Appendix A

Coding into Structures and Theories

Several of our applications of local degree theory have relied on our ability to code certain information into structures and theories. Thus we needed to associate a lattice with each degree, and we accomplish this by fixing a set of a given degree and *coding* it into a lattice in such a way that the set can be recovered recursively from any presentation of the lattice. Also, one of the methods which we use to prove undecidability results is to code one theory T_0 into another theory T_1. We accomplish this by describing a recursive translation which takes any sentence θ_0 of the language for T_0 into a sentence θ_1 in the language for T_1 so that $\theta_0 \in T_0 \Leftrightarrow \theta_1 \in T_1$. The undecidability of T_1 will then imply the undecidability of T_0. The major theories in which we have an interest are the theories of true first and second order arithmetic, the theory of (distributive) lattices, and the theory of graphs.

1. Degrees of Presentations of Lattices

Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ be a countable lattice. A *presentation* of $\mathscr{L}$ is an isomorphic copy $\mathscr{P} = \langle P, \leqslant_P, \vee_P, \wedge_P \rangle$ of $\mathscr{L}$ such that P is a recursive set. The *degree* of $\mathscr{P}$ is then the join of the degrees of $\leqslant_P$, $\vee_P$ and $\wedge_P$.

We wish to prove a result used in Chap. VIII.2 which assigns a lattice $\mathscr{L}_{\mathbf{a}}$ to each degree $\mathbf{a}$. Thus given $\mathbf{a} \in \mathbf{D}$, we choose a set A of degree $\mathbf{a}$ and code A into a lattice $\mathscr{L}_{\mathbf{a}}$. We show that $\mathscr{L}_{\mathbf{a}}$ has a presentation of degree $\mathbf{a}$, and that A can be recovered recursively from any presentation of $\mathscr{L}_{\mathbf{a}}$.

1.1 Theorem. *For any degree* $\mathbf{a}$, *there is a countable lattice* $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ *such that*:

(i) $\langle L, \leqslant \rangle$ *has a presentation of degree* $\mathbf{a}$.
(ii) *Any presentation of* $\langle L, \leqslant \rangle$ *has degree* $\geqslant \mathbf{a}$.

Proof. For each $n \in N$, let $\mathscr{L}_n$ be the lattice of Fig. 1.1. Thus $\mathscr{L}_n$ is the lattice with $2n + 9$ elements, its universe is $L_n = \{d^n, e^n, c_0^n, c_1^n, a_1^n, \ldots, a_{n+2}^n, b_0^n, \ldots, b_{n+2}^n\}$ which is ordered by specifying that exactly the following relations hold:

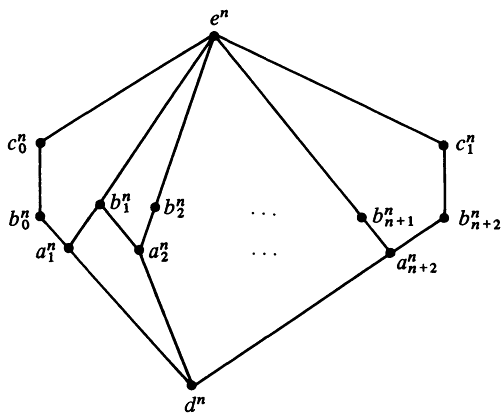

Fig. 1.1

(1) $\forall x(d^n \leqslant x)$.

(2) $\forall i(1 \leqslant i \leqslant n+2 \rightarrow a_i^n \leqslant b_{i-1}^n \,\&\, a_i^n \leqslant b_i^n)$.

(3) $a_1^n \leqslant c_0^n \,\&\, a_{n+2}^n \leqslant c_1^n$.

(4) $\forall x(x \leqslant e^n)$.

(5) $b_0^n \leqslant c_0^n \,\&\, b_{n+2}^n \leqslant c_1^n$.

We assume that for all $m, n \in N$, if $m \neq n$ then $L_m \cap L_n = \emptyset$. Given a set A of degree **a**, let $\mathscr{L}_A = \langle L_A, \leqslant, \vee, \wedge \rangle$ be the lattice which satisfies the following conditions, where $d \notin \bigcup\{L_i : i \in N\}$:

(6) $L_A = \{d\} \cup \bigcup\{L_{2n} : n \in A\} \cup \bigcup\{L_{2n+1} : n \notin A\}$.

(7) $\forall n \in N(\mathscr{L}_{2n}$ is a segment of $\mathscr{L}_A \Leftrightarrow n \in A)$.

(8) $\forall n \in N(\mathscr{L}_{2n+1}$ is a segment of $\mathscr{L}_A \Leftrightarrow n \notin A)$.

(9) $\forall m, n \in N$ (if $\mathscr{L}_m$ and $\mathscr{L}_n$ are segments of $\mathscr{L}_A$ and $m < n$, then $\forall x \in L_m \forall y \in L_n(x \leqslant y)$).

(10) $\forall x \in L_A(x \leqslant d)$.

It is easily checked that properties (6)–(10) can be used to obtain a presentation of $\langle L_A, \leqslant \rangle$ which has degree $\leqslant$ **a**. (In fact, we can obtain a presentation of $\mathscr{L}_A$ which has degree $\leqslant$ **a**, although this fact is not needed.) Furthermore, it is easily checked that a subposet of $\langle L_A, \leqslant \rangle$ is isomorphic to $\langle L_{2n}, \leqslant \rangle$ if and only if $n \in A$, and a subposet of $\langle L_A, \leqslant \rangle$ is isomorphic to $\langle L_{2n+1}, \leqslant \rangle$ if and only if $n \notin A$. (ii) follows easily from this fact. ▯

1.2 Remarks. Theorem 1.1 is due to Richter [1979]. Epstein [1979] presents a different proof using distributive lattices.

2. Interpreting Theories within Other Theories

In this section, we present recursive translations which enable us to interpret one theory within another, and so to transfer undecidability results from one theory to another. We will be interested in undecidability results for fragments of theories on which we place restrictions on the number of alternations of quantifier. Thus our translation will have to be very delicate, designed to keep the number of alternations of quantifier and the number of occurrences of negation to a minimum.

2.1 Definition. Let σ be a formula of a language $\mathscr{L}$. We assume that σ is in prenex normal form, so $\sigma = \forall x_1 \cdots \exists x_n(R(x_1, \ldots, x_n, y_1, \ldots, y_k))$ where R is quantifier free. The formula σ is said to be *positive* if all logical connectives in R are either $\vee$ or & (no negations can appear). σ is said to be *negative* if its negation, $\neg\sigma$, is logically equivalent to a positive formula.

We begin by considering an arbitrary finitely axiomatizable theory T_0 in a language $\mathscr{L}_m$ consisting of m relation symbols $R_1, \ldots, R_m$. We indicate how to interpret this theory within the theory T_b in the language $\mathscr{L}_b$ of a single binary relation symbol R. (We assume that we are working in the pure predicate calculus with relation symbols for equality and the negation of equality.) Let $\mathscr{X} = \langle X, R_1, \ldots, R_m\rangle$ be an $\mathscr{L}_m$-structure. We build an $\mathscr{L}_b$-structure $\mathscr{B} = \langle B, R\rangle$ and an effective translation of $\mathscr{L}_m$ into $\mathscr{L}_b$ such that for any sentence σ of $\mathscr{L}_m$, the translation takes σ into σ_b and $\mathscr{X} \models \sigma \Leftrightarrow \mathscr{B} \models \sigma_b$.

We define the binary relation R below pictorially, letting $R(x, y)$ hold exactly when an arrow goes from x to y. $X = \{x_i : 1 \leqslant i \leqslant n\}$. If R_i is an n-ary relation and $\mathscr{X} \models R_i(x_1, \ldots, x_n)$, then we draw the following picture:

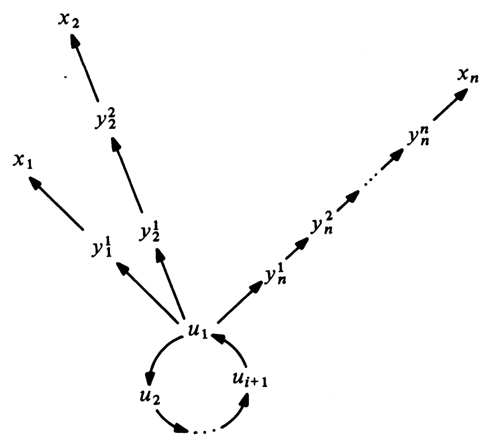

Fig. 2.1

The original structure is now interpreted within this new structure with universe expanded from X to include the u_k and y_r^s introduced, all of which are distinct. The $i + 1$ cycle $\{u_1, \ldots, u_{i+1}\}$ tells us that we are coding the relation R_i. The number of elements y_r^s between u_1 (the only element from which more than one arrow emanates) and x_r tells us the place of x_r in the relation R_i. And the elements from which no arrows emanate code the domain X.

We now describe a uniform effective translation of $\mathscr{L}_m$ into $\mathscr{L}_b$ which takes the sentence θ to the sentence θ_b. We define the universe of interpretation, B^*, as the set of elements from which no arrows emanate. Thus

$$(1) \qquad x \in B^* \Leftrightarrow \forall y(\neg xRy).$$

We obtain θ_b from θ by restricting all quantifiers to B^* and replacing $R_i(x_1, \ldots, x_n)$ with the formula

$$(2) \qquad \exists u_1, \ldots, u_{i+1} \exists y_1^1 \exists y_2^1 \exists y_2^2 \cdots \exists y_n^1 \cdots \exists y_n^n \left(\left(\bigwedge_{j=1}^{i} R(u_j, u_{j+1}) \right) \& \right.$$

$$R(u_{i+1}, u_1) \& \left(\bigwedge_{j=1}^{n} R(u_1, y_j^1) \right) \& \cdots \& \left(\bigwedge_{j=k<n}^{n} R(y_j^{k-1}, y_j^k) \right) \& \cdots \&$$

$$\left. \left(\bigwedge_{j=1}^{n} R(y_j^j, x_j) \right) \right).$$

Since T_0 is finitely axiomatizable, there is a single sentence σ of $\mathscr{L}_0$ which is the conjunction of all the axioms for T_0. Under the above translation, σ is carried to a sentence σ^+. Let $\mathbb{B}$ be the class of all $\mathscr{L}_b$-structures satisfying σ^+. Then for all sentences θ of $\mathscr{L}_0$,

$$(3) \qquad \theta \in T_0 \Leftrightarrow \forall \mathscr{X} \text{ (If } \mathscr{X} \text{ is an } \mathscr{L}_0\text{-structure and } \mathscr{X} \models \sigma \text{ then } \mathscr{X} \models \theta)$$

$$\Leftrightarrow \forall \mathscr{B} \in \mathbb{B}(\mathscr{B} \models \theta_b) \Leftrightarrow \sigma^+ \to \theta_b \in T_b.$$

For later applications, we need an alternate definition of B^*. The faithfulness of the definition depends on whether T_0 has the following property.

2.2 Definition. T_0 is *accessible* if given that for each $j \leqslant m$, R_j is an n_j-ary relation then:

(i) $\forall j \leqslant m(n_j \leqslant n_m)$.

(ii) For all models $\mathscr{M}$ of T_0 with universe M and all $y \in M$ there are $y_1, \ldots, y_{n_m - 1} \in M$ such that $R_m(y_1, \ldots, y_{n_m - 1}, y)$.

If T_0 is accessible, then we note that

$$(4) \qquad x \in B^* \leftrightarrow \exists z_1, \ldots, z_{m+n_m+2} \left(\left(\bigwedge_{\substack{i,j \leqslant m+n_m+2 \\ i \neq j}} z_i \neq z_j \right) \& x = z_{m+n_m+2} \right.$$

$$\left. \& \left(\bigwedge_{i=1}^{m+n_m+1} R(z_i, z_{i+1}) \right) \right).$$

We note that both B^* and R_i have positive $\exists_1$ definitions if T_0 is accessible. Furthermore, in this case, we modify our translation to use (4) instead of (1) to define B^*, and we replace σ^+ with the conjunction of σ^+ with the sentence which asserts that the right-hand side of (1) is equivalent to the right-hand side of (4).

We summarize the facts which we will need about the translation in the following remark.

2.3 Remark. Assume that T_0 is accessible. Then there is a translation of $\mathscr{L}_0$ into $\mathscr{L}_b$ which satisfies (3) and which has the property that every model of T_0 corresponds to a model of T_b in which the interpretation of the universe of the model of T_0 and of all atomic relations on that model are given by positive $\exists_1$ formulas. A translation having this last property is called *positive* $\exists_1$.

The next step will be to translate T_0 into the theory of graphs. We accomplish this by translating T_b into the theory of graphs. We will need the following property of iterated translations.

2.4 Proposition. *Let T_1, T_2, and T_3 be theories, and suppose that there is a positive $\exists_1$ translation of $\mathscr{L}_i$ (the underlying language for T_i) into $\mathscr{L}_{i+1}$ for $i = 1, 2$. Then there is a positive $\exists_1$ translation of $\mathscr{L}_1$ into $\mathscr{L}_3$.*

Proof. Write down the obvious definitions for the interpretations of the universe and atomic relations of a model of T_1. The result follows from the positivity of the translation. ▯

The next step will be the interpretation of the theory of a single binary relation within the theory of graphs.

2.5 Definition. A *graph* is a structure $\mathscr{A} = \langle A, S\rangle$ where S is a symmetric irreflexive binary relation on A, i.e., S satisfies:

(i) (Symmetry) $\forall x, y \in A(xSy \rightarrow ySx)$.

(ii) (Irreflexivity) $\forall x \in A(\neg xSx)$.

Let $\mathscr{B} = \langle B, R\rangle$ be a structure in which R is a binary relation on the universe B. We will define a graph $\mathscr{A} = \langle A, S\rangle$ on which we will be able to interpret $\mathscr{B}$. S is defined pictorially in Fig. 2.2. In that figure, x and y are elements of $B \cap A$ and the remaining elements mentioned are in $A - B$.

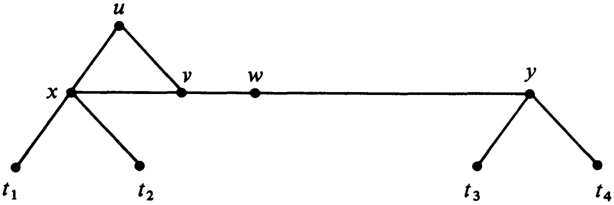

Fig. 2.2

$S(x, y)$ holds exactly when a line segment connects x and y. The original structure is now interpreted within this new structure with universe expanded to include the additional elements mentioned in the copies of Fig. 2.2. A copy of Fig. 2.2 will be placed in S exactly when $\mathscr{B} \models R(x, y)$. The letters u, v, w and t_i are different for each copy of Fig. 2.2 inserted to define S. t_1, t_2, t_3 and t_4 are tags which are distinguished by the fact that they are connected to exactly one element. The elements of B are exactly those connected to two tags. And $R(x,y)$ holds in $\langle B, R\rangle$ if and only if there are elements u, v, w connecting x to y as in Fig. 2.2.

We now describe a uniform effective translation of $\mathscr{L}_b$ into $\mathscr{L}_b$ which takes the sentence θ to the sentence θ^*. We define the universe of interpretation, A^*, as those elements connected to two tags;

$$(5)\qquad x \in A^* \Leftrightarrow \exists t_1, t_2(t_1 \neq t_2 \,\&\, S(x,t_1) \,\&\, S(x,t_2) \,\&\, \forall y(S(y,t_1) \,\&\, S(y,t_2) \rightarrow y = x)).$$

We obtain θ^* from θ by restricting all quantifiers to A^* and replacing $R(x,y)$ with the formula

$$(6)\qquad \exists u, v, w(S(x,u) \,\&\, S(x,v) \,\&\, S(u,v) \,\&\, S(v,w) \,\&\, S(w,y) \,\&\, u \neq v \,\&\, u \neq w \,\&\, v \neq w).$$

Since T_0 is finitely axiomatizable, there is a single sentence σ of $\mathscr{L}_0$ which is the conjunction of all the axioms for T_0. By (3), $\theta \in T_0 \Leftrightarrow \sigma^+ \rightarrow \theta_b \in T_b$ where σ^+ and θ_b were previously described. Let $\tau = \sigma^+$, and let τ^+ be the conjunction of τ^* with the sentence which asserts that S is a graph. Let $\mathbb{B}$ be the class of all $\mathscr{L}_b$-structures satisfying σ^+ and let $\mathbb{A}$ be the class of all graphs satisfying τ^+. Then for all sentences θ of $\mathscr{L}_0$,

$$(7)\qquad \theta \in T_0 \Leftrightarrow \sigma^+ \rightarrow \theta_b \in T_b \Leftrightarrow \forall \mathscr{B} \in \mathbb{B}(\mathscr{B} \models \theta_b) \Leftrightarrow \forall \mathscr{A} \in \mathbb{A}(\mathscr{A} \models \theta_b^*) \Leftrightarrow \tau^+ \rightarrow \theta_b^* \in T_G$$

where T_G is the theory of graphs.

For later applications we will need an $\exists_1$ definition of A^*. Note that in the graph described in Fig. 2.2, the elements of A^* are those which are connected to at least three distinct elements. Hence in this model,

$$(8)\qquad x \in A^* \Leftrightarrow \exists t_1, t_2, z(t_1 \neq t_2 \,\&\, t_1 \neq z \,\&\, t_2 \neq z \,\&\, S(x,t_1) \,\&\, S(x,t_2) \,\&\, S(x,z)).$$

We note that both A^* and R have positive $\exists_1$ definitions for these models. Furthermore, if we modify our translation to use (8) instead of (5) to define A^*, and we replace τ^+ with the conjunction of τ^+ with the sentence which asserts that the right-hand side of (5) is equivalent to the right-hand side of (8), then our translation is positive $\exists_1$.

We summarize the facts which we will need about the translation in the following remark, noting that Proposition 2.4 is being applied.

2.6 Remark. Assume that T_0 is accessible. Then there is a positive $\exists_1$ translation of $\mathscr{L}_0$ into $\mathscr{L}_b$ which satisfies (7).

We now show how to pass from the theory of graphs to the theory T_d of distributive lattices with least and greatest elements. The language $\mathscr{L}_d$ used for these lattices has symbols $\leqslant$ for the ordering, $\vee$ and $\wedge$ for the join and meet respectively, and 0 and 1 for the least and greatest elements, respectively.

Let $\mathscr{A} = \langle A, S \rangle$ be a graph. We will define a distributive lattice $\mathscr{C} = \langle C, \leqslant, \vee, \wedge, 0, 1 \rangle$ on which we will be able to interpret $\mathscr{A}$. The elements of A will be interpreted as the *atoms* of $\mathscr{C}$, i.e., those elements x which are immediate successors of 0 (see Fig. 2.3). We will place a new *join irreducible* element z directly above $x \vee y$ (see Fig. 2.3) exactly when $\mathscr{A} \models S(x, y)$. (An element is *join irreducible* if it cannot be expressed as the join of two smaller elements.) $\mathscr{C}$ will be the distributive lattice generated by these atoms and join irreducible elements, together with a new greatest element.

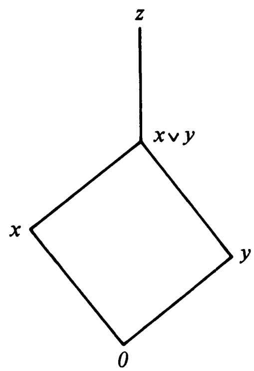

Fig. 2.3

We now describe a uniform effective translation of $\mathscr{L}_b$ into $\mathscr{L}_d$ which takes the sentence θ to the sentence θ_d. We describe the universe of interpretation, C^*, as the set of atoms of $\mathscr{C}$;

$$(9) \qquad x \in C^* \Leftrightarrow x \neq 0 \,\&\, \forall y (y \leqslant x \rightarrow y = x \text{ or } y = 0).$$

We obtain θ_d from θ by restricting all quantifiers to C^* and replacing $S(x, y)$ with the formula

$$(10) \qquad \exists c (c \geqslant x \vee y \,\&\, c \neq x \vee y \,\&\, \forall d (d \leqslant c \rightarrow d = c \text{ or } d \leqslant x \vee y)).$$

Since T_0 is finitely axiomatizable, there is a single sentence σ of $\mathscr{L}_0$ which is the conjunction of all the axioms for T_0. By (7), $\theta \in T_0 \Leftrightarrow \tau^+ \rightarrow \theta_b^* \in T_G$ where τ^+ and θ_b^* were previously described. Let $\xi = \tau^+$ and let ξ^+ be the conjunction of ξ_d with the sentence which asserts that $\mathscr{C}$ is a distributive lattice with least and greatest elements. Let $\mathbb{A}$ be the class of all graphs satisfying τ^+ and let $\mathbb{C}$ be the class of all lattices satisfying ξ^+. Then for all sentences θ of $\mathscr{L}_0$,

$$(11) \qquad \theta \in T_0 \Leftrightarrow \tau^+ \rightarrow \theta_b^* \in T_G \Leftrightarrow \forall \mathscr{A} \in \mathbb{A} (\mathscr{A} \models \theta_b^*) \Leftrightarrow$$
$$\forall \mathscr{C} \in \mathbb{C} (\mathscr{C} \models (\theta_b^*)_d) \Leftrightarrow \xi^+ \rightarrow (\theta_b^*)_d \in T_d.$$

We now note that $\vee$, $\wedge$, 0 and 1 are definable in $\mathscr{L}_b$ by $\forall_1$ formulas over any lattice. Hence the right-hand sides of (9) and (10) can be expressed as $\exists_2$ formulas of

$\mathscr{L}_b$. We thus summarize the facts which we will need about the translation in the following remark, noting that its truth follows from Remark 2.6.

2.7 Remark. Assume that T_0 is accessible. Then there is a translation of $\mathscr{L}_0$ into $\mathscr{L}_b$ which satisfies (11) and which has the property that every model of T_0 corresponds to a model of T_d in which the interpretations of the universe of the model of T_0 and of all atomic relations on that model are given by $\exists_2$ formulas.

The preceding results will be used in the next section to relate the theory of second order arithmetic to the theory of distributive lattices with least and greatest elements. Similar methods are used to prove the result of Chap. VII that $\forall_3 \cap \mathrm{Th}(\mathscr{D})$ is undecidable. We begin by translating the theory of graphs into the theory of lattices.

Let $\mathscr{A} = \langle A, S\rangle$ be a graph. In order to avoid special cases, we assume that $|A| \geqslant 3$. We build a lattice $\mathscr{C} = \langle C, \leqslant\rangle$ (viewed as a poset) and an effective translation of $\mathscr{L}_b$ into $\mathscr{L}_b$ such that for any sentence θ of $\mathscr{L}_b$, the translation takes θ into $\theta^{\#}$ and $\mathscr{A} \models \theta \Leftrightarrow \mathscr{C} \models \theta^{\#}$.

We define the relation $\leqslant$ pictorially by means of the two figures below, letting $x \leqslant y$ hold exactly when a sequence of arrows goes from x to y. Let $A = \{a_i\}$ be the universe of $\mathscr{A}$. Then the configuration in Fig. 2.4 is inserted into the lattice if $\mathscr{A} \models S(a_i, a_j)$, and the configuration of Fig. 2.5 is inserted into the lattice otherwise.

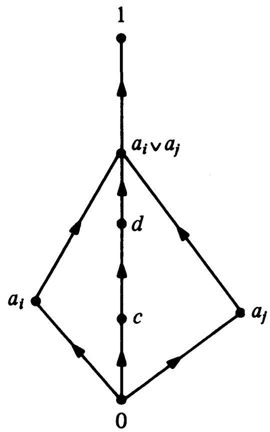

Fig. 2.4

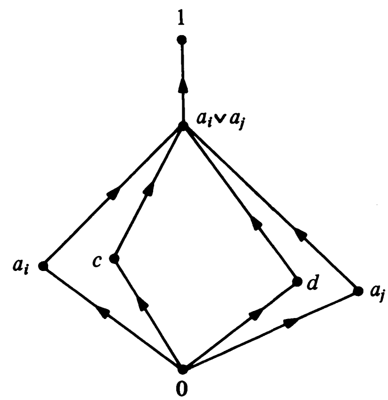

Fig. 2.5

The original structure can now be interpreted within this new structure, with universe expanded from A to include 0, 1, $a_i \vee a_j$, c and d. Furthermore, different elements are used to represent c and d for different choices of the pair $\langle a_i, a_j\rangle$. Since $|A| \geqslant 3$, we can pick out the elements of A as those atoms of C which lie below more than one join $\neq 1$. Thus if C^* is the set of interpretations of elements of A, then

$$(12) \qquad x \in C^* \Leftrightarrow \exists z, y_0, y_1 (z < x \,\&\, x < y_0 \,\&\, x < y_1 \,\&\, y_0 \mid y_1).$$

We now describe a uniform effective translation of $\mathscr{L}_b$ into $\mathscr{L}_b$. $\theta^{\#}$ is obtained from θ by restricting all quantifiers of θ to C^* and replacing all positive occurrences of $S(x, y)$ with a formula asserting that Fig. 2.5 does not lie in $\mathscr{C}$ for this choice of x and y, i.e.,

(13) $$x \neq y \,\&\, \forall y_0, y_1, y_2, y_3, y_4(y_0 < x \,\&\, y_0 < y_1 \,\&\, y_0 < y_2 \,\&\, y_0 < y \,\& \\ y_0 < y_3 \,\&\, y_0 < y_4 \,\&\, x \mid y_1 \,\&\, x \mid y_2 \,\&\, x \mid y_3 \,\&\, x \mid y \,\&\, x < y_3 \,\&\, x < y_4 \,\& \\ y_1 \mid y_2 \,\&\, y_1 \mid y \,\&\, y_1 < y_3 \,\&\, y_1 < y_4 \,\&\, y_2 \mid y \,\&\, y_2 < y_3 \,\&\, y_2 < y_4 \,\& \\ y < y_3 \,\&\, y < y_4 \,\&\, y_3 < y_4)$$

and replacing all negative occurrences of $S(x, y)$ with a formula asserting that Fig. 2.4 lies in $\mathscr{C}$ for this choice of x and y, i.e.,

(14) $$\exists y_0, y_1, y_2, y_3, y_4(y_0 < x \,\&\, y_0 < y_1 \,\&\, y_0 < y \,\&\, y_0 < y_2 \,\&\, y_0 < y_3 \,\& \\ y_0 < y_4 \,\&\, x \mid y_1 \,\&\, x \mid y \,\&\, x \mid y_2 \,\&\, x < y_3 \,\&\, x < y_4 \,\&\, y_1 < y_2 \,\& \\ y_1 \mid y \,\&\, y_1 < y_3 \,\&\, y_1 < y_4 \,\&\, y \mid y_2 \,\&\, y < y_3 \,\&\, y < y_4 \,\&\, y_2 < y_3 \,\& \\ y_2 < y_4 \,\&\, y_3 < y_4).$$

We note that this translation takes $\exists_2$ sentences to $\exists_2$ sentences. In order to evaluate this translation, we wish to restrict the class of posets considered. We let $\mathbb{P}$ be the class of all posets satisfying the following conditions:

(15) The poset axioms.

(16) Every chain has length $\leqslant 5$.

(17) There is a unique minimal element 0 and a unique maximal element 1.

(18) $\forall x(x \in C^* \rightarrow x$ has exactly one predecessor & any chain of elements $> x$ has at most two elements).

(19) $\forall x, y(x \in C^* \,\&\, y \in C^* \,\&\, x \neq y \rightarrow \exists z(x < z \,\&\, y < z \,\&\, z < 1))$.

(20) If z has exactly one successor, then the initial segment determined by z is isomorphic to either Fig. 2.4 or Fig. 2.5 with 1 deleted, with certain elements specified to be in C^* as stipulated by the definition of those figures.

Note that the conjunction of (15)–(20) can be expressed as a single sentence β of $\mathscr{L}_b$. We note that we can use (17) to write (16) as an $\forall_2$ sentence, so that β can be expressed as an $\forall_2$ sentence. Furthermore, all posets satisfying β arise from some graph. Hence

(21) $$\theta \in T_G \Leftrightarrow \forall \mathscr{A} \in \mathbb{A}(\mathscr{A} \models \theta) \Leftrightarrow \forall \mathscr{P} \in \mathbb{P}(\mathscr{P} \models \theta^{\#}) \Leftrightarrow \vdash \beta \rightarrow \theta^{\#}.$$

Strongly undecidable sets of sentences play a role in the statement and proof of the undecidability of $\forall_3 \cap \mathrm{Th}(\mathscr{D})$.

2.8 Definition. Let V be the set of all logically valid sentences. A set Σ of sentences is *strongly undecidable* if there is no recursive set R such that $V \cap \Sigma \subseteq R \subseteq \Sigma$.

2.9 Theorem. *The set of all* $\exists_2$ *sentences of* $\mathscr{L}_b$ *which are true in all finite lattices is strongly undecidable.*

Proof. We note that by Ershov and Taitslin [1963], the set of all $\exists_2$ sentences of $\mathscr{L}_b$ which are true in all finite graphs is strongly undecidable. Since all posets which satisfy β are lattices, and our interpretation passes from finite graphs to finite lattices, it suffices to show that $\Sigma^* = \{\beta \to \sigma^\# : \sigma$ is an $\exists_2$ sentence of $\mathscr{L}_b$ and $\beta \to \sigma^\#$ is true in all finite posets$\}$ is strongly undecidable. We assume that this is not the case, and obtain a contradiction.

Let R be a recursive set of sentences of $\mathscr{L}_b$ and suppose that $V \cap \Sigma^* \subseteq R \subseteq \Sigma^*$. Let $S = \{\sigma : \beta \to \sigma^\# \in R\}$. Then S is recursive. Let Σ be the set of $\exists_2$ sentences of $\mathscr{L}_b$ which are true in all finite graphs. Let $\sigma \in V \cap \Sigma$ be given. Then for all finite graphs $\mathscr{A}$, $\mathscr{A} \models \sigma$. By (21), $\beta \to \sigma^\# \in V$. Since $\beta \to \sigma^\# \in \Sigma^*$, we conclude that $\beta \to \sigma^\# \in R$ and so that $\sigma \in S$. Now assume that $\sigma \in S$. Then $\beta \to \sigma^\# \in R \subseteq \Sigma^*$ and so $\beta \to \sigma^\#$ is true in all finite posets. By the correspondence between graphs and lattices, σ must be true in all finite graphs, so $\sigma \in \Sigma$. Hence $V \cap \Sigma \subseteq S \subseteq \Sigma$, yielding the desired contradiction. ▯

2.10 Remarks. Remarks 2.3, 2.6 and 2.7 can be found in Nerode and Shore [1979], but the results are due to Rabin and Scott. Theorem 2.9 is due to Schmerl.

3. Second Order Arithmetic

The preliminary steps for coding second order arithmetic into the degrees are described in this section. We make use of some results from the previous section.

We will need to work with a finitely axiomatizable theory of arithmetic, so the theory we work with will be very weak. Since we will eventually be able to talk about the theory of a standard model of arithmetic, however, we will eventually be able to work with true arithmetic.

3.1 Definition. The language $\mathscr{L}_a$ is the language of the pure predicate calculus together with a binary relation symbol $\leqslant$, and ternary relation symbols $+$ and $\times$. By the *theory of arithmetic* we mean the deductive closure in the language $\mathscr{L}_a$ of the axioms which assert that we have a discretely ordered commutative semiring with unity. (We note that this theory is finitely axiomatizable.) A *model of arithmetic* is a structure $\mathscr{M} = \langle M, \leqslant, +, \times \rangle$ where $+$, $\times$, and $\leqslant$ have the obvious interpretation such that $\mathscr{M}$ satisfies all the sentences of the theory of arithmetic. A *standard model of arithmetic* is one in which the posets $\langle M, \leqslant \rangle$ and $\langle N, \leqslant \rangle$ are isomorphic. We define *second order arithmetic* to be $\mathrm{Th}(\langle N, 2^N, \leqslant, +, \times, \in \rangle)$ where quantifiers are introduced to range over 2^N, the set of subsets of N, and $\in \subseteq N \times 2^N$ is interpreted as the binary relation *is an element of*.

3.2 Remark. It follows from Definition 2.1 that $\mathscr{M}$ is a model of arithmetic if and only if $\mathscr{M}$ satisfies a first order sentence σ_a in the language $\mathscr{L}_a$. Also, since $1 \times y = y$ for any y in the universe of any model of arithmetic, we note that the theory of arithmetic is accessible. Hence Remark 2.7 applies to this theory.

By Remark 2.7, there is a sentence σ_d of $\mathscr{L}_b$ such that for all sentences θ of $\mathscr{L}_a$, θ holds in all models of arithmetic if and only if the translation of θ into $\mathscr{L}_b$ holds in all distributive lattices which satisfy σ_d.

3.3 Definition. Let $\mathscr{L} = \langle L, \leqslant \rangle$ be a poset which is order isomorphic to a lattice. We say that $\mathscr{L}$ *codes a model of arithmetic* if $\mathscr{L} \models \sigma_d$.

We now turn to second order arithmetic. Let $\mathscr{L}_I$ be the language $\mathscr{L}_b$ augmented with a binary relation symbol $\in$ to be interpreted as a subset of $L \times \mathscr{I}_L$ (L is the universe of a lattice $\mathscr{L}$ and $\mathscr{I}_L$ is a set of countable ideals of $\mathscr{L}$) by *is an element of*, with second order quantifiers $\forall I$ and $\exists I$ which range over $\mathscr{I}_L$. (We show in Chap. VIII.3 how to translate a sentence of $\mathscr{L}_I$ interpreted on distributive lattices into an equivalent first order sentence about $\mathscr{D}$.) We will be able to define $\mathscr{L}$ *codes a standard model of arithmetic* in $\mathscr{L}_I$. We wish to interpret the quantifiers $\forall A \subseteq N$ and $\exists A \subseteq N$ of second order arithmetic by $\forall I$ and $\exists I$ respectively, and for $A \subseteq N$, the formula $x \in A$ in the language of second order arithmetic by $x \in L^* \cap I$, where I is a countable ideal of $\mathscr{L}$ and L^* is the universe of interpretation of the model of arithmetic given by the translation of Remark 2.7. Thus we need a one-one correspondence between subsets of N and countable ideals of $\mathscr{L}$ containing those elements of L^* corresponding to elements of A, and no other elements of L^*. Since $\mathscr{L}$ will be a distributive lattice and L^* will consist of atoms of $\mathscr{L}$, this correspondence must exist.

A sentence of $\mathscr{L}_I$ which asserts that $\mathscr{L}$ codes a standard model of arithmetic, is obtained as follows. We note that the original model $\mathscr{M}$ of arithmetic is a standard model exactly when $\langle M, \leqslant \rangle$ and $\langle N, \leqslant \rangle$ are isomorphic. Thus we need to be able to say that if $\leqslant_L$ is the interpretation in $\mathscr{L}$ of the ordering $\leqslant$ in $\mathscr{M}$, then any subset of L^* which is bounded under $\leqslant_L$ has a greatest element. The following sentence σ^* asserts this fact:

$$\forall I((\exists z \in L^* \, \forall x \in L^*(x \in I \rightarrow x \leqslant_L z)) \rightarrow \exists z \in L^*(z \in I \,\&$$
$$\forall x \in L^*(x \in I \rightarrow x \leqslant_L z))).$$

We now see that if a lattice $\mathscr{L}$ satisfies σ^*, then second order arithmetic is reducible to Th($\mathscr{L}$) in the language $\mathscr{L}_I$. We thus summarize the results of this section.

3.4 Theorem. *There is a sentence σ^* of $\mathscr{L}_I$ and an effective translation taking any sentence θ of second order arithmetic into the sentence θ_I of $\mathscr{L}_I$ such that*

$$\langle N, 2^N, \leqslant, +, \times, \in \rangle \models \theta \Leftrightarrow \forall \mathscr{L}(\mathscr{L} \models \sigma^* \rightarrow \mathscr{L} \models \theta_I).$$

Under this translation, the integers are interpreted by an $\exists_2$-definable subset $L^* \subseteq L$ and $\leqslant_L$ is interpreted by an $\exists_2$ formula of $\mathscr{L}_b$. Furthermore, there is a recursive $\mathscr{L}$ which satisfies σ^* for which L^* is recursive. (*An $\mathscr{L}$ which satisfies σ^* is said to code a standard model of arithmetic.*)

Proof. Since all standard models of arithmetic are isomorphic, they have the same elementary theory which is complete. The theorem follows from Remark 2.7 and the fact that the correspondence between models of T_0 and models of T_d given in Sect. 2 is recursive. Since the standard model of arithmetic is recursive, there must therefore be a recursive $\mathscr{L}$ which codes a standard model of arithmetic. ▯

3.5 Remark. The results of this section are due to Nerode and Shore [1979].