cdot\{e\}(x, \tau)) = 1 \wedge t = 1). \end{aligned}$$

By 3.1.6 we have a selection function $q^*(e, \tau)$, and we see that

$$q^*(e, \tau) = 0 \quad \text{iff} \quad \exists x \in S[\{e\}(x, \tau) \simeq 0],$$

and

$$q^*(e, \tau) = 1 \quad \text{iff} \quad \forall x \in S \quad \exists y \neq 0[\{e\}(x, \tau) \simeq y].$$

We may therefore set $\mathbf{E}_S(\lambda x\cdot\{e\}(x, \tau)) = q^*(e, \tau)$.

3.1.11 Corollary. *If Θ is p-normal, then N is strongly Θ-finite iff it is weakly Θ-finite.*

The proof is immediate since p-normality of Θ gives us selection operators over N, hence the validity of (*).

Let us at this point make the following *methodological remark*: From the above corollary we see that it does not really matter whether we use $\mathbf{E}_N$ or $\mathbf{E}'_N = {}^2E$ in defining recursion over N.

We shall include one more "useful" technical result. Let Θ be a computation theory on a domain $\mathfrak{A}$ and $\mathbf{R} = R_1, \ldots, R_n$ a list of relations on A. As in Definition 2.2.1 we can construct a theory $\Theta[\mathbf{E}_A, \mathbf{R}]$, where $\mathbf{E}_A$ is the strong quantifier on A.

Since $\mathbf{E}_A$ is a consistent partial functional on the domain, it is better to use the length function instead of the subcomputation relation, for reasons discussed in connection with Definition 2.2.3. We therefore assume that we have given a theory $\langle\Theta, |\ \ |_\Theta\rangle$, and we construct the set $\Theta[\mathbf{E}_A, \mathbf{R}]$ with the naturally associated length function.

From the general theory of Chapter 2 we know that the relations $\mathbf{R}$ are $\Theta[\mathbf{E}_A, \mathbf{R}]$-computable and that the domain A is strongly $\Theta[\mathbf{E}_A, \mathbf{R}]$-finite. Θ is imbeddable in $\Theta[\mathbf{E}_A, \mathbf{R}]$, and the imbedding function $r(a, n)$ can be chosen such that

$$|a, \sigma, z|_\Theta = |r(a, \mathrm{lh}(\sigma)), \sigma, z|_{\Theta[\mathbf{E}_A, \mathbf{R}]},$$

whenever $(a, \sigma, z) \in \Theta$.

We add the following complement to these results.

3.1.12 Proposition. *If Θ is p-normal, then so is $\Theta[\mathbf{E}_A, \mathbf{R}]$.*

Note that if Θ = ORT over ω, then this result shows that Kleene-recursion in ${}^2E$ over ω is a p-normal theory, hence by Theorem 3.1.6 has selection operators.

The idea behind the proof is described in example (2) of 3.1.3. We must analyze the construction of $\Theta[\mathbf{E}_A, \mathbf{R}]$ from Θ. The function p will be defined by cases via the fixed-point theorem. It will be convenient to omit the argument z from a computation tuple (a, σ, z). This implies no loss of information since our theories are single-valued. We comment below on why the omission of z is convenient, even necessary.

As a typical example let $x = (x_0, \hat{f}, \hat{g}, \sigma)$ be an "abbreviated" substitution and $y = (y_0, h)$ an application of $\mathbf{E}_A$.

We define two auxiliary functions for this case

$$\varphi_1(\hat{p}, x, y) \simeq \mathbf{E}_A(\lambda t \cdot \{\hat{p}\}((\hat{g}, \sigma), (\hat{h}, t)))$$
$$\varphi_2(\hat{p}, x, y) \simeq \mathbf{E}_A(\lambda t \cdot \{\hat{p}\}((\hat{f}, \{\hat{g}\}(\sigma), \sigma), (\hat{h}, t))).$$

(Here is one reason for the abbreviated computation tuple. If in φ_1 we had a subpart $(\hat{g}, \sigma, \{\hat{g}\}(\sigma))$, $\varphi_1(\hat{p}, x, y)$ would be undefined if $\{\hat{g}\}(\sigma)\uparrow$. But $\varphi_1(\hat{p}, x, y)$ shall be defined if y is defined.)

Define a function φ_3 by primitive recursion

$$\varphi_3(0, \hat{p}, x, y) \simeq \varphi_2(\hat{p}, x, y)$$
$$\varphi_3(n + 1, \hat{p}, x, t) \simeq 1.$$

Finally, let

$$\psi(\hat{p}, x, y) \simeq \varphi_3(\varphi_1(\hat{p}, x, y), \hat{p}, x, y).$$

ψ must also be defined for all other possibilities. This done we apply the recursion theorem to obtain a function p with $\Theta[\mathbf{E}_A, \mathbf{R}]$-code $\hat{p}$ such that $p(x, y) \simeq \psi(\hat{p}, x, y)$.

By induction on $\min\{|x|, |y|\}$ one shows that p satisfies the requirement in the definition of p-normality. As an example let us verify that

$$x \in \Theta[\mathbf{E}_A, \mathbf{R}] \wedge |x| \leqslant |y| \Rightarrow p(x, y) = 0,$$

in the case considered above.

From the assumption $x \in \Theta[\mathbf{E}_A, \mathbf{R}]$ we conclude that $(\hat{g}, \sigma), (\hat{f}, \{\hat{g}\}(\sigma), \sigma) \in \Theta[\mathbf{E}_A, \mathbf{R}]$ and have lengths less than $|x|$. By the induction hypothesis

$$p((\hat{g}, \sigma), (\hat{h}, t))\downarrow, \qquad p((\hat{f}, \{\hat{g}\}(\sigma), \sigma), (\hat{h}, t))\downarrow$$

for all t. Since $|x| \leqslant |y|$ we must further have

$$\exists t \cdot |\hat{g}, \sigma| \leqslant |\hat{h}, t| \quad \text{and} \quad \exists t \cdot |\hat{f}, \{\hat{g}\}(\sigma), \sigma| \leqslant |\hat{h}, t|,$$

i.e. $\exists t \cdot p((\hat{g}, \sigma), (\hat{h}, t)) \simeq 0$ and $\exists t \cdot p((\hat{f}, \{\hat{g}\}(\sigma), \sigma), (\hat{h}, t)) \simeq 0$. But then $\varphi_1(\hat{p}, x, y) \simeq 0$ and $\varphi_2(\hat{p}, x, y) \simeq 0$, and

$$p(x, y) \simeq \psi(\hat{p}, x, y) \simeq \varphi_3(0, \hat{p}, x, y) \simeq 0.$$

Thus the proposition is verified in this case.

This concludes our general discussion of the prewellordering property. Theorem 3.1.6 is the important result. The rest are necessary and sometimes useful housecleaning results. But now on to more substantial matters.

3.2 Spector Theories

The prewellordering property and finiteness of the computation domain come together in the notion of a *Spector theory*. This important class of computation theories was introduced by Y. Moschovakis in [113]. The name was chosen as a tribute to Clifford Spector's many and important contributions to hyperarithmetic theory, of which these theories is an appropriate general version. (We shall, as remarked above, return to Spector's work in connection with the imbedding theorem of Chapter 5.)

In this section we develop some basic "internal" results about Spector theories which lead to a general representation Theorem, 3.2.9. This theorem comes as a natural continuation of the Representation Theorems 1.6.3 and 2.7.3 of Part A and represents a theme which will be taken up again at several points in the further development of the theory, see, in particular, the discussions in Sections 5.4, 7.2, 7.3, and 8.3.

As a preliminary we shall introduce some suitable notations and terminology for the Θ-computable and Θ-semicomputable relations.

3.2.1 Definition. Let Θ be a computation theory on $\mathfrak{A}$.

$\mathrm{sc}^*(\Theta) = \{S \subseteq A$: there exists an index e and constants $a_1, \ldots, a_n \in A$ such that $\lambda x \cdot \{e\}_\Theta(x, a_1, \ldots, a_n)$ is the characteristic function for $S\}$.

$\mathrm{sc}(\Theta) = \{S \subseteq A$: there exists an index e such that $\lambda x \cdot \{e\}_\Theta(x)$ is the characteristic function for $S\}$.

Note, that if the constant functions are computable in Θ, then $\mathrm{sc}^*(\Theta) = \mathrm{sc}(\Theta)$.

$\mathrm{en}^*(\Theta) = \{S \subseteq A$: there exists an index e and constants $a_1, \ldots, a_n \in A$ such that $x \in S$ iff $\{e\}_\Theta(x, a_1, \ldots, a_n) \simeq 0\}$.

$\mathrm{en}(\Theta) = \{S \subseteq A$: there exists an index e such that $x \in S$ iff $\{e\}_\Theta(x) \simeq 0\}$.

One question we may ask is to what extent the *section* $\mathrm{sc}^*(\Theta)$ and the *envelope* $\mathrm{en}^*(\Theta)$ of a theory Θ determines the theory. We show in this section that the envelope determines the theory for the following class.

3.2.2 Definition. Let Θ be a computation theory on the domain $\mathfrak{A}$. Θ is called a *Spector theory* if

(1) $A = C$,
(2) $\mathbf{E}_A$ is Θ-computable, i.e. A is strongly Θ-finite,
(3) Θ is p-normal.

There are a number of remarks to make. First, from (1) it follows that $=_A$ is Θ-computable and that all constant functions are Θ-computable. We could, however, for the results of this section, only require that $=_A$ belongs to $\mathrm{sc}(\Theta)$ and that A might differ from C.

Next, the assumption that A is strongly Θ-finite means that $\mathrm{en}(\Theta)$ is closed under $\wedge$, $\vee$, $\exists_A$, and $\forall_A$.

Finally, p-normality implies that Θ has a selection operator over N. This means that we have a "good" notion of Θ-finiteness and that $\mathrm{sc}(\Theta) = \mathrm{en}(\Theta) \cap \neg\mathrm{en}(\Theta)$, where $\neg\mathrm{en}(\Theta)$ consists of the complements of sets in $\mathrm{en}(\Theta)$.

We now spell out in more detail the properties of $\mathrm{en}(\Theta)$:

3.2.3 Definition. A class Γ of relations on A is called a *Spector class* if it satisfies the following six conditions:

1. $=_A \in \Gamma \cap \neg\Gamma$.

2. Γ is closed under substitution, i.e. if $R(x_1, \ldots, x_i, \ldots, x_n) \in \Gamma$ and $a \in A$, then $R(x_1, \ldots, a, \ldots, x_n) \in \Gamma$.

3. Γ is closed under $\wedge$, $\vee$, $\exists_A$, and $\forall_A$.

4. A has a Γ-*coding scheme*. This means that there is a coding scheme $\langle N, \leqslant, \langle\,\rangle\rangle$ such that $\langle N, \leqslant\rangle$ is isomorphic to the natural numbers with the usual ordering and $\langle\,\rangle$ is an injection of $\bigcup_n A^n \to A$. Associated with $\langle N, \leqslant, \langle\,\rangle\rangle$ is a relation Seq which is the range of $\langle\,\rangle$ and functions lh, q where

$$\mathrm{lh}(x) = \begin{cases} 0, & \text{if } x \notin \mathrm{Seq} \\ n, & \text{if } x = \langle x_1, \ldots, x_n\rangle, \end{cases}$$

and

$$q(x, i) = \begin{cases} 0, & \text{if } \neg(x \in \mathrm{Seq} \wedge i \in N \wedge 1 \leqslant i \leqslant \mathrm{lh}(x)) \\ x_i, & \text{if } x = \langle x_1, \ldots, x_n \rangle \wedge 1 \leqslant i \leqslant n. \end{cases}$$

The coding scheme $\langle N, \leqslant, < > \rangle$ is called a Γ-coding scheme if the relations $x \in N$, $x \leqslant y$, $\mathrm{Seq}(x)$, $\mathrm{lh}(x) = y$, and $q(x, i) = y$ all belong to $\Gamma \cap \neg\Gamma$.

5. Γ is *parametrizable*, i.e. for each n there is an $n + 1$-ary relation $U_n \in \Gamma$ such that if $R(x_1, \ldots, x_n) \in \Gamma$, there is an $a \in A$ such that

$$R(x_1, \ldots, x_n) \quad \text{iff} \quad U_n(a, x_1, \ldots, x_n).$$

6. Γ is *normed*, i.e. every relation $R \in \Gamma$ has a Γ-norm, where $\sigma: R \to On$ is a Γ-norm on R if the associated prewellorderings:

$$\begin{aligned} x \leqslant_\sigma y \quad &\text{iff} \quad x \in R \wedge (y \in R \Rightarrow \sigma(x) \leqslant \sigma(y)) \\ x <_\sigma y \quad &\text{iff} \quad x \in R \wedge (y \in R \Rightarrow \sigma(x) < \sigma(y)) \end{aligned}$$

belong to Γ.

Note in connection with **6** that if $y \in R$ and $\neg(x <_\sigma y)$, then $(y \leqslant_\sigma x)$. We also note that the weak substitution property in **2** is sufficient to show, in conjunction with **1** and **3**, that Γ is closed under "trivial combinatorial substitutions" in the sense of Moschovakis [115, p. 165].

3.2.4 Proposition. *If* Θ *is a Spector theory, then* $\mathrm{en}(\Theta)$ *is a Spector class.*

The proof is immediate from the remarks made in connection with Definition 3.2.2.

3.2.5 Remark. The notion of a Spector class is a natural companion to the notion of a Spector theory, and was introduced by Y. Moschovakis in Chapter 9 of [115], to which we refer the reader for the elementary structure theory of these classes.

Here we only list a few of the classic, but elementary, consequences of the prewellordering property **6** of Definition 3.2.3. Assume that Γ is a Spector class on A.

(A) *Reduction Property:* Let P and Q be sets in Γ, there exist P_1, Q_1 in Γ such that $P_1 \subseteq P$, $Q_1 \subseteq Q$, $P_1 \cap Q_1 = \varnothing$, and $P_1 \cup Q_1 = P \cup Q$.

(B) *Separation Property:* For any disjoint pair of sets P, Q in $\neg\Gamma$ there is an $S \in \Delta = \Gamma \cap \neg\Gamma$ such that $P \subseteq S$ and $S \cap Q = \varnothing$.

(C) *Selection Principle:* Let $R(x, y)$ be in Γ. There is a set $R^* \subseteq R$ in Δ such that

$$\exists y R(x, y) \Rightarrow \exists y R^*(x, y).$$

Note that R^* is not necessarily the graph of a function.

We now prove the converse to Proposition 3.2.4.

3.2.6 Theorem. *Let Γ be a Spector class on A. There exists a Spector theory Θ on A such that $\Gamma = \mathrm{en}(\Theta)$.*

The proof should by now be entirely standard and we restrict ourselves to some brief remarks. The theory Θ will be defined by the usual kind of inductive clauses, not forgetting to include the appropriate clauses for the functional $\mathbf{E}_A$.

In this inductive definition the class Γ will be built in as follows: For each $n > 0$, let $U_n \in \Gamma$ be an $n + 1$-ary relation enumerating all the n-ary relations in Γ. Let σ_n be a Γ-norm on U_n. The appropriate inductive clause is now

$$(*) \qquad \text{If } (a, \sigma) \in U_n \text{ and } \forall a', \sigma'[(a', \sigma') <_{\sigma_n} (a, \sigma) \Rightarrow (\langle n_0, a'\rangle, \sigma', 0) \in X], \\ \text{then } (\langle n_0, a\rangle, \sigma, 0) \in \Lambda(X),$$

where Λ is the inductive operator being defined, and n_0 is some suitable natural number index for U_n. Note that this clause allows us to prove in the end that

$$\begin{aligned} (a, \sigma) \in U_n \quad &\text{iff} \quad (\langle n_0, a\rangle, \sigma, 0) \in \Theta \\ (a', \sigma') <_{\sigma_n} (a, \sigma) \quad &\text{iff} \quad |\langle n_0, a'\rangle, \sigma', 0| < |\langle n_0, a\rangle, \sigma, 0|. \end{aligned}$$

Θ is now the least fixed-point for Λ, and comes with the length function inherited from the inductive definition.

The inclusion $\Gamma \subseteq \mathrm{en}(\Theta)$ is immediate from the construction of Θ. The converse inclusion follows from the first recursion theorem for Spector classes:

3.2.7 First Recursion Theorem for Spector Classes. *Let Φ be a monotone operator and assume that Γ is uniformly closed under Φ. Then $\Phi^\infty \in \Gamma$.*

This is proved in Moschovakis [114]. Γ is *uniformly closed* under the operator Φ if the relation

$$Q(x, y) \quad \text{iff} \quad x \in \Phi(\{x' : P(x', y)\}),$$

is in Γ whenever P is in Γ. Since from our point of view a Spector class always is the envelope of some Spector theory and we do have the first recursion theorem for Spector theories (Theorem 2.3.1), we omit the proof of 3.2.7.

With 3.2.7 at hand it is easy to see that the set

$$B = \{x : \mathrm{Seq}(x) \wedge \exists n[n = \mathrm{lh}(x) \wedge x = \langle x_1, \ldots, x_n\rangle \wedge (x_1, \ldots, x_n) \in \Theta]\}$$

is in Γ, hence $\mathrm{en}(\Theta) \subseteq \Gamma$.

The proof of p-normality of Θ follows in the same way as in Proposition 3.1.12. There is, however, one new and important point to observe. In clause (*) we seem to refer to the relation $<_\sigma$ negatively. But this can be circumvented by the remark immediately following Definition 3.2.3, i.e. whenever $(a, \sigma) \in U_n$, then $\neg((a', \sigma') <_{\sigma_n} (a, \sigma))$ can be replaced by $(a, \sigma) \leqslant_{\sigma_n} (a', \sigma')$.

A final remark on the proof: The reader may be worried by the infinity of clauses introduced through the scheme (*), one clause for each U_n, $n = 1, 2, 3, \ldots$. But Γ has a coding scheme, so we may in (*) just take U_1.

This ends our remarks on Theorem 3.2.6.

Proposition 3.2.4 and Theorem 3.2.6 show that Spector theories and Spector classes are for most purposes interchangeable. The following result shows that the Spector class determines the theory up to equivalence.

3.2.8 Theorem. *Let* Θ_1 *and* Θ_2 *be Spector theories on* A. *Then* $\mathrm{en}(\Theta_1) = \mathrm{en}(\Theta_2)$ *iff* $\Theta_1 \sim \Theta_2$.

Equivalence obviously implies equality of envelopes. A proof of the converse can be based on the representation Theorem 2.7.3. Adapted to Spector theories with length instead of subcomputations we now have a Θ-computable functional F such that

$$\begin{aligned}&F(f, \langle a, \sigma\rangle) \simeq z \quad \text{iff} \quad (a, \sigma, z) \in \Theta \;\wedge\\ &\forall(b, \tau, w) \in S^*_{(a,\sigma,z)}[f(\langle b, \tau\rangle) \simeq w],\end{aligned}$$

where $(b, \tau, w) \in S^*_{(a,\sigma,z)}$ now means that $|b, \tau, w|_\Theta < |a, \sigma, z|_\Theta$. The same proof now shows that $\Theta \sim \mathrm{PR}[F]$.

Let us now start with Θ_1 and construct the corresponding F. One now shows that since Θ_2 is a Spector theory and $\mathrm{en}(\Theta_1) \subseteq \mathrm{en}(\Theta_2)$, F will also be Θ_2-computable. Hence, $\Theta_1 \leqslant \Theta_2$.

We conclude the discussion of this section by developing a general representation theorem for Spector theories. Since a Spector theory is s-normal we know from Theorem 2.7.3 that it can be written in the form $\mathrm{PR}[F^\#]$, where $F^\#$ is a consistent partial functional of type 2 over the domain. But this result is not entirely satisfactory for not every consistent partial $F^\#$ is s-normal. In the Spector case we can go one step further.

Let us start by analyzing the proof of 2.7.3. In the case of domain ω we define the representing $F^\#$ by the equation

$$\begin{aligned}F^\#(f, \langle a, \sigma, z\rangle) \simeq 0 \quad \text{iff} \quad &(a, \sigma, z) \in \Theta \wedge \forall(b, \tau, w)[(b, \tau, w) \in \mathbf{S}_{(a,\sigma z)}\\ &\Rightarrow f(\langle b, \tau, w\rangle) \simeq 0].\end{aligned}$$

$F^\#$ is an inductive operator Φ, viz.

(1) $\qquad c \in \Phi(X) \quad \text{iff} \quad c \in \Theta \wedge \forall y[y <_\Theta x \Rightarrow y \in X].$

It is well known that Φ has an associated monotone quantifier $\mathbf{Q}$,

(2) $\qquad X \in \mathbf{Q} \quad \text{iff} \quad \exists a[\exists b \cdot \langle a, b\rangle \in X \wedge a \in \Phi(X_a)],$

where $X_a = \{b : \langle a, b\rangle \in X\}$. Conversely, Φ can be recovered from $\mathbf{Q}$,

(3) $\qquad a \in \Phi(X) \quad \text{iff} \quad \langle a, X\rangle \in \mathbf{Q}.$

According to (4) of Example 3.1.3 we can pass from $\mathbf{Q}$ to a consistent partial functional $F_{\mathbf{Q}}^{\#}$:

$$(4) \qquad F_{\mathbf{Q}}^{\#}(f) \simeq \begin{cases} 0 & \text{if } \{x : f(x) = 0\} \in \breve{\mathbf{Q}} \\ 1 & \text{if } \{x : f(x) > 0\} \in \mathbf{Q}. \end{cases}$$

We know that $\mathrm{PR}[{}^2E, F_{\mathbf{Q}}^{\#}]$ is a Spector theory. Could we not obtain a satisfactory representation theorem for Spector theories by starting with the $F_{\mathbf{Q}}^{\#}$ provided by 2.7.3 and going through the construction above? It is, indeed, trivial to see that from a Spector theory Θ we get a $F_{\mathbf{Q}}^{\#}$ such that $\Theta \leqslant \mathrm{PR}[{}^2E, F_{\mathbf{Q}}^{\#}]$, but we have problems in proving the converse reduction, that $F_{\mathbf{Q}}^{\#}$ is Θ-computable. Our trouble stems from the $\breve{\mathbf{Q}}$-clause in the definition of $F_{\mathbf{Q}}^{\#}$.

Using a construction due to L. Harrington it is possible to get around this difficulty. In the construction of 2.7.3 we cared only about the "positive" part, building up the computation set Θ in stages through the inductive operator $F^{\#} = \Phi$. Now we have to be more careful.

Let Θ be a Spector theory, let $\Gamma = \mathrm{en}(\Theta)$ be the associated Spector class, and let P be universal in Γ, e.g. let P be the coded computation tuples. Define a set

$$R = \{\langle 0, a, b\rangle : b \in P \wedge a \leqslant_{\Theta} b\} \cup \{\langle 1, a, b\rangle : b \in P \wedge b <_{\Theta} a\}.$$

As usual let $R_i = \{\langle a, b\rangle : \langle i, a, b\rangle \in R\}$, $i = 0, 1$. From a norm on P we can introduce a norm $|\ \ |$ on R. We now introduce the following inductive operator Φ,

$$(5) \qquad c \in \Phi(X) \text{ iff } [X_0 \cap X_1 \neq \varnothing] \vee [c \in R \wedge \forall c' \in R(|c'| < |c| \to c' \in X)].$$

Note the similarity to (1).

From Φ we can construct a functional $F_{\mathbf{Q}}^{\#}$, see (2) and (4). The only difficult point in proving that $\Theta \sim \mathrm{PR}[\mathbf{E}_A, F_{\mathbf{Q}}^{\#}]$ is to verify the Θ-computability of $F_{\mathbf{Q}}^{\#}$. This reduces to an analysis of the $\breve{\mathbf{Q}}$-clause of (4), or, equivalently, to an analysis of the dual operator $\breve{\Phi}$ of Φ. By definition

$$\begin{aligned} (6) \qquad c \in \breve{\Phi}(Y) &\text{ iff } c \notin \Phi(A - Y) \\ &\text{ iff } Y_0 \cup Y_1 = A \wedge [c \notin R \vee \exists c' \in R(|c'| < |c| \wedge c' \in Y)]. \end{aligned}$$

(Recall the meaning of Y_0 and Y_1.)

Let $Y \in \Gamma$ be such that $Y_0 \cup Y_1 = A$. If we can in a Θ-computable way obtain from a code of Y an element $d \in P$ such that

$$(7) \qquad c \in R \wedge \forall c' \in R[|c'| < |c| \to c' \notin Y] \text{ implies } |c| \leqslant |d|,$$

then we can replace the clause $c \notin R$ in (6) by $c \notin R^{|d|}$ and conclude that $\breve{\Phi}(Y)$ is Θ-semicomputable. This will take care of the $\breve{\mathbf{Q}}$-clause in (4) and prove the following result.

3.2.9 Theorem. *Let Θ be a Spector theory on A. Then Θ is of the form* $\mathrm{PR}[\mathbf{E}_A, F_{\mathbf{Q}}^{\#}]$

for some monotone quantifier $\mathbf{Q}$ *on* A. *Conversely*, $\mathrm{PR}[\mathbf{E}_A, F^\#_{\mathbf{Q}}]$ *is always a Spector theory.*

The last part follows from (4) of Example 3.1.3. It remains to prove (7).

Let $c = \langle i, a, b\rangle$ be an element of R satisfying the premiss of (7). Introduce the sets

$$T^c_0 = \{\langle a', b'\rangle : a' \leqslant_\Theta b' \leqslant_\Theta b\},$$
$$T^c_1 = \{\langle a', b'\rangle : b' <_\Theta b, b' <_\Theta a'\}.$$

Remembering that $Y \in \Gamma$ is such that $Y_0 \cup Y_1 = A$, it is not difficult to verify (i) $T^c_0 \subseteq Y_1$, (ii) $T^c_1 \subseteq Y_0$, and (iii) $T^c_0 \cap T^c_1 = \varnothing$. We may now use the reduction property, 3.2.5 (A), to obtain sets X_0, X_1 such that $X_0 \cup X_1 = Y_0 \cup Y_1 = A$, $X_0 \cap X_1 = \varnothing$, and such that

$$\text{(8)} \qquad T^c_0 \subseteq X_0 \subseteq Y_1 \quad \text{and} \quad T^c_1 \subseteq X_1 \subseteq Y_0.$$

Note that both X_0, X_1 are Θ-computable and that an index for X_0 can be Θ-effectively computed from an index of Y as a Θ-computable set. Further, note that (8) is true of all c satisfying the premiss of (7). It remains to compute an element from X_0 which can serve as a bound for the conclusion of (7).

To this end introduce a set

$$W(X_0, b_1) = \{\langle a', b'\rangle : \langle b', a'\rangle \notin X_0 \ \wedge \ \langle b', b_1\rangle \in X_0\}.$$

$W(X_0, b_1)$ is Θ-computable, and it is easily seen that if $b_1 <_\Theta b$, then $W(X_0, b_1) = \{\langle a', b'\rangle : a' \leqslant_\Theta b' <_\Theta b_1\}$ is a well-founded set. From this we conclude that

$$\text{(9)} \qquad |b| \leqslant \sup\{|W(X_0, b_1)| : W(X_0, b_1) \text{ is well-founded}\}.$$

Here we seem to run against a serious obstacle, the notion of well-foundedness is not Θ-computable in every Spector theory Θ; recall that over ω well-foundedness means the Θ-computability of the functional E_1 which is the restriction to total arguments of the functional $E^\#_1$ discussed in Example 3.3.7 below.

However, independently of Theorem 3.2.9, we will prove in Section 5.4 that a Spector theory is of the form $\mathrm{PR}[^2G]$ for a total, normal type-2 G iff Θ is not Θ-Mahlo (see Definition 5.4.6). So we may assume in the proof of Theorem 3.2.9 that Θ is Θ-Mahlo. If this is the case the notion of well-foundedness is weakly Θ-computable. Thus we can compute within Θ from the inequality in (9) an element $d \in P$ giving a suitable bound for (7).

This completes the proof of Theorem 3.2.9. We should perhaps add an explanation as to why well-foundedness can be handled in the Θ-Mahlo case. Given a relation S we can easily construct a consistent partial functional $F^\#_S$, uniformly in S, such that if f_S is the least fixed-point of $F^\#_S$ then S is well-founded iff $f_S(0) \simeq 0$. (*Hint:* Brouwer-Kleene ordering.) If S is Θ-computable, which is the case if $S = W(W_0, b_1)$, then $F^\#_S$ is weakly Θ-computable, hence by Θ-Mahloness $F^\#_S$ is

Θ'-computable in some theory $\Theta' <_1 \Theta$ (see Definition 5.4.1). But then the graph f_S of the least fixed-point of $F_S^{\#}$ is Θ-computable, since $\mathrm{en}(\Theta') \subseteq \mathrm{sc}(\Theta)$. This is exactly what is needed to continue from (*g*).

3.2.10 Remark. We have regarded Theorem 3.2.9 as a natural extension of Theorem 2.7.3. Historically, this is not correct. Theorem 3.2.9 is due to L. Harrington who, working in the context of inductive definability and Spector classes, proved that *every Spector class is of the form* IND(**Q**). We see from Example 3.3.7 below that this is equivalent to the computation-theoretic version presented in 3.2.9. Harrington did not publish his proof, we have followed the exposition in A. Kechris [75].

3.3 Spector Theories and Inductive Definability

In recent years there have been several proposals to develop definability theory (descriptive set theory) and generalized recursion theory on the basis of a general theory of inductive definability. Strong advocates for this approach have been Robin Gandy, see e.g. his [39], and Yiannis Moschovakis, see in particular his book [115] and the papers [116] and [117]. The reader should also consult the work of Peter Aczel [4, 6], and [7].

We discussed in the introduction *computations* versus *inductive definability* as a foundation for general recursion theory, and shall not repeat that discussion here. Our aim in this section is to make a connection between computation theories, in particular Spector theories, and inductive definability.

3.3.1 Remark. For a general account of descriptive set theory the reader should consult the recent book of Moschovakis [118]. It follows from the results of this and the next chapter that the theory can be developed in the framework of computation theories, i.e. as finite theories on one or two types.

We assume that the reader is familiar with the basic facts of the theory of inductive definability. However, to fix notation we recall some of the definitions.

Let A be a set and Γ an operator on A, i.e. a map from 2^A to 2^A. Γ defines inductively a set $\Gamma_\infty \subseteq A$ by the following equation, where $\alpha \in \mathrm{O}n$:

$$\Gamma_\alpha = \bigcup_{\xi < \alpha} \Gamma(\Gamma_\xi),$$

so that $\Gamma_\infty = \bigcup_{\xi \in \mathrm{On}} \Gamma_\xi$. The Γ_α's are called the *stages* of the inductive definition, and

$$|\Gamma| = \text{least } \alpha(\Gamma_{\alpha+1} = \Gamma_\alpha),$$

is called *the ordinal of the inductive definition*. The operator Γ is called *monotone* if $X \subseteq Y$ implies that $\Gamma(X) \subseteq \Gamma(Y)$.

An inductive definition is classified by the complexity of the relation $x \in \Gamma(X)$. In order to obtain a reasonable theory over arbitrary structures $\mathfrak{A}$ we assume that $\mathfrak{A}$ has a coding scheme (see clause 4 of Definition 3.2.3) and that the language of $\mathfrak{A}$ includes the relations and functions of the coding scheme. In this way we have the usual Σ_m^n, Π_m^n, and Δ_m^n classification of the relation $x \in \Gamma(X)$.

3.3.2 Definition. Let $\mathbf{C}$ be a class of operators on a structure $\mathfrak{A}$. Let $\mathbf{C}_\infty$ denote the class of fixed-points Γ_∞ of operators $\Gamma \in \mathbf{C}$.

$$\mathrm{IND}(\mathbf{C}) = \{R : \exists \mathbf{a} \exists \Gamma \in \mathbf{C}[x \in R \text{ iff } (\mathbf{a}, x) \in \Gamma_\infty]\}.$$

$\mathrm{IND}(\mathbf{C})$ is the class of $\mathbf{C}$-inductive relations. The associated ordinal is

$$|\mathbf{C}| = \sup\{|\Gamma| : \Gamma \in \mathbf{C}\}.$$

For every operator Γ on A we define $\neg\Gamma$ by $\neg\Gamma(X) = A \backslash \Gamma(X)$ and $\neg\mathbf{C} = \{\neg\Gamma : \Gamma \in \mathbf{C}\}$. We set

$$\mathrm{IND}(\neg\mathbf{C}) = \{A - R : R \in \mathrm{IND}(\mathbf{C})\},$$

and call this the class of $\mathbf{C}$-coinductive relations. Finally, set

$$\mathrm{HYP}(\mathbf{C}) = \mathrm{IND}(\mathbf{C}) \cap \mathrm{IND}(\neg\mathbf{C}),$$

this is called the class of $\mathbf{C}$-hyperdefinable relations on $\mathfrak{A}$.

We list some of the basic examples from which the general theory has been abstracted and developed.

3.3.3 Example. Positive Σ_1^0 inductive definitions over ω.

This is a well-known way of developing ORT over ω, due to E. Post. We know that recursively enumerable is the same as Σ_1^0. Hence the first recursion theorem gives the equivalence between recursively enumerable and positive Σ_1^0 inductive definability. This is a theme to which we will often return.

3.3.4 Example. Π_1^0 inductive definitions over ω.

General, i.e. non-monotone, Π_1^0 inductive definability gives us already the class of Π_1^1 sets. This was first proved by R. Gandy, but remained unpublished by him (see, however, [39]). A proof in the setting of α-recursion theory was published in W. Richter [134]. A proof is also contained in Grilliot's [49]. Grilliot's work clearly brings out the important role of *reflection principles* in the theory of inductive definability. We shall in many connections return to this theme, e.g. in the study of finite theories on two types (e.g. recursion in ${}^3E$). The present version reads as follows.

3.3.4.1 Σ_2^0-Reflection. *Let B be a Π_1^1 set and $\Gamma(X)$ a Σ_2^0 relation. If $\Gamma(B)$, then there exists a hyperarithmetic $B_0 \subseteq B$ such that $\Gamma(B_0)$.*

We indicate a proof assuming that the reader has some basic knowledge of Π_1^1-theory. First we note that B can be obtained by an approximation B_α, $\alpha < \omega_1$ (= the first non-recursive ordinal), where each B_α is hyperarithmetic. Let $\Gamma(X)$ be of the form $\exists y \forall z \Gamma_0(y, z, X)$. Since we have $\Gamma(B)$, fix a parameter b such that $\forall z \Gamma_0(b, z, B)$.

The heart of the argument is a boundedness property: From $\forall z \Gamma_0(b, z, B)$ it follows that $\forall z \exists \beta < \omega_1 \Gamma_0(b, z, B_\beta)$. We want to conclude that there is an $\alpha < \omega_1$ such that $\forall z \Gamma_0(b, z, B_\alpha)$.

The details are as follows. Naively we would like to set $\alpha = \sup_z \mu\beta[\Gamma_0(b, z, B_\beta)]$. But $\Gamma_0(b, z, B_\beta)$ for some $\beta < \alpha$ does not necessarily imply $\Gamma_0(b, z, B_\alpha)$. Hence, we have to complicate the construction a bit. Choose the α_n's such that

$$\alpha_{n+1} \geqslant \sup_z \mu\beta[\beta \geqslant \alpha_n \wedge \Gamma_0(b, z, B_\beta)].$$

Then for every z there is a sequence β_n such that $\alpha_n \leqslant \beta_n \leqslant \alpha_{n+1}$ and $\Gamma_0(b, z, B_{\beta_n})$. Let $\alpha = \sup \alpha_n$. Now we get $\Gamma_0(b, z, B_\alpha)$ for *all* z, since $B_\alpha = \bigcup B_{\beta_n}$.

A recursion-theoretic analysis of the construction (including notations for ordinals etc.) and an application of the Spector boundedness principle, shows that $\alpha < \omega_1$.

The application to IND(Π_1^0) is immediate. Let Γ_∞ be a fixed-point for a Π_1^0 operator Γ. It is not difficult to show that $\Gamma_{\omega_1} \in \Pi_1^1$. Consider the relation $x \in \Gamma_{\omega_1+1} = \Gamma(\Gamma_{\omega_1})$. By Lemma 3.3.5 there is some $\alpha < \omega_1$ such that $x \in \Gamma(\Gamma_\alpha) = \Gamma_{\alpha+1}$. Hence $\Gamma_\infty = \Gamma_{\omega_1}$, from which we conclude that IND(Π_1^0) = Π_1^1 and $|\Pi_1^0| = \omega_1$.

3.3.5 Example. Π_1^1 monotone inductive definitions over ω.

The basic analysis of this case is due to C. Spector [161]. Today this part of the theory is best viewed from the standpoint of admissibility theory. Monotone Π_1^1 is the same as positive. And Π_1^1 on ω corresponds to Σ_1 on the next admissible set, in this case L_{ω_1}. Σ_1 positive inductive operators have fixed-points and the length of the inductive definition is at most the ordinal of the admissible set. Translating back one sees that if Γ is Π_1^1-monotone, then $\Gamma_\infty \in \Pi_1^1$ and $|\Gamma| \leqslant \omega_1$.

If one wants to prove this in the setting of hyperarithmetic theory, the basic fact to verify is that if Γ is Π_1^1-monotone and $B \in \Pi_1^1$, then $x \in \Gamma(B)$ iff there is some hyperarithmetic (i.e. "finite") $B_0 \subseteq B$ such that $x \in \Gamma(B_0)$.

3.3.6 Example. Positive elementary inductive definitions on a structure $\mathfrak{A}$.

Over ω we know that Π_1^0 positive and Π_1^1 monotone give the same class of relations, viz. the Π_1^1 relations. Hence the class of positive elementary operators also gives the same inductively defined relations. (*Elementary* here means first-order in the language of the structure.)

Moschovakis developed in his book *Elementary Induction on Abstract Structures* [115] the theory of positive elementary inductive definability over arbitrary structures $\mathfrak{A}$ (equipped with a suitable coding scheme) as a general approach to definability. Barwise obtained the same theory in his book *Admissible Sets and*

Structures [11] from the standpoint of the theory of HYP$\mathfrak{M}$, the "next admissible" set.

In our approach we note that if $\mathbf{C}$ = the class of positive elementary operators, then IND($\mathbf{C}$) is the "least" Spector class on the structure. Hence we view the theory as part of the development of Spector theories, see Section 3.2 of this chapter.

The HYP$\mathfrak{M}$—or *imbedding* aspect will be treated in full in Chapter 5.

3.3.7 Example. Σ_1^1 monotone inductive definitions and generalized quantifiers.

This example is due to P. Aczel [3] and has been the source of much of the work on generalized quantifiers in the context of general recursion theory; see Aczel [4, 6], Moschovakis [115], Barwise [13], and Kolaitis [87].

Let $E_1^\#$ be the partial functional given by the equation

$$E_1^\#(f) \simeq \begin{cases} 0 & \text{if} \quad \forall\alpha\exists n\cdot f(\bar{\alpha}(n)) = 0 \\ 1 & \text{if} \quad \exists\alpha\forall n\cdot f(\bar{\alpha}(n)) > 0, \end{cases}$$

where f is a partial function from ω to ω. Aczel's main result in [3] states that if $A \subseteq \omega$, then A is semicomputable in $E_1^\#$ iff $A \in \text{IND}(\Sigma_1^1\text{-mon})$.

The proof is not difficult. An analysis of recursion in $E_1^\#$ shows that it is given by a Σ_1^1 monotone operator. Conversely, if Γ is a Σ_1^1 monotone operator, i.e.

$$x \in \Gamma(\{n : \alpha(n) = 0\}) \quad \text{iff} \quad R(\alpha, n),$$

where R is Σ_1^1, then we can write, using the monotonicity of Γ,

$$x \in \Gamma(X) \quad \text{iff} \quad \exists\alpha\forall n[(\alpha(n) = 0 \to n \in X) \wedge R(\alpha, x)],$$

which has the form

$$x \in \Gamma(X) \quad \text{iff} \quad \exists\alpha\forall n[(R_1(\bar{\alpha}(n)) \vee g_1(\bar{\alpha}(n)) \in X) \wedge S_1(\bar{\alpha}(n), x)],$$

where R_1, S_1, g_1 are recursive.

Introduce the functional F by

$$F(f, x) \simeq 1 \quad \text{iff} \quad \exists\alpha\forall n[(R_1(\bar{\alpha}(n)) \vee f(g_1(\bar{\alpha}(n))) = 1) \wedge S_1(\bar{\alpha}(n), x)].$$

Then one sees that F is $E_1^\#$-computable and monotone. Let g be the least fixed-point of F, g is semicomputable in $E_1^\#$ by the first recursion theorem. Hence Γ_∞ is also semicomputable in $E_1^\#$, since evidently $x \in \Gamma_\infty$ iff $g(x) \simeq 1$.

This example can be generalized to arbitrary monotone quantifiers $\mathbf{Q}$. A *monotone quantifier* $\mathbf{Q}$ is a family of subsets of ω (or some other suitable domain in the generalized versions) such that $A \in \mathbf{Q}$ and $A \subseteq B$ implies $B \in \mathbf{Q}$. The dual of $\mathbf{Q}$ is defined as $\breve{\mathbf{Q}} = \{\omega - X; X \in \mathbf{Q}\}$. In the example above $\mathbf{Q} = \{A \subseteq \omega;$

$\exists\alpha\forall n \cdot \bar{\alpha}(n) \in A\}$. Analogous to $E_1^\#$ we have an associated functional $F_{\mathbf{Q}}^\#$ defined by

$$F_{\mathbf{Q}}^\#(f) \simeq \begin{cases} 0 & \text{if } \{x : f(x) = 0\} \in \breve{\mathbf{Q}} \\ 1 & \text{if } \{x : f(x) > 0\} \in \mathbf{Q}. \end{cases}$$

The main result above generalizes: Given $A \subseteq \omega$, then A is semicomputable in $F_{\mathbf{Q}}^\#$ iff $A \in \mathrm{IND}(\mathbf{Q})$, see Aczel [3].

The reader will have noticed the dramatic difference between the strength of non-monotone versus monotone inductive operators, e.g. both $\mathrm{IND}(\Pi_1^1\text{-mon})$ and $\mathrm{IND}(\Pi_0^1)$ leads to the same class, viz. the Π_1^1 sets over ω.

We shall present a few basic results on the connection between Spector classes/theories and non-monotone inductive definability. Some of the first and basic results are due to Grilliot [49]. Aczel and Richter in their joint paper [135] emphasized the importance of reflection principles and developed a general theory over ω. Aanderaa [1] solved a fundamental and long-standing open problem about the size of ordinals of inductive definitions. And in the paper [116] Moschovakis brought the various developments together and presented a unified approach. The reader should also consult Cenzer [18]. Finally, we should mention the work of Harrington and Kechris [57] on the relationship between monotone and non-monotone induction.

Here is the appropriate definition to get the theory off the ground (see Moschovakis [116]).

3.3.8 Definition. $\mathbf{C}$ is a *typical non-monotone class* of operators if it satisfies the following six conditions:

A. $\mathbf{C}$ contains all second-order relations on A which are definable by (first-order) universal formulas of the trivial structure $\langle A \rangle$.

B. $\mathbf{C}$ is closed under $\wedge$ and $\vee$.

C. $\mathbf{C}$ is closed under trivial, combinatorial substitutions.

Remark. From these conditions we already have a large part of the structure theory for Spector classes, see Moschovakis [116, §3].

D. $\mathbf{C}$ contains all second-order relations definable by existential formulas of the trivial structure $\langle A \rangle$.

E. There is an ordering $\leqslant \subseteq A \times A$ isomorphic to the ordering on ω and a 1-1 function $f: A \times A \to A$ which belong to $\mathrm{HYP}(\mathbf{C})$.

F. For each $n \geqslant 1$ the n-ary $\mathrm{IND}(\mathbf{C})$ relations are parametrized by an $n + 1$-ary $\mathrm{IND}(\mathbf{C})$ relation.

From these conditions on $\mathbf{C}$ it is not surprising that the following result holds.

3.3.9 Theorem. *If* $\mathbf{C}$ *is a typical non-monotone class of operators on* A, *then* $\mathrm{IND}(\mathbf{C})$ *is a Spector-class.*

We can add various refinements, e.g. $|\mathbf{C}|$ = supremum of the length of computations in the associated Spector theory. And every second-order relation in $\mathbf{C}$ is "Δ on Δ".

The notion "Δ on Δ" was introduced by Moschovakis in the setting of Spector classes. Working with the associated Spector theory seems to simplify the conceptual set-up.

Let $\mathbf{C}$ be a typical non-monotone class of operators. Theorem 3.3.9 says that $\mathrm{IND}(\mathbf{C})$ is a Spector class. Associated with $\mathrm{IND}(\mathbf{C})$ is a Spector theory $\Theta(\mathbf{C})$, see Theorem 3.2.6. With every operator Γ in $\mathbf{C}$ there is associated a functional F_Γ defined as follows

$$F_\Gamma(\alpha, x) = \begin{cases} 1 & \text{if } x \in \Gamma(\mathrm{set}_\alpha) \\ 0 & \text{if } x \notin \Gamma(\mathrm{set}_\alpha). \end{cases}$$

α is here supposed to be total, and $\mathrm{set}_\alpha = \{x : \alpha(x) = 0\}$. It is immediate that F_Γ is $\Theta(\mathbf{C})$-computable for every operator $\Gamma \in \mathbf{C}$.

We further notice that in the Spector theory $\Theta(\mathbf{C})$ we have $A = C$ and the relation $\mathrm{set}_\alpha \in \mathrm{sc}(\Theta(\mathbf{C}))$ is $\Theta(\mathbf{C})$-semicomputable. We say that $\mathbf{C}$ is Θ-computable if F_Γ is Θ-computable for every operator $\Gamma \in \mathbf{C}$. This is our notion "Δ on Δ".

The crucial point in proving that $\mathrm{IND}(\Pi_1^0) = \Pi_1^1$ was to establish the Reflection Property 3.3.4.1. We state a general definition.

3.3.10 Definition. Let $\mathbf{C}$ be a class of operators and Θ a Spector theory. Θ has the $\mathbf{C}$-*reflection property* if whenever $R \in \mathrm{en}(\Theta)$, $\Gamma \in \mathbf{C}$, and $R_0 \subseteq R$ belongs to $\mathrm{sc}(\Theta)$,

$$\Gamma(R) \Rightarrow \text{there exists } R^* \in \mathrm{sc}(\Theta),\ R_0 \subseteq R^* \subseteq R, \text{ and } \Gamma(R^*).$$

We showed in 3.3.4.1 that Π_1^1 as a Spector theory has the Σ_2^0-reflection property.

The following result of Moschovakis [116] is a general statement of these facts and provides a converse to Theorem 3.3.9.

3.3.11 Theorem. *Let Θ be a Spector theory and $\mathbf{C}$ a typical non-monotone class of operators such that*

(i) *Θ is $\mathbf{C}$-reflecting, and*
(ii) *$\mathbf{C}$ is Θ-computable.*

Then $\mathrm{IND}(\mathbf{C}) \subseteq \mathrm{en}(\Theta)$.

The idea of the proof is simple. We use the fact that $\mathbf{C}$ is Θ-computable to carry out the inductive definition inside $\mathrm{en}(\Theta)$, and conclude from the $\mathbf{C}$-reflecting property of Θ that the inductive definition closes at a stage $\leqslant \|\Theta\|$.

3.3.12 Remark. We should supplement Theorem 3.3.9 by noting that the Spector theory $\Theta(\mathbf{C})$ associated to a typical non-monotone class $\mathbf{C}$ is $\mathbf{C}$-reflecting: For simplicity set $R_0 = \varnothing$ in 3.3.10 and assume $R \in \mathrm{en}(\Theta) = \mathrm{IND}(\mathbf{C})$, i.e. $x \in R$ iff $(a, z) \in \Gamma_\infty$, for some $\Gamma \in \mathbf{C}$. Let $\Delta \in \mathbf{C}$ and assume that $\Delta(R)$ is valid.

Choose an element b and let

$$\begin{aligned}\Psi(X) &= \Gamma(X),\\ \Phi(X) &= \{b : \Delta(\{x : (a, x) \in X\})\}.\end{aligned}$$

This is a simultaneous inductive definition in the class $\mathbf{C}$. Obviously, $\Psi_\alpha = \Gamma_\alpha$; hence, $\{x : (a, x) \in \Psi_\infty\} = R$ and thus $b \in \Phi_\infty$. But then there is some $\xi < |\mathbf{C}|$ such that $b \in \Phi_{\xi+1}$. This means that we have $\Delta(R^*)$ for $R^* = \{x : (a, x) \in \Psi_\xi\}$. But the latter set belongs to $\mathrm{sc}(\Theta) = \mathrm{HYP}(\mathbf{C})$ by the usual boundedness argument.

We mention one result on the relative size of the ordinals of inductive definitions. The reader should recall the notations of Definition 3.3.2. The following general result is due to S. Aanderaa [1].

3.3.13 Theorem. *Let* $\mathbf{C}$ *be a typical non-monotone class of operators. If* $\mathbf{C}$ *has the prewellordering property, then*

$$|\mathbf{C}| < |\neg\mathbf{C}|.$$

In particular, $|\Pi^1_1| < |\Sigma^1_1|$ and $|\Sigma^1_2| < |\Pi^1_2|$.

Recall that a class $\mathbf{C}$ has the prewellordering property if every relation in $\mathbf{C}$ has a prewellordering in $\mathbf{C}$.

Note that beyond the second level of the analytic hierarchy the prewellordering property depends upon the axioms of set theory, see Aanderaa [1] for complete statements.

We shall give a brief outline of the proof, following the original exposition closely, but correcting several minor mistakes along the way.

Fact 1. There exists $\Gamma \in \mathbf{C}$ such that $|\Gamma| = |\mathbf{C}|$.

This is an easy consequence of the ω-parametrization property of the class $\mathbf{C}$. So let us start with a $\Gamma \in \mathbf{C}$ such that $|\Gamma| = |\mathbf{C}|$. We must produce a $\Lambda \in \neg\mathbf{C}$ such that $|\Gamma| < |\Lambda|$.

Fact 2. Given $\Gamma \in \mathbf{C}$ and $\check{\Gamma} \in \neg\mathbf{C}$ such that whenever $\Gamma(S) - S \neq \varnothing$, then $\varnothing \neq \check{\Gamma}(S) - S \subseteq \Gamma(S)$. Then there exists $\Lambda \in \neg\mathbf{C}$ such that $|\Gamma| < |\Lambda|$. (In fact, we construct a $\Lambda \in \neg\mathbf{C}$ such that $|\Lambda| = |\Gamma| + 1$.)

Let us postpone the proof of fact 2 for a moment and see how we produce a suitable $\check{\Gamma} \in \neg\mathbf{C}$ from the Γ given in fact 1.

Fact 3. Let $\Gamma \in \mathbf{C}$ and assume $\mathrm{PWO}(\mathbf{C})$. Then there exist $\hat{\Gamma} \in \mathbf{C}$ and $\check{\Gamma} \in \neg\mathbf{C}$ such that whenever $\Gamma(S) - S \neq \varnothing$, then $\varnothing \neq \hat{\Gamma}(S) = \check{\Gamma}(S) \subseteq \Gamma(S)$.

This proves the theorem.

Let us first give the proof of fact 3. Let $\Gamma'(S) = \Gamma(S) - S$, then $\Gamma' \in \mathbf{C}$. From the assumption PWO(**C**), let $\|\cdot\|$ be a norm on the class $A = \{(x, S) : x \in \Gamma'(S)\}$. Define

$$\hat{\Gamma}(S) = \{x : x \in \Gamma'(S) \wedge \forall y(\|y, S\| \leqslant \|x, S\| \\ \rightarrow (\|x, S\| \leqslant \|y, S\| \wedge x \leqslant y))\}.$$

Clearly $\hat{\Gamma} \in \mathbf{C}$. And if $\Gamma'(S) \neq \varnothing$, let λ_S be the least ordinal of the form $\|x, S\|$, where $x \in \Gamma'(S)$. Finally, let x_S be the least element x of ω such that $\|x, S\| = \lambda_S$. Then we see that $\hat{\Gamma}(S) = \{x_S\}$. Let now

$$x \in \check{\Gamma}(S) \quad \text{iff} \quad \forall y(y \in \hat{\Gamma}(S) \rightarrow y = x).$$

$\check{\Gamma}$ clearly satisfies the requirements of fact 3.

The construction of Λ in fact 2 uses the operator $\check{\Gamma}$ and the fact that $\varnothing \neq \check{\Gamma}(S) - S \subseteq \Gamma(S)$. This is an exercise in constructing an inductive definition. We outline one possible way.

Let

$$H(S) = \{u : \exists v(\langle v + 2, u\rangle \notin S \wedge \langle 0, v\rangle \in S)\},$$

and define

$$\begin{aligned}\Lambda(S) = \{&\langle 0, x\rangle : x \in \check{\Gamma}(H(S))\} \\ &\cup \{\langle z + 2, x\rangle : z \in \check{\Gamma}(H(S)) \wedge z \notin H(S) \wedge x \notin \Gamma(H(S)) \\ &\qquad \wedge x \notin H(S)\} \\ &\cup \{\langle 1, 0\rangle : \forall x(x \in \Gamma(H(S)) \rightarrow x \in H(S))\}.\end{aligned}$$

Finally let $f(y) = \langle 0, y\rangle$. By simultaneous induction on the ordinal λ one may now prove

(i) $\Gamma^{\lambda} = H(\Lambda^{\lambda})$ and $f^{-1}(\Lambda^{\lambda}) \subseteq \Gamma^{\lambda}$

(ii) $\Gamma(H(\Lambda^{\lambda})) \subseteq \Gamma^{\lambda+1}$

(iii) $\Gamma^{\lambda+1} - \Gamma^{\lambda} \neq \varnothing \Rightarrow \varnothing \neq \check{\Gamma}(H(\Lambda^{\lambda})) - H(\Lambda^{\lambda}) \subseteq \Gamma^{\lambda+1}$

(iv) $\Gamma^{\lambda+1} = H(\Lambda^{\lambda+1})$ and $f^{-1}(\Lambda^{\lambda+1}) \subseteq \Gamma^{\lambda+1}$.

Clearly, $\Lambda \in \neg\mathbf{C}$, and $|\Lambda| = |\Gamma| + 1$ since $\langle 1, 0\rangle \in \Lambda^{|\Gamma|+1} - \Lambda^{|\Gamma|}$.

We started our discussion of non-monotone inductive definability by pointing to the striking difference between IND(Π^1_1-mon) = IND(Π^0_1) = Π^1_1 and IND(Π^1_1). The ordinal e.g. of the latter class is enormously larger than $|\Pi^1_1\text{-mon}| = \omega_1$, the first non-recursive ordinal. This is the situation over the integers. There are, however, cases where the difference disappears. If the notion WF of wellfoundedness is elementary over a structure $\mathfrak{A}$ (i.e. is first-order definable in the relations of the structure), then the classes of elementary monotone and elementary non-monotone coincide. This is a corollary of a more general result due to L. Harrington and A. Kechris [57].

To obtain a sufficiently general result they call a class **C** of operators on a domain A *adequate* if it contains all operators defined by a universal formula of the structure, is closed under $\wedge$, $\vee$, $\exists_A$ and trivial combinatorial substitutions and contains a coding scheme. The reader may want to compare this in detail with Definition 3.3.8.

3.3.14 Theorem. *Let* **C** *be an adequate class of operators on* A. *If* $\neg\mathrm{WF} \in \mathbf{C}$ *and* $\neg\mathbf{C} \subseteq \mathrm{IND}(\mathbf{C}\text{-mon})$, *then* $\mathrm{IND}(\mathbf{C}) = \mathrm{IND}(\mathbf{C}\text{-mon})$.

Here $\neg\mathrm{WF}$ is the negation of the relation of wellfoundedness, and $\neg\mathbf{C} \subseteq \mathrm{IND}(\mathbf{C}\text{-mon})$ means the usual thing: if $\Gamma(\mathbf{a}, \mathrm{S})$ is an operator in **C**, then there exists in **C** an operator $\Gamma^*(\mathbf{e}, \mathbf{a}, R, \mathrm{S})$ which is monotone in R such that for a suitable choice of parameters $\mathbf{e}_0$

$$\neg\Gamma(\mathbf{a}, \mathrm{S}) \quad \text{iff} \quad \Gamma^*_\infty(\mathbf{e}_0, \mathbf{a}, \mathrm{S}).$$

We conclude our excursion into the theory of definability by relating the classes $\mathrm{IND}(\Sigma^0_2)$ and $\mathrm{IND}(\Pi^0_1)$ over an acceptable structure $\mathfrak{A}$ to the notions of *strong* and *weak* finiteness, respectively. (Recall that a structure $\mathfrak{A}$ is acceptable if it has an elementary coding scheme, see Moschovakis [115].)

Let us call a theory Θ a *weak Spector theory* if we relax the condition of strong finiteness of the domain to weak finiteness (see condition (2) of Definition 3.2.2). The following results are essentially due to Grilliot [49].

3.3.15 Theorem. *Let* $\mathfrak{A}$ *be an acceptable structure.*

(1) *Let* Θ *be a Spector theory on* $\mathfrak{A}$:

a $\mathrm{IND}(\Sigma^0_2) \subseteq \mathrm{en}(\Theta)$,
b $\Theta \sim \mathrm{PR}[\mathbf{E}_A, =]$ *iff* $\mathrm{en}(\Theta) = \mathrm{IND}(\Sigma^2_0)$.

(2) *Let* Θ *be a weak Spector theory on* $\mathfrak{A}$:

c $\mathrm{IND}(\Pi^0_1) \subseteq \mathrm{en}(\Theta)$,
d $\Theta \sim \mathrm{PR}[\mathbf{E}'_A, =]$ *iff* $\mathrm{en}(\Theta) = \mathrm{IND}(\Pi^0_1)$.

(*Here* $\mathbf{E}'_A$ *is the functional of Definition* 3.1.2.)

The proof of (1) is essentially contained in the proof of 3.3.4.1, rephrasing the argument inside an arbitrary Spector class on $\mathfrak{A}$ rather than in terms of Π^1_1. Recall also from 3.2.7 and 3.2.8 that there is a one-to-one correspondence between Spector theories and Spector classes, and that a Spector theory is determined by its envelope.

The proof for Π^0_1 is a bit more subtle. Obviously, we cannot substitute Π^0_1-

reflection for Σ_2^0-reflection in the proof for (1), since Π_1^0-reflection implies Σ_2^0-reflection and there exist weak Spector theories which are not strong, e.g. Kleene recursion in ${}^3E$ over the reals $\mathbb{R}$.

The difference can be traced to another fact, which we shall elaborate on in later chapters, that whereas Spector theories correspond to admissible structure *weak* Spector theories do not. Any weak Spector theory Θ has an associated family of (non-transitive) admissible sets, Spec(Θ), see Chapter 8 in particular. The reflection argument to follow is really the argument for case (1) localized to components of Spec(Θ).

We give a sketch of the non-trivial part of (2), adapted from Grilliot [49]: Let S be a Π_1^0 fixed-point, defined from an inductive operator

$$x \in \Gamma(X) \quad \text{iff} \quad \forall z P(x, z, X),$$

where P is quantifier-free. Let $\pi(\sigma)$ be the supremum of lengths of computations in Θ with σ as input. We want to show that $x \in S$ implies $x \in S_{\pi(x)}$, which yields that

$$S = \bigcup_x S_{\pi(x)} = S_\kappa,$$

where κ is the ordinal of Θ, i.e. the supremum of all computations in Θ. A recursion-theoretic analysis will show that $S = S_k \in \text{en}(\Theta)$ (similar to the analysis in 3.3.4 showing that $\Gamma_{\omega_1} \in \Pi_1^1$). Thus $\text{IND}(\Pi_1^0) \subseteq \text{en}(\Theta)$, proving c of (2).

It remains to verify that $x \in S$ implies $x \in S_{\pi(x)}$. For $x \in S$ let $|x|$ be the first stage at which x occurs in S. As induction hypothesis we assume that whenever $|y| < |x|$, then $y \in S_{\pi(y)}$. We also assume that $\pi(x) \leqslant |x|$, since otherwise $x \in S_{|x|+1} \subseteq S_{\pi(x)}$.

As in the proof of 3.3.4.1 we can show that $\forall z P(x, z, S_\alpha)$ is true when $\alpha = \sup \alpha_n$, where the α_n's satisfy

$$\alpha_{n+1} \leqslant \sup_z \mu\beta[\beta \geqslant \alpha_n \wedge P(x, z, S_\beta)].$$

It remains to get $\alpha < \pi(x)$. This is where "admissibility" enters the proof in 3.3.4.1. Here we must argue more carefully.

Suppose that we can compute in Θ an ordinal α_n, uniformly in n. We show how to compute in Θ an ordinal uniformly in n and z larger than

$$\mu\beta[\beta \geqslant \alpha_n \wedge P(x, z, S_\beta)].$$

Case 1: $|x| < \pi(x, z)$. This is immediate since $|x| \geqslant \alpha_n$ and $P(x, z, S_{|x|})$.

Case 2: $\pi(x, z) \leqslant |x|$. We first note that if a clause $b \in X$ occurs in $P(x, z, X)$, then $b \in S_{|x|}$ iff $b \in S_{\pi(x,z)}$. (If $b \in S_{|x|}$, then $|b| < |x|$, hence by induction hypothesis $b \in S_{\pi(b)} \subseteq S_{\pi(x,z)}$. The last inclusion follows since b is computable in x, z.)

From the assumption $P(x, z, S_{|x|})$ we may therefore conclude $P(x, z, S_{\pi(x,z)})$, hence also $P(x, z, S_\beta)$ for some β immediately below $\pi(x, z)$. This follows since P is Π_0^0 and $\pi(x, z)$ is a limit number.

In both cases we can compute some γ_{nz} larger than $\mu\beta[\beta \geqslant \alpha_n \wedge P(x, z, S_\beta)]$. Simple recursion theoretic properties give us an $\alpha_{n+1} \geqslant \sup_z \gamma_{nz}$ and finally $\alpha = \sup_n \alpha_n$. Since the constructions are computable in x, $\alpha < \pi(x)$. This completes the proof of c.

We make a remark on how to prove the converse inclusion that $\mathrm{en}(\Theta) \subseteq \mathrm{IND}(\Pi_1^0)$ if $\Theta \sim \mathrm{PR}[\mathbf{E}'_A, =]$. This follows from a careful analysis of the inductive definition of the theory $\mathrm{PR}[\mathbf{E}'_A, =]$.

The clauses of the "usual" inductive definition of the relations $\{e\}_{\mathrm{PR}[\mathbf{E}'_A,=]}(x)\downarrow$ and $\{e\}_{\mathrm{PR}[\mathbf{E}'_A,=]} \simeq z$ are all of the Π_0^1 form except substitution and application of $\mathbf{E}'_A$. The trick is to replace these parts of the inductive definition by non-monotonic clauses. We take first the case of substitution, $\{e\}(x) \simeq \{e_1\}(\{e_2\})x), x)$. We will introduce a new inductive operator Γ such that in the end $(0, e, x) \in \Gamma_\infty$ iff $\{e\}(x)\downarrow$ and $(1, e, x, z) \in \Gamma_\infty$ iff $\{e\}(x) \simeq z$. For substitution this is obtained by the non-monotonic clauses:

$$\begin{aligned}
(0, e, x) \in \Gamma(X) \quad &\text{if} \quad (0, e_2, x) \in X \wedge \forall u[(1, e_2, x, u) \in X \\
&\qquad\qquad \rightarrow (0, e_1, u, x) \in X]. \\
(1, e, x, z) \in \Gamma(X) \quad &\text{if} \quad (0, e_2, x) \in X \wedge \forall u[(1, e_2, x, u) \in X \\
&\qquad\qquad \rightarrow (1, e_1, u, x, z) \in X].
\end{aligned}$$

Application of $\mathbf{E}'_A$ is a bit more involved. In this case we have to build the function

$$\{e\}(x) \simeq \mathbf{E}'_A(\lambda y \cdot \{e'\}(y, x)),$$

into the operator Γ. This will be done by the following clauses using two auxiliary tuples $(a_1, \ldots)$ and $(a_2, \ldots)$

$$\begin{aligned}
(a_1, e', x) \in \Gamma(X) \quad &\text{if} \quad \forall y \cdot (0, e', y, x) \in X \\
(0, e, x) \in \Gamma(X) \quad &\text{if} \quad (a_1, e', x) \in X. \\
(a_2, e', x) \in \Gamma(X) \quad &\text{if} \quad \forall y \cdot (0, e', y, x) \in X \;\wedge\; \forall y \cdot (1, e', y, x, 0) \notin X. \\
(1, e, x, 1) \in \Gamma(X) \quad &\text{if} \quad (a_2, e', x) \in X \\
(1, e, x, 0) \in \Gamma(X) \quad &\text{if} \quad (a_1, e', x) \in X \;\wedge\; (a_2, e', x) \notin X.
\end{aligned}$$

(The definition is arranged so that the ordinals of $(0, e, x)$ and $(1, e, x, z)$ are the same. This is necessary for the inductive proof that $(0, e, x) \in \Gamma_\infty$ iff $\{e\}(x)\downarrow$ and $(1 \cdot e \cdot x \cdot z) \in \Gamma_\infty$ iff $\{e\}(x) \simeq z$.)

3.3.16 Remark. In Corollary 3.1.11 we saw that ω is strongly Θ-finite iff it is weakly Θ-finite. Hence on ω $\mathrm{IND}(\Sigma_0^2) = \mathrm{IND}(\Pi_1^0) = \Pi_1^1$.