Chapter XI

Bounding Minimal Degrees with Recursively Enumerable Degrees

The constructions of minimal degrees which have been presented to this point have been oracle constructions. Most of the theorems in Part C, however, were originally proved using *full approximation* constructions. Although different constructions have different features, the common thread in full approximation constructions is that both the set of minimal degree and the trees on which this set lies are simultaneously constructed through recursive approximations.

In this chapter, we prove that every non-zero recursively enumerable degree bounds a minimal degree. The proof we give involves a full approximation construction.

1. Trees Permitted by Recursively Enumerable Sets

Let C be a non-recursive, recursively enumerable set, and let $h: N \to N$ be a one-one recursive function enumerating C. Let $C^s = \{h(x): x \leqslant s\}$. We construct a set $A \leqslant_T C$ by recursive approximation. Thus we define a recursive sequence of strings $\{\alpha_s: s \in N\}$ and let $A(x) = \lim_s \alpha_s(x)$ for all $x \in N$. (Recall that in oracle constructions, A was more simply defined by $A = \cup\{\alpha_s: s \in N\}$.) C will control the recursive approximation $\{\alpha_s: s \in N\}$ by subjecting the construction of this approximation to the following constraint:

(1) $\qquad \forall x, s \in N(C^s \upharpoonright x = C \upharpoonright x \to \alpha_s \upharpoonright x \subset A).$

Condition (1) will guarantee that $A \leqslant_T C$, as is shown in the following lemma.

1.1 Yates Permitting Lemma. *Let* $\{\alpha_s: s \in N\}$ *be a recursive sequence of elements of* $\mathscr{S}_2$. *Define* $A \subseteq N$ *by* $A(x) = \lim_s \alpha_s(x)$ *for all* $x \in N$. *Assume that* (1) *holds. Then* $A \leqslant_T C$.

Proof. To compute $A(x)$, search for the least $s \in N$ such that $C^s \upharpoonright x + 1 = C \upharpoonright x + 1$ and $\mathrm{lh}(\alpha_s) > x$. Since h is a one-one recursive function which enumerates C, such an s can be found through the use of a C oracle. By (1), $A(x) = \alpha_s(x)$. $\square$

We will not worry, during the construction, about forcing the standard requirements whose purpose is to guarantee that A is not recursive. Posner's Lemma will allow us to show that all such requirements are automatically satisfied. Thus it will suffice to satisfy the following requirements for all $e \in N$:

$$P_e\text{: If } \Phi_e^A \text{ is total, then either } \Phi_e^A \text{ is recursive or } A \leqslant_T \Phi_e^A.$$

Such requirements were previously satisfied through the use of e-splitting trees, and in this construction, we will also try to place A on an e-splitting tree for each $e \in N$. Suppose that a partial recursive tree T is given, and we attempt to construct an e-splitting subtree T^* of T. Let σ be terminal on T_s^*, the approximation to T^* at stage s. Since no appeal to an oracle of degree $\mathbf{0}'$ is allowed, we cannot determine, at stage s, whether or not there is an e-splitting of σ on T. While such an e-splitting is being sought, we are defining $\{\alpha_s : s \in N\}$, so if such an e-splitting is eventually found and $\sigma \subset A$, then it is possible that the e-splitting has been found too late, as erecting it on T^* may violate (1); (1) has priority over all other requirements, so C must act to prevent the erection of unsuitable e-splittings on T^*. We will show that if there are infinitely many $\sigma \subset A$ which have e-splitting extensions on T, then C will permit one of these e-splittings to be erected on T^*, and so we will be able to prove a computation lemma. Otherwise, we will find a $\sigma \subset A$ such that $\sigma \subset T$ and σ has no e-splitting extensions on T; so if $A \subset T$ then Φ_e^A is recursive.

We begin the construction of e-splitting trees with the above motivation in mind. The trees are constructed by recursive approximation. The first step, described in the next definition, tells us how to erect one new e-splitting on T^*. The parameter α in this definition represents the approximation, at stage s, to the set of minimal degree.

1.2 Definition. Let T and T^* be finite trees such that $T^* \subseteq T$. Let $e, s \in N$ and $\alpha \in \mathscr{S}_2$ be given. Define the tree $T^+ = \mathrm{PSp}_2(T, T^*, \alpha, e, s)$ as follows: Let $T^+(\xi) = T^*(\xi)$ for all $\xi \in \mathscr{S}_2$ such that $T^*(\xi)\downarrow$. If $T^*(\emptyset)\uparrow$, let $T^+(\emptyset) = T(\emptyset)$ if $T(\emptyset)\downarrow$, and $T^+(\emptyset)\uparrow$ otherwise. Assume that $T^*(\xi^-)\downarrow$ and is terminal on T^*. Search for the least $\langle \sigma_0, \sigma_1, x\rangle \in \mathscr{S}_2^2 \times N$ (under some fixed recursive one-one correspondence of N with $\mathscr{S}_2^2 \times N$) such that:

(2) $\quad \forall i \leqslant 1(T^*(\xi^-) \subset \sigma_i \subset T).$

(3) $\quad \forall y \in N(y < h(s) \,\&\, y \leqslant \mathrm{lh}(\alpha) \rightarrow \sigma_0(y) = \sigma_1(y) = \alpha(y)).$

(4) $\quad \forall i \leqslant 1(\mathrm{lh}(\sigma_i) \leqslant s) \,\&\, x \leqslant s.$

(5) $\quad \langle \sigma_0, \sigma_1 \rangle$ e-splits on x.

Note that if we are presented with T and T^* as finite trees whose domains are known, then the existence of such a triple can be determined uniformly and recursively in e, x, and s. If no such triple exists, then $T^+(\xi)\uparrow$. Otherwise, $T^+(\xi) = \sigma_i$ where $\xi = \xi^- * i$. For all other strings δ, $T^+(\delta)\uparrow$.

$\mathrm{PSp}_2(T, T^*, \alpha, e, s)$ provides the basic building block for the construction of an e-splitting tree $T^* = \cup\{T_s^* : s \in N\}$. The tree T^* is defined in terms of the sequence

$\{\alpha_s: s \in N\}$, with α_s used to defined T^*_{s+1} which, in turn, is used to determine α_{s+1}. It will also be necessary to specify the stage t at which the construction of T^* begins.

1.3 Definition. Let T be a partial recursive tree with recursive approximation $\{T_s: s \in N\}$ of finite trees such that $T = \bigcup\{T_s: s \in N\}$ and $\{\langle\sigma, s\rangle \in \mathcal{S}_2 \times N: \sigma \in \mathrm{dom}(T_s)\}$ is recursive. Let $A \subseteq N$ and $e \in N$ be given, and let $\{\alpha_s: s \in N\}$ be a recursive sequence of strings such that $A = \lim_s \alpha_s$. Assume that $A \subset T$. Define the tree $T^* = \mathrm{PSp}_2(T, e, t, \{\alpha_s: s \in N\}) = \bigcup\{T^*_s: s \geqslant t\}$ by $T^*_t = \emptyset$ and $T^*_{s+1} = \mathrm{PSp}_2(T_{s+1}, T^*_s, \alpha_s, e, s+1)$.

1.4 Remark. It is easily verified that for all $s \in N$, $T^*_s \subseteq T_s$ and T^*_{s+1} extends T^*_s, where $T^* = \mathrm{PSp}_2(T, e, t, \{\alpha_s: s \in N\})$. Furthermore, T^* is partial recursive, and $\{\langle\sigma, s\rangle \in \mathcal{S}_2 \times N: \sigma \in \mathrm{dom}(T^*_s)\}$ is recursive.

We now prove a computation lemma for the splitting trees which have just been defined.

1.5 Computation Lemma. *Let T be a partial recursive tree, and let $e, t \in N$ be given. Let $T^* = \mathrm{PSp}_2(T, e, t, \{\alpha_s: s \in N\})$ where $\{\alpha_s: s \in N\}$ is a recursive sequence of binary strings with limit $A \subseteq N$. Then:*

(i) *If $A \subset T^*$ and Φ_e^A is total, then $A \equiv_T \Phi_e^A$.*

(ii) *If $A \subset T$ and there is a terminal string $\sigma \subset T^*$ such that $\sigma \subset A$, Φ_e^A is total and* (1) *holds, then Φ_e^A is recursive.*

Proof. For all $A \subseteq N$, if Φ_e^A is total, then $\Phi_e^A \leqslant_T A$. By the proof of Computation Lemma V.2.6, since T^* is an e-splitting tree, for all branches A of T^*, $A \leqslant_T \Phi_e^A$. Hence (i) holds.

The verification of (ii) also refers to Computation Lemma V.2.6. By that lemma and since $A \subset T$ and Φ_e^A is total, it suffices to show that there is a $\tau \subset T$ such that $\tau \subset A$ and there are no e-splittings of τ on T. We assume that no such τ exists, and obtain a contradiction by showing that C is recursive.

Fix σ as in (ii). For each $y \in N$, search for $s(y) = s \in N$, $x(y) = x \in N$ and $\tau, \sigma_0, \sigma_1 \in \mathcal{S}_2$ such that (4) and (5) hold, $\sigma \subseteq \tau \subseteq \sigma_i \subset T_s$ for $i \leqslant 1$ (T is specified through a recursive approximation $\{T_s: s \in N\}$), $\mathrm{lh}(\tau) > y$, $\sigma \subset T^*_s$ and $\tau \upharpoonright y+1 = \alpha_s \upharpoonright y+1$. Such s, x, τ, σ_0 and σ_1 must exist by the assumption which has been made, and can be found recursively. It suffices to show that for each $y \in N$, $h(t) > y$ for all $t \geqslant s(y)+1$. For then $C(y) = C^{s(y)+1}(y)$, and since s is a recursive function, we will then have a recursive computation of C, yielding the desired contradiction. We may therefore assume that for some $y \in N$, there is a least $r > s(y) = s$ such that $h(r) \leqslant y$. But then by (1), $\alpha_r \upharpoonright y+1 = \alpha_s \upharpoonright y+1$, so (2)–(5) hold for σ in place of $T^*(\xi^-)$ and $r+1$ in place of s. Thus σ is not terminal on T^*, yielding the desired contradiction. ▯

The trees which were introduced in this section will be used in the next section to construct a minimal degree below the degree of C. Other problems which will be encountered during the construction will be discussed in the next section.

1.6 Remarks. Permitting in the form presented here was developed by Friedberg [1957b] and Yates [1965]. A simultaneous construction of e-splitting trees permitted by C and $\{\alpha_s: s \in N\}$ can be found in Yates [1970a], and another such construction in the style of Cooper [1973], [1974] which is carefully motivated and presented appears in Epstein [1975].

1.7 Exercise. Let T be a partial recursive tree, let $\{\alpha_s : s \in N\}$ be a recursive sequence of binary strings with limit $A \subseteq N$, and let $e \in N$ be given. Construct a partial recursive subtree $T^* = \mathrm{PTot}_2(T, e, \{\alpha_s : s \in N\})$ of T such that:

(i) For every branch $B \subset T^*$, Φ_e^B is total.

(ii) If $A \subset T$, $A \not\subset T^*$, and (1) holds, then there is a terminal string $\sigma \subset A$ of T^*, a string τ such that $\sigma \subseteq \tau \subset T$ and $x \in N$ such that for all $\rho \subset T$, if $\tau \subseteq \rho$ then $\Phi_e^\rho(x)\uparrow$.

2. *Minimal Degrees and Recursively Enumerable Permitting*

Fix C and h as in Section 1. A set A of minimal degree is constructed such that $A \leqslant_T C$.

In order to make use of the Permitting Lemma, we cannot allow any appeal to an oracle during the construction of A. Hence it will not be possible for us to decide at stage s of the construction of A whether or not a string α_s in the approximation to A is terminal on $T = \cup\{T_t : t \in N\}$. Thus, at stage s, we may be forced to guess that α_s is terminal on T (because α_s is terminal on T_s), in which case we cease the attempt to construct A on T; but we may later discover, at stage $t > s$, that α_s is not terminal on T_t, and hence α_s is not terminal on T. We must then resume our attempt to construct A on T. When we return to the finite tree T_t which extends T_s, we must modify α_{t-1} to obtain α_t such that $\alpha_s \subseteq \alpha_t$, and this modification must be permitted by C. As we cannot control C, we insure that this modification will be possible by imposing constraints on $\{\alpha_r : s \leqslant r \leqslant t\}$ which require that $\alpha_r \supseteq \alpha_s$ unless we make the decision never again to return to T. Such a decision will sometimes be made, and when this happens, it will be due to certain priority considerations.

Since we can never know if we are deserting a tree forever, it will not be possible to construct A on a sequence of trees. Rather, we choose a path through a tree of trees on which we construct A. That path is the one of highest priority which we follow infinitely often during the course of the construction. *e-states* are assigned to paths in order to facilitate the definition of priority of paths.

2.1 Definition. Let $\lambda \in \mathcal{S}_2$ be given such that $\mathrm{lh}(\lambda) = e + 1$. The *e-state* of λ is $E(\lambda) = \Sigma\{2^{e-i} : \lambda(i) = 1\}$.

The properties of the binary representation of natural numbers imply the following facts about e-states. We leave the verification of these facts to the reader.

(1) $$\lambda_1 \neq \lambda_2 \in \mathcal{S}_2 \,\&\, \mathrm{lh}(\lambda_1) = \mathrm{lh}(\lambda_2) = e + 1 \to E(\lambda_1) \neq E(\lambda_2).$$

(2) $$\forall \lambda_0, \lambda_1, \delta_0, \delta_1 \in \mathcal{S}_2(\mathrm{lh}(\lambda_0) = \mathrm{lh}(\lambda_1) = e + 1 \,\&\, \mathrm{lh}(\delta_0) = \mathrm{lh}(\delta_1) = n + 1$$
$$\&\, e < n \,\&\, E(\lambda_0) < E(\lambda_1) \,\&\, \lambda_0 \subset \delta_0 \,\&\, \lambda_1 \subset \delta_1 \to E(\delta_0) < E(\delta_1)).$$

(3) $$\forall \lambda_0, \lambda_1, \delta_0, \delta_1 \in \mathcal{S}_2(\mathrm{lh}(\lambda_0) = \mathrm{lh}(\lambda_1) = e + 1 \,\&\, \mathrm{lh}(\delta_0) = \mathrm{lh}(\delta_1) = n + 1$$
$$\&\, e < n \,\&\, E(\delta_0) \leqslant E(\delta_1) \,\&\, \lambda_0 \subset \delta_0 \,\&\, \lambda_1 \subset \delta_1 \to E(\lambda_0) \leqslant E(\lambda_1)).$$

The tree of trees is used as follows. For each $\delta \in \mathscr{S}_2$ and $s \in N$, we will define $T_{\delta,s}$ such that $\{\delta: T_{\delta,s} \neq \emptyset\}$ is finite. $T_{\delta,s}$ will be a finite approximation to what we hope will be a tree T_δ. We begin, for $\delta = \emptyset$, by setting $T_{\emptyset,s} = \mathrm{Id}_{2,s}$ (see Section 10.1). Given $T_{\delta,s}$, we try to construct $T_{\delta*1,s}$ as an e-splitting tree for $e = \mathrm{lh}(\delta)$, and if we are forced to leave $T_{\delta*1,s}$, there will be a terminal string $\sigma \subset A$ on $T_{\delta*1,s}$ such that $\sigma = T_{\delta,s}(\xi)$. We then set $T_{\delta*0} = \mathrm{PExt}_2(T_\delta, \xi)$ and try to build A on $T_{\delta*0}$. As indicated earlier, if we discover, at stage $t > s$, that σ is not terminal on $T_{\delta*1,t}$ and $T_{\delta*1,t}$ extends $T_{\delta*1,s}$, then we may choose to return to $T_{\delta*1}$. In this case, $T_{\delta*0}$ is cancelled, and we later begin new attempts at building $T_{\delta*0}$. If $T_{\delta*0}$ is cancelled only finitely often, then there will be a stage r such that $T_{\delta*0} = \bigcup\{T_{\delta*0,s}: s \geqslant r\}$ is a well-defined tree. Otherwise, there will be a $\lambda \in \mathscr{S}_2$ such that $\mathrm{lh}(\lambda) = \mathrm{lh}(\delta) + 1$, λ has higher priority than $\delta * 0$, and we choose to follow $T_{\lambda,s}$ at infinitely many stages of the construction.

At stage s of the construction, we define $\gamma_s \in \mathscr{S}_2$ and choose to continue the construction of the minimal degree at stage s along the sequence of trees $\{T_{\gamma,s}: \gamma \subseteq \gamma_s\}$. We will define $\Gamma = \limsup_s \gamma_s$ as a path through the tree of trees, and will show that for all $\gamma \subseteq \Gamma$, T_γ is cancelled at only finitely many stages. The minimal degree requirements will then be forced by $\{T_\gamma: \gamma \subset \Gamma\}$.

A minimal degree permitted by C is now constructed.

2.2 Theorem. *Let* $\mathbf{c} \neq \mathbf{0}$ *be a recursively enumerable degree. Then there is a minimal degree* $\mathbf{a} \leqslant \mathbf{c}$.

Proof. Let C and h be as in Section 1, and let $\mathbf{c}$ be the degree of C. We will construct recursive sequences of elements of $\mathscr{S}_2$ $\{\alpha_s: s \in N\}$ and $\{\gamma_s: s \in N\}$, together with a recursive array of trees $\{T_{\delta,s}: s \in N \,\&\, \mathrm{lh}(\delta) \leqslant \mathrm{lh}(\gamma_s)\}$. At stage s, T_δ will be designated either as the identity tree, an extension tree or a splitting tree, and $T_{\delta,s}$ will be the recursive approximation to T_δ as specified either by Definition 1.3, X.1.6 or X.1.8. $A = \lim_s \alpha_s$ will be the set of minimal degree, and $\Gamma = \limsup_s \gamma_s$ will pick out the path through the tree of trees on which A lies. Thus $A \subset T_{\Gamma \restriction n,s}$ for all $n \in N$ and all sufficiently large s. The following induction hypotheses will be satisfied at the end of stage s:

(4) $s > 0 \rightarrow \forall x < h(s)(\alpha_{s-1}(x)\downarrow \rightarrow \alpha_s(x)\downarrow = \alpha_{s-1}(x))$. (Thus we make sure that (1) is satisfied, and so can apply the Permitting Lemma.)

(5) $s > 0 \,\&\, h(s) > \mathrm{lh}(\alpha_{s-1}) \rightarrow \mathrm{lh}(\alpha_s) > \mathrm{lh}(\alpha_{s-1})$. (This condition will insure that $\lim_s \mathrm{lh}(\alpha_s) = \infty$ and so that $A \subseteq N$.)

(6) $\forall \delta \in \mathscr{S}_2(\mathrm{lh}(\delta) \leqslant \mathrm{lh}(\gamma_s) \,\&\, E(\delta) > E(\gamma_s \restriction \mathrm{lh}(\delta)) \rightarrow \exists \sigma \in \mathscr{S}_2$ (σ is terminal on $T_{\delta,s}$ & σ is terminal on $T_{\delta,s-1}$ & $\sigma \subset \alpha_s$)). (Thus if δ has higher priority than γ_s, then α_s extends a terminal string of $T_{\delta,s}$. This condition will allow us to apply the Computation Lemma.)

(7) $\forall \delta \in \mathscr{S}_2(\mathrm{lh}(\delta) \leqslant \mathrm{lh}(\gamma_s) \,\&\, E(\delta) \leqslant E(\gamma_s \restriction \mathrm{lh}(\delta)) \rightarrow \alpha_s \subset T_{\delta,s})$. (This condition allows us to return to higher priority trees later in the construction if we desert them now, by insuring that all strings on trees of lower priority than T_{γ_s} extend α_s. This is the consideration mentioned prior to Definition 2.1.)

(8) $\mathrm{lh}(\gamma_s) = s$. (Thus $\mathrm{lh}(\Gamma)$ will be infinite.)

(9) $T_{\emptyset,s} = \mathrm{Id}_{2,s}$.

(10) $\forall\delta\in\mathcal{S}_2(0<\mathrm{lh}(\delta)\leqslant\mathrm{lh}(\gamma_s)\to T_{\delta,s}\subseteq T_{\delta^-,s})$. (The subtree condition.)

(11) $\forall\delta\in\mathcal{S}_2(0<\mathrm{lh}(\delta)\leqslant\mathrm{lh}(\gamma_s)\,\&\,\delta=\delta^-*1\,\&\,s>0\,\&\,E(\delta)\geqslant E(\gamma_s\restriction\mathrm{lh}(\delta))\to T_{\delta,s}=\mathrm{PSp}_2(T_{\delta^-,s},T^*_{\delta,s-1},\alpha_{s-1},\mathrm{lh}(\delta^-),s))$. (Splitting subtrees are constructed in the high priority direction.)

(12) $\forall\delta\in\mathcal{S}_2(0<\mathrm{lh}(\delta)\leqslant\mathrm{lh}(\gamma_s)\,\&\,\delta=\delta^-*0\,\&\,E(\delta)\geqslant E(\gamma_s\restriction\mathrm{lh}(\delta))\to\exists\xi\in\mathcal{S}_2(T_{\delta,s}=\mathrm{PExt}_2(T_{\delta^-,s},\xi)\,\&\,T_{\delta^-,s}(\xi)$ is terminal on $T_{\delta^-*1,s}))$. (Thus we leave splitting trees via terminal strings for lower priority trees. The Computation Lemma will show that if $\lambda*0\subset\Gamma$ then some $\sigma\subset A$ will have no $\mathrm{lh}(\lambda)$-splittings on $T_\lambda=\cup\{T_{\lambda,s}: s\geqslant r\}$ for some r.)

The construction proceeds as follows.

Stage 0. $\alpha_0=\gamma_0=\emptyset$ and $T_{\emptyset,0}=\mathrm{Id}_{2,0}$.

Stage $s+1$. Let $k(s)=\min(\{h(s+1),\mathrm{lh}(\alpha_s)\})$ and let $\beta_{s+1}=\alpha_s\restriction k(s)$. Thus we cut α_s back to as short a string as C will permit. We now proceed by substages $\{m: 0\leqslant m<2^{s+1}\}$ each of which is carried out in steps $\{n: 0\leqslant n\leqslant s+1\}$. At each substage, we define a tree $T_{\delta,s+1}$ with $\mathrm{lh}(\delta)=s+1$, higher priority trees being defined first. The steps in each substage lead up to the definition of $T_{\delta,s+1}$, by defining $T_{\lambda,s+1}$ for $\lambda\subseteq\delta$, with trees for which λ is shorter defined first.

Substage m. Fix $\delta_m\in\mathcal{S}_2$ such that $\mathrm{lh}(\delta_m)=s+1$ and $E(\delta_m)=2^{s+1}-m-1$. Let $\delta=\delta_m$.

Step $n=0$. Let $T_{\emptyset,s+1}=\mathrm{Id}_{2,s+1}$.

Step n; $0<n\leqslant s+1$. Let $\lambda_{m,n}\in\mathcal{S}_2$ be given such that $\mathrm{lh}(\lambda_{m,n})=n$ and $\lambda_{m,n}\subseteq\delta_m$. If $T_{\lambda_{m,n},s+1}$ has already been defined, proceed to the next step. Otherwise, there are three cases. Let $\lambda=\lambda_{m,n}$.

Case 1. $\lambda=\lambda^-*1$ and α_{s+1} has not yet been defined. Let

$$T^*_{\lambda,s}=\begin{cases}T_{\lambda,s} & \text{if } n<s+1\\ \emptyset & \text{if } n=s+1\end{cases}$$

and

$$T_{\lambda,s+1}=\mathrm{PSp}_2(T_{\lambda^-,s+1},T^*_{\lambda,s},\alpha_s,\mathrm{lh}(\lambda^-),s+1).$$

Case 2. $\lambda=\lambda^-*0$ and α_{s+1} has not yet been defined. In this case, we will be able to fix $\xi=\xi(m,n,s)\in\mathcal{S}_2$ such that $\beta_{s+1}\supseteq T_{\lambda^-,s+1}(\xi)$ and $T_{\lambda^-,s+1}(\xi)$ is terminal on $T_{\lambda^-*1,s+1}$. Let $T_{\lambda,s+1}=\mathrm{PExt}_2(T_{\lambda^-,s+1},\xi)$.

Case 3. α_{s+1} has been defined. Let

$$T_{\lambda,s+1}(\eta)=\begin{cases}\alpha_{s+1} & \text{if } \eta=\emptyset\\ \uparrow & \text{otherwise.}\end{cases}$$

If $n=s+1$ and α_{s+1} has not yet been defined, search for $\eta\in\mathcal{S}_2$ such that $\beta_{s+1}\subset T_{\delta,s+1}(\eta)\downarrow$. If such an η exists, fix such an η of shortest length with the lexicographically least η preferred, and let $\alpha_{s+1}=T_{\delta,s+1}(\eta)$ and $\gamma_{s+1}=\delta_m$. Proceed to substage $m+1$ if $m+1<2^{s+1}$ and to stage $s+2$ otherwise.

This completes the construction. Note that $\{E(\delta)\colon \mathrm{lh}(\delta) = s+1\} = [0, 2^{s+1})$ so for each $\delta \in \mathscr{S}_2$ such that $\mathrm{lh}(\delta) = s+1$, there is a substage during which $T_{\delta,s+1}$ is defined. The following lemma will be used to show that α_{s+1} and γ_{s+1} exist, and that α_{s+1} is compatible with all trees defined at stage $s+1$.

2.3 Lemma. *Let $s \in N$ and $\delta \in \mathscr{S}_2$ be given such that $\mathrm{lh}(\delta) \leqslant s+1$, and*

(i) $\gamma_{s+1}\downarrow \leftrightarrow E(\gamma_{s+1} \upharpoonright \mathrm{lh}(\delta)) \leqslant E(\delta)$.

Then the following conditions hold:

(ii) *β_{s+1} is compatible with $T_{\delta,s+1}$ (i.e., there is a $\xi \in \mathscr{S}_2$ such that either $\beta \subseteq T_{\delta,s+1}(\xi)$ or $T_{\delta,s+1}(\xi) \subset \beta$ and $T_{\delta,s+1}(\xi)$ is terminal on $T_{\delta,s+1}$).*

(iii) *If $\delta = \emptyset$, or if $\delta = \delta^- * 0$ and $\exists \sigma \in \mathscr{S}_2(\beta_{s+1} \subset \sigma \subset T_{\delta^-,s+1})$, then $\exists \sigma \in \mathscr{S}_2(\beta_{s+1} \subset \sigma \subset T_{\delta,s+1})$.*

Proof. We proceed inductively, following the length and e-state of δ. Note that since $T_{\emptyset,t} = \mathrm{Id}_{2,t}$ for all t, (ii) and (iii) follow easily for $\delta = \emptyset$. Fix s and δ as in the hypothesis of the lemma, and let $t = \mathrm{lh}(\delta)$. Let m be the first substage of stage $s+1$ such that $\delta \subseteq \delta_m$. By (i), $T_{\delta,s+1}$ is defined through Case 1 or Case 2 at step t, substage m of stage $s+1$ depending, respectively, on whether $\delta = \delta^- * 1$ or $\delta = \delta^- * 0$.

Assume first that $\delta = \delta^- * 1$. By Case 1 of the construction, $T_{\delta,s+1} = \mathrm{PSp}_2(T_{\delta^-,s+1}, T^*_{\delta,s}, \alpha_s, \mathrm{lh}(\delta^-), s+1)$. Furthermore, $\beta_{s+1} = \alpha_s \upharpoonright h(s+1)$, $T^*_{\delta,s} = T_{\delta,s}$ if $\mathrm{lh}(\delta) \leqslant s$, and $T^*_{\delta,s} = \emptyset$ if $\mathrm{lh}(\delta) = s+1$; hence by Definition 1.2, if β_{s+1} is compatible with $T^*_{\delta,s}$ then β_{s+1} will be compatible with $T_{\delta,s+1}$. This latter fact is clear if $\mathrm{lh}(\delta) = s+1$, and follows inductively from (6) and (7) if $\mathrm{lh}(\delta) \leqslant s$ since α_s is compatible with $T_{\delta,s}$. Hence (ii) follows. (iii) is vacuous in this case.

Assume that $\delta = \delta^- * 0$. By induction on e-states, β_{s+1} is compatible with $T_{\delta^- * 1,s+1}$. There can be no $\sigma \in \mathscr{S}_2$ such that $\beta_{s+1} \subset \sigma \subset T_{\delta^-,s+1}$, else applying (iii) repeatedly by induction, we conclude that $\gamma_{s+1} \supseteq \lambda^- * 1$ and hence that $E(\gamma_{s+1} \upharpoonright \mathrm{lh}(\delta)) > E(\delta)$. Hence we can fix ξ as in Case 2. Thus $T_{\delta,s+1} = \mathrm{PExt}_2(T_{\delta^-,s+1}, \xi)$ where $T_{\delta^-,s+1}(\xi)$ is terminal on $T_{\delta^- * 1,s+1}$ and $T_{\delta^-,s+1}(\xi) \subseteq \beta_{s+1}$. By induction on $\mathrm{lh}(\delta)$, β_{s+1} is compatible with $T_{\delta^-,s+1}$ and hence with $T_{\delta,s+1}$. Furthermore, if $\exists \sigma \in \mathscr{S}_2(\beta_{s+1} \subset \sigma \subset T_{\delta^-,s+1})$, then since $\beta_{s+1} \supseteq T_{\delta^-,s+1}(\xi)$, $\sigma \subset T_{\delta,s+1}$. Hence (iii) follows. ▯

Lemma 2.3 is used to show that α_{s+1} and γ_{s+1} are defined during some substage of stage $s+1$. Let $0_0 = \emptyset$ and $0_{j+1} = 0_{j*0}$ for all $j \in N$. Assume that α_{s+1} and γ_{s+1} are not defined before substage $2^{s+1} - 1$ of stage $s+1$. It suffices to show that α_{s+1} and γ_{s+1} are then defined at substage $m = 2^{s+1} - 1$ of stage $s+1$. Note that $\delta_m = 0_{s+1}$ and that $\beta_{s+1} \subseteq \alpha_s$. By the definition of $T_{\emptyset,s+1} = \mathrm{Id}_{2,s+1}$, $T_{\emptyset,s+1}$ extends $T_{\emptyset,s}$ and no $\sigma \subset T_{\emptyset,s}$ is terminal on $T_{\emptyset,s+1}$. By (7) and (9), $\beta_{s+1} \subset T_{\emptyset,s+1}$ so $\exists \sigma \in \mathscr{S}_2(\beta_{s+1} \subset \sigma \subset T_{\emptyset,s+1})$. Since α_{s+1} and γ_{s+1} are not defined before substage $2^{s+1} - 1$, Lemma 2.3(iii) can be applied repeatedly by induction to show that $\exists \sigma \in \mathscr{S}_2(\beta_{s+1} \subset \sigma \subset T_{0_{s+1},s+1})$. Fix such a σ and fix $\eta \in \mathscr{S}_2$ such that $\sigma = T_{0_{s+1},s+1}(\eta)$. α_{s+1} and γ_{s+1} are now defined during step $s+1$, Case 2 of substage $2^{s+1} - 1$.

The induction hypotheses are now verified. (4) and (5) follow from the definition of β_{s+1} and since $\beta_{s+1} \subset \alpha_{s+1}$.

Fix $\delta \in \mathscr{S}_2$ such that $\mathrm{lh}(\delta) \leqslant \mathrm{lh}(\gamma_{s+1})$ and $E(\delta) > E(\gamma_{s+1} \upharpoonright \mathrm{lh}(\delta))$. Fix $v \in \mathscr{S}_2$ of greatest length such that $v \subseteq \delta$ and $v \subseteq \gamma_{s+1}$. Since $E(\delta) > E(\gamma_{s+1} \upharpoonright \mathrm{lh}(\delta))$, $v * 1 \subseteq \delta$ and $v * 0 \subseteq \gamma_{s+1}$. By 2.3(ii), there is a $\sigma \in \mathscr{S}_2$ such that $\sigma \subset T_{v*1,s}$ and either $\beta_{s+1} \subset \sigma$, or σ is terminal on $T_{v*1,s}$ and $\sigma \subseteq \alpha_s$. Fix such a σ. By Definition 1.2, any proper extension σ^* of σ on $T_{v*1,s+1}$ extends β_{s+1}. Hence either there is a $\sigma^* \in \mathscr{S}_2$ such that $\beta_{s+1} \subset \sigma^* \subset T_{v*1,s+1}$, or $\sigma \subseteq \beta_{s+1}$ and σ is terminal on $T_{v*1,s+1}$. There cannot be a $\sigma^* \in \mathscr{S}_2$ such that $\beta_{s+1} \subset \sigma^* \subset T_{v*1,s+1}$; else the repeated use of 2.3(iii) (as in the proof that $\gamma_{s+1}\downarrow$) will imply that $\gamma_{s+1} \supseteq v * 1$, a contradiction. Hence $\sigma \subseteq \beta_{s+1}$ and σ is terminal on $T_{\delta,s+1}$. Since $\beta_{s+1} \subset \alpha_{s+1}$, (6) now follows for $s + 1$ in place of s.

(8) and (9) follow easily for $s + 1$ in place of s.

(10) is immediate for $s + 1$ in place of s whenever $E(\delta) \geqslant E(\gamma_{s+1} \upharpoonright \mathrm{lh}(\delta))$ since Case 1 or Case 2 is then used to define $T_{\delta,s+1}$. Suppose that $E(\delta) < E(\gamma_{s+1} \upharpoonright \mathrm{lh}(\delta))$. Fix $v \in \mathscr{S}_2$ of greatest length such that $v \subseteq \delta$ and $v \subseteq \gamma_{s+1}$. Since $E(\delta) < E(\gamma_{s+1} \upharpoonright \mathrm{lh}(\delta))$, $v * 0 \subseteq \delta$ and $v * 1 \subseteq \gamma_{s+1}$. Since $v \subseteq \gamma_{s+1}$, $E(v) = E(\gamma_{s+1} \upharpoonright \mathrm{lh}(v))$, so by (10), $\alpha_{s+1} \subset T_{\gamma_{s+1},s+1} \subseteq T_{v,s+1}$. Since $v * 0 \subseteq \delta$ and $v * 1 \subseteq \gamma_{s+1}$, $E(v * 0) < E(\gamma_{s+1} \upharpoonright \mathrm{lh}(v * 0))$ so by Case 3 of the construction, $T_{\delta,s+1}(\eta)\downarrow \Leftrightarrow \eta = \emptyset$ and $T_{\delta,s+1}(\emptyset) = \alpha_{s+1}$ so either $T_{\delta,s+1} = T_{\delta^-,s+1}$ or $\delta^- = v$ and $T_{\delta,s+1} \subseteq T_{\delta^-,s+1}$. Hence (10) follows for $s + 1$ in place of s.

We now verify (7). Fix $\delta \in \mathscr{S}_2$ such that $\mathrm{lh}(\delta) \leqslant \mathrm{lh}(\gamma_{s+1})$ and $E(\delta) \leqslant E(\gamma_{s+1} \upharpoonright \mathrm{lh}(\delta))$. If $E(\delta) = E(\gamma_{s+1} \upharpoonright \mathrm{lh}(\delta))$, then since $\alpha_{s+1} \subset T_{\gamma_{s+1},s+1}$, $\alpha_{s+1} \subset T_{\delta,s+1}$ by (10). And if $E(\delta) < E(\gamma_{s+1} \upharpoonright \mathrm{lh}(\delta))$, then Case 3 is used to define $T_{\delta,s+1}$. Hence (7) follows for $s + 1$ in place of s.

(11) follows immediately from Definition 1.2 and Case 1 of the construction. And (12) follows immediately from Definition 1.2 and Case 2 of the construction and the proof of Lemma 2.3. Hence all the induction hypotheses are valid at the end of stage $s + 1$.

Let $\Gamma = \limsup_s \gamma_s$, i.e., $\Gamma \upharpoonright x + 1 = \gamma$ if for infinitely many $s \in N$, $\gamma_s \upharpoonright x + 1 = \gamma$ and for all $\delta \in \mathscr{S}_2$ such that $\mathrm{lh}(\delta) = x + 1$ and $E(\delta) > E(\gamma)$, there is an $s \in N$ such that for all $t \geqslant s$, $\gamma_t \upharpoonright x + 1 \neq \delta$. By induction, Γ must exist, since once $\Gamma \upharpoonright x = \gamma$ is defined, there are infinitely many $s \in N$ such that $\gamma_s \upharpoonright x = \gamma$, and by (8), for each such $s > \mathrm{lh}(\gamma)$, either $\gamma * 0 \subseteq \gamma_s$ or $\gamma * 1 \subseteq \gamma_s$. Hence $\Gamma \upharpoonright x + 1$ must also be defined. By (5), $\lim_s \mathrm{lh}(\alpha_s) = \infty$, so by (4), $\lim_s \alpha_s = A \subseteq N$ must exist. Furthermore, by (4) and the Permitting Lemma, $A \leqslant_T C$.

The next lemma is used to show that A is a set of minimal degree.

2.4 Lemma. *Let $\delta \in \mathscr{S}_2$ be given such that $E(\delta) \geqslant E(\Gamma \upharpoonright \mathrm{lh}(\delta))$. Then there is a stage s such that for all $t > s$, $T_{\delta,t+1}$ extends $T_{\delta,t}$. Furthermore, if $\mathrm{lh}(\delta) > 0$, then if $\delta = \delta^- * 1$, $\lim_t T_{\delta,t} = \mathrm{PSp}_2(\lim_t T_{\delta^-,t}, \mathrm{lh}(\delta^-), s, \{\alpha_t\colon t \in N\})$ and if $\delta = \delta^- * 0$, then $\lim_t T_{\delta,t} = \mathrm{PExt}_2(\lim_t T_{\delta^-,t}, \xi)$ for some $\xi \in \mathscr{S}_2$ such that $\lim_t T_{\delta^-,t}(\xi)$ is terminal on $\lim_t T_{\delta^-*1,t}$.*

Proof. We proceed by induction on $\mathrm{lh}(\delta)$. If $\mathrm{lh}(\delta) = 0$, then $\delta = \emptyset$ and the lemma follows from (9). Assume that the lemma holds for all δ such that $\mathrm{lh}(\delta) < n$ where $n > 0$. Fix $\delta \in \mathscr{S}_2$ such that $\mathrm{lh}(\delta) = n$ and $E(\delta) \geqslant E(\Gamma \upharpoonright \mathrm{lh}(\delta))$. By the definition of Γ, there is a least $s \in N$ such that for all $t \geqslant s$, $\mathrm{lh}(\gamma_t) \geqslant \mathrm{lh}(\delta)$ and

$E(\gamma_t \upharpoonright \mathrm{lh}(\delta)) \leqslant E(\Gamma \upharpoonright \mathrm{lh}(\delta))$. If $\delta = \delta^- * 1$, then $T_{\delta,t}$ is defined by Case 1 for all $t > s$, so $T_{\delta,t+1}$ extends $T_{\delta,t}$ for all $t > s$, and by (11), induction, and Definition 1.3, $\lim_t T_{\delta,t} = \mathrm{PSp}_2(\lim_t T_{\delta^-,t}, \mathrm{lh}(\delta^-), s, \{\alpha_t\colon t \in N\})$. And if $\delta = \delta^- * 0$, then $T_{\delta,t}$ is defined by Case 2 for all $t > s$. By (12), for each such t, there is a $\sigma(t) = T_{\delta^-,t}(\xi(t))$ such that $T_{\delta,t} = \mathrm{PExt}_2(T_{\delta^-,t}, \xi(t))$ and $\sigma(t)$ is terminal on $T_{\delta^- * 1,t}$. Since $E(\delta^- * 1) > E(\delta^- * 0) \geqslant E(\gamma_t \upharpoonright \mathrm{lh}(\delta))$, it follows from (6) that $\sigma(r) = \sigma(t)$ for all $r, t > s$. The lemma now follows by induction. ▯

By Lemma 2.4, for all $\delta \in \mathscr{S}_2$ such that $E(\delta) \geqslant E(\Gamma \upharpoonright \mathrm{lh}(\delta))$, $T_\delta = \lim_t T_{\delta,t}$ exists, and there is an $s(\delta) \in N$ such that $T_\delta = \cup\{T_{\delta,t}\colon t \geqslant s(\delta)\}$. By (7), if $\delta \subset \Gamma$, then $A \subset T_\delta$.

Fix $e \in N$ and $\delta \in \mathscr{S}_2$ such that $\mathrm{lh}(\delta) = e + 1$. If $\delta = \delta^- * 1$, then by Lemma 2.4, $T_\delta = \mathrm{PSp}_2(T_{\delta^-}, e, t, \{\alpha_s\colon s \in N\})$ for some $t \in N$, so T_δ is an e-splitting tree. By the Computation Lemma, if Φ_e^A is total, then $A \equiv_T \Phi_e^A$. If $\delta = \delta^- * 0$ and Φ_e^A is total, then by Lemma 2.4, $T_\delta = \mathrm{PExt}_2(T_{\delta^-}, \xi)$ for some $\xi \in \mathscr{S}_2$ such that $T_{\delta^-}(\xi)$ is terminal on $T_{\delta^- * 1} = \mathrm{PSp}_2(T_{\delta^-}, e, t, \{\alpha_s\colon s \in N\})$ for some $t \in N$. So by the Computation Lemma, Φ_e^A is recursive.

We complete the proof of the theorem by showing that A is not recursive.

2.5 Posner's Lemma. *A is not recursive.*

Proof. We obtain a contradiction under the assumption that A is recursive. Given $B \subseteq N$, we define a function θ^B partial recursive in B uniformly in B as follows:

$$\theta^B(x) = \begin{cases} B(x) & \text{if } \exists \sigma \subset B(\sigma \not\subset A \,\&\, \mathrm{lh}(\sigma) > x) \\ \uparrow & \text{otherwise.} \end{cases}$$

Since A is recursive, there is an $e \in N$ such that for all $B \subseteq N$, $\theta^B = \Phi_e^B$. Fix such an e. Fix $\delta \in \mathscr{S}_2$ such that $\delta \subset I$ and $\mathrm{lh}(\delta) = e + 1$.

Assume first that $\delta = \delta^- * 1$. Then by Lemma 2.4, $\langle T_\delta(0), T_\delta(1)\rangle$ e-splits $T_\delta(\emptyset)$ and $A \supset T_\delta(0)$ or $A \supset T_\delta(1)$. Hence for some $x \in N$, $\Phi_e^{T_\delta(0)}(x)\downarrow$ and $\Phi_e^{T_\delta(1)}(x)\downarrow$, so $\Phi_e^A(x)\downarrow$. But $\Phi_e^A(x) = \theta^A(x)$ and $\theta^A(x)\uparrow$ for all $x \in N$, a contradiction.

Assume that $\delta = \delta^- * 0$. By Lemma 2.4 and the proof of the Computation Lemma, there is a $\sigma \subset A$ for which $\sigma \subset T_{\delta^-}$ and there are no e-splittings of σ on T_{δ^-}. Let $\sigma = T_{\delta^-}(\xi)$. Then $T_{\delta^-}(\xi * 0)\downarrow$ and $T_{\delta^-}(\xi * 1)\downarrow$ since $\sigma \subset A \subset T_{\delta^-}$. Fix the $j \leqslant 1$ such that $T_{\delta^-}(\xi * j) \subset A$. Then $T_{\delta^-}(\xi * j * 0)\downarrow$ and $T_{\delta^-}(\xi * j * 1)\downarrow$. Fix $k \in N$ such that $T_{\delta^-}(\xi * j * k) \not\subset A$, and fix the least $x < \mathrm{lh}(T_{\delta^-}(\xi * 0))$ such that $T_{\delta^-}(\xi * 0)(x)\downarrow \neq T_{\delta^-}(\xi * 1)(x)\downarrow$. By the definition of θ^B, $\langle T_{\delta^-}(\xi * j * k), T_{\delta^-}(\xi * (1 - j))\rangle$ e-splits $T_{\delta^-}(\xi)$ on x, a contradiction. ▯

This completes the proof of the theorem. ▯

2.6 Remarks. Theorem 2.2 was proved by Yates [1970a]. A different proof can be found in Epstein [1975]. Posner's Lemma appears in Epstein and Posner [1978].

2.7 Exercise. Use Theorem III.7.4 and Theorem 2.2 to show that there is a minimal degree $\mathbf{a} \in \mathbf{L}_1$.