Chapter IV
High/Low Hierarchies

Hierarchies on both $\mathbf{D}$ and the arithmetical degrees are introduced. Properties of sets which lie in certain classes of this hierarchy are examined, and results obtained are used to find automorphism bases for certain classes of degrees.

1. High/Low Hierarchies

Post's Theorem implies that the arithmetical hierarchy gives rise to a hierarchy $\{\mathbf{D_n}: n \in N\}$ for the arithmetical degrees, where $\mathbf{D_n} = \{\mathbf{d} \in \mathbf{D}: \mathbf{d} \leqslant \mathbf{0^{(n)}}\}$. While this hierarchy has its uses, it is not a very good hierarchy for studying the degrees. Two of its drawbacks are that there are degrees which are not classified by this hierarchy, and that the classes of the hierarchy have little to do with the properties of the degrees in the classes. We therefore introduce new hierarchies which are better suited to classifying properties of degrees. We prove basic facts about these hierarchies in this section, and study some classes of the hierarchies in subsequent sections.

1.1 Definition. Let $n \geqslant 0$ be given. Define $\mathbf{L_n}$, the class of *low$_n$ degrees* by $\mathbf{L_n} = \{\mathbf{d} \leqslant \mathbf{0'}: \mathbf{d^{(n)}} = \mathbf{0^{(n)}}\}$, and $\mathbf{H_n}$, the class of *high$_n$ degrees* by $\mathbf{H_n} = \{\mathbf{d} \leqslant \mathbf{0'}: \mathbf{d^{(n)}} = \mathbf{0^{(n+1)}}\}$. Define $\mathbf{I}$, the class of *intermediate degrees* by $\mathbf{I} = \{\mathbf{d} \leqslant \mathbf{0'}: \forall n \in N(\mathbf{0^{(n)}} < \mathbf{d^{(n)}} < \mathbf{0^{(n+1)}})\}$. $\{\mathbf{H_n}: n \in N\} \cup \{\mathbf{L_n}: n \in N\} \cup \{\mathbf{I}\}$ is the set of classes of the *high/low* hierarchy.

The high/low hierarchy induces a partition of $\mathbf{D[0, 0']}$. The low$_n$ degrees are the degrees below $\mathbf{0'}$ whose nth jump is as small as possible, and the high$_n$ degrees are the degrees below $\mathbf{0'}$ whose nth jump is as large as possible. A similar hierarchy can be defined on $\mathbf{D[a, a']}$ for each $\mathbf{a} \in \mathbf{D}$.

1.2 Definition. Fix $\mathbf{a} \in \mathbf{D}$ and $n \in N$. Define $\mathbf{L_n(a)}$, the class of **a**-*low$_n$ degrees* by $\mathbf{L_n(a)} = \{\mathbf{d} \in \mathbf{D[a, a']}: \mathbf{d^{(n)}} = \mathbf{a^{(n)}}\}$, and $\mathbf{H_n(a)}$, the class of **a**-*high$_n$ degrees* by $\mathbf{H_n(a)} = \{\mathbf{d} \in \mathbf{D[a, a']}: \mathbf{d^{(n)}} = \mathbf{a^{(n+1)}}\}$. Define $\mathbf{I(a)}$, the class of **a**-*intermediate degrees* by $\mathbf{I(a)} = \{\mathbf{d} \in \mathbf{D[a, a']}: \forall n \in N(\mathbf{a^{(n)}} < \mathbf{d^{(n)}} < \mathbf{a^{(n+1)}})\}$. $\{\mathbf{L_n(a)}: n \in N\} \cup \{\mathbf{H_n(a)}: n \in N\} \cup \{\mathbf{I(a)}\}$ is the set of classes of the **a**-*high/low hierarchy.*

Figure 1.1 below gives a pictorial description of the **a**-high/low hierarchy for $\mathbf{a} \in \mathbf{D}$. The lower the degree in Fig. 1.1, the smaller it is in the ordering of $\mathbf{D}$.

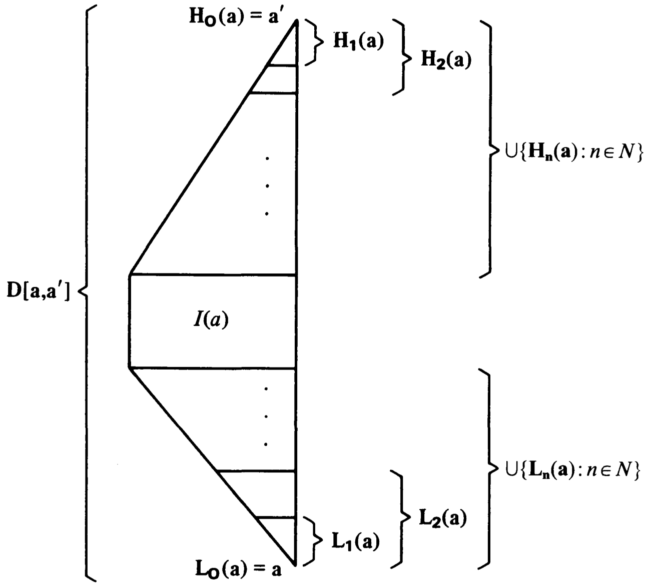

Fig. 1.1

The next proposition specifies some properties of the **a**-high/low hierarchy. The proposition follows easily from Theorem III.2.3(v), and its proof is left to the reader (Exercise 1.13).

1.3 Proposition. *Fix* $\mathbf{a} \in \mathbf{D}$. *Then*

(i) $\forall m, n \in N(m < n \rightarrow \mathbf{L_m(a)} \subseteq \mathbf{L_n(a)} \,\&\, \mathbf{H_m(a)} \subseteq \mathbf{H_n(a)})$.

(ii) $\forall m, n \in N(\mathbf{L_m(a)} \cap \mathbf{H_n(a)} = \mathbf{L_m(a)} \cap \mathbf{I(a)} = \mathbf{H_n(a)} \cap \mathbf{I(a)} = \emptyset)$.

(iii) $\forall m, n \in N\, \forall \mathbf{b}, \mathbf{c} \in \mathbf{D}(m < n \,\&\, \mathbf{b} \in \mathbf{L_m(a)} \,\&\, \mathbf{c} \in \mathbf{L_n(a)} - \mathbf{L_m(a)} \rightarrow \mathbf{c} \nleq \mathbf{b})$.

(iv) $\forall m, n \in N\, \forall \mathbf{b}, \mathbf{c} \in \mathbf{D}(\mathbf{b} \in \mathbf{L_m(a)} \,\&\, \mathbf{c} \in \mathbf{H_n(a)} \cup \mathbf{I(A)} \rightarrow \mathbf{c} \nleq \mathbf{b})$.

(v) $\forall m \in N\, \forall \mathbf{b}, \mathbf{c} \in \mathbf{D}(\mathbf{b} \in \mathbf{I(a)} \,\&\, \mathbf{c} \in \mathbf{H_m(a)} \rightarrow \mathbf{c} \nleq \mathbf{b})$.

(vi) $\forall m, n \in N\, \forall \mathbf{b}, \mathbf{c} \in \mathbf{D}(m < n \,\&\, \mathbf{c} \in \mathbf{H_m(a)} \,\&\, \mathbf{b} \in \mathbf{H_n(a)} - \mathbf{H_m(a)} \rightarrow \mathbf{c} \nleq \mathbf{b})$.

(vii) $\mathbf{L_0(a)} = \mathbf{a} \,\&\, \mathbf{H_0(a)} = \mathbf{a'}$.

The **a**-high/low hierarchy partitions $\mathbf{D[a, a']}$ into the classes $\{\mathbf{I(a)}, \mathbf{H_0(a)}, \mathbf{L_0(a)}, \mathbf{H_{n+1}(a)} - \mathbf{H_n(a)}, \mathbf{L_{n+1}(a)} - \mathbf{L_n(a)} : n \in N\}$. Proposition 1.3(vii) shows that $\mathbf{L_0(a)}$ and $\mathbf{H_0(a)}$ are non-empty. The next theorem shows that the remaining classes of this partition are also non-empty.

1.4 Theorem. *Let* $\mathbf{a} \in \mathbf{D}$ *and* $n \in N$ *be given. Then*:

(i) $\mathbf{L_{n+1}(a)} - \mathbf{L_n(a)} \neq \emptyset$.

(ii) $\mathbf{H_{n+1}(a)} - \mathbf{H_n(a)} \neq \emptyset$.

(iii) $\mathbf{I(a)} \neq \emptyset$.

Proof. Relativize Theorems III.7.9 and III.7.11. We note that the proofs of those theorems are actually proofs of the relativized versions. ▯

The **a**-high/low hierarchies are useful for studying local properties of **D**, but have little connection with global properties of **D**. We are thus motivated to introduce new hierarchies subsuming the ones already discussed which are more closely related to global properties of **D**.

1.5 Definition. Let $\mathbf{GL_0} = \{\mathbf{0}\}$. For $n > 0$, define $\mathbf{GL_n}$, the class of *generalized low$_n$ degrees* by $\mathbf{GL_n} = \{\mathbf{d} \in \mathbf{D} : \mathbf{d^{(n)}} = (\mathbf{d} \cup \mathbf{0}')^{(\mathbf{n}-\mathbf{1})}\}$. For $n \geqslant 0$, define $\mathbf{GH_n}$, the class of *generalized high$_n$ degrees* by $\mathbf{GH_n} = \{\mathbf{d} \in \mathbf{D} : \mathbf{d^{(n)}} = (\mathbf{d} \cup \mathbf{0}')^{(\mathbf{n})}\}$. Define $\mathbf{GI}$, the class of *generalized intermediate degrees* by $\mathbf{GI} = \{\mathbf{d} \in \mathbf{D} : \forall n > 0((\mathbf{d} \cup \mathbf{0}')^{(\mathbf{n}-\mathbf{1})} < \mathbf{d^{(n)}} < (\mathbf{d} \cup \mathbf{0}')^{(\mathbf{n})})\}$. $\{\mathbf{GL_n} : n \in N\} \cup \{\mathbf{GH_n} : n \in N\} \cup \{\mathbf{GI}\}$ is the set of classes of the *generalized high/low hierarchy*.

The generalized high/low hierarchy can be relativized to the degrees above **a** as follows:

1.6 Definition. Let $\mathbf{a} \in \mathbf{D}$ be given. Define $\mathbf{GL_0(a)} = \mathbf{a}$, and for $n > 0$ define $\mathbf{GL_n(a)}$, the class of *generalized* **a**-*low$_n$ degrees* by

$$\mathbf{GL_n(a)} = \{\mathbf{d} \in \mathbf{D} : \mathbf{d} \geqslant \mathbf{a} \;\&\; \mathbf{d^{(n)}} = (\mathbf{d} \cup \mathbf{a}')^{(\mathbf{n}-\mathbf{1})}\}.$$

For $n \geqslant 0$, define $\mathbf{GH_n(a)}$, the class of *generalized* **a**-*high$_n$ degrees* by

$$\mathbf{GH_n(a)} = \{\mathbf{d} \in \mathbf{D} : \mathbf{d} \geqslant \mathbf{a} \;\&\; \mathbf{d^{(n)}} = (\mathbf{d} \cup \mathbf{a}')^{(\mathbf{n})}\}.$$

Define $\mathbf{GI(a)}$, the class of *generalized* **a**-*intermediate degrees* by

$$\mathbf{GI(a)} = \{\mathbf{d} \in \mathbf{D} : \mathbf{d} \geqslant \mathbf{a} \;\&\; \forall n > 0((\mathbf{d} \cup \mathbf{a}')^{(\mathbf{n}-\mathbf{1})} < \mathbf{d^{(n)}} < (\mathbf{d} \cup \mathbf{a}')^{(\mathbf{n})})\}.$$

$\{\mathbf{GL_n(a)} : n \in N\} \cup \{\mathbf{GH_n(a)} : n \in N\} \cup \{\mathbf{GI(a)}\}$ is the set of classes of the *generalized* **a**-*high/low hierarchy*.

The generalized high/low hierarchies are indeed extensions of the high/low hierarchies, as is shown in the next theorem.

1.7 Theorem. *Fix* $\mathbf{a} \in \mathbf{D}$ *and* $n \in N$. *Then* $\mathbf{L_n(a)} = \mathbf{GL_n(a)} \cap \mathbf{D[a, a']}$, $\mathbf{H_n(a)} = \mathbf{GH_n(a)} \cap \mathbf{D[a, a']}$, *and* $\mathbf{I(a)} = \mathbf{GI(a)} \cap \mathbf{D[a, a']}$.

Proof. If $\mathbf{d} \in \mathbf{D[a, a']}$, then $\mathbf{d} \cup \mathbf{a}' = \mathbf{a}'$. The theorem now follows from the definitions of the respective hierarchies. ▯

If $\mathbf{a} \in \mathbf{GL_1}$ then we can replace the generalized **a**-high/low hierarchy with the generalized high/low hierarchy in the statement of Theorem 1.7.

1.8 Theorem. *Fix* $\mathbf{a} \in \mathbf{GL_1}$ *and* $n > 0$. *Then* $\mathbf{L_n(a)} = \mathbf{GL_n} \cap \mathbf{D[a, a']}$, $\mathbf{H_n(a)} = \mathbf{GH_n} \cap \mathbf{D[a, a']}$, *and* $\mathbf{I(a)} = \mathbf{GI} \cap \mathbf{D[a, a']}$.

Proof. Fix $\mathbf{a} \in \mathbf{GL_1}$ and $\mathbf{c} \in \mathbf{D[a, a']}$. Then $\mathbf{c} \cup \mathbf{a}' = \mathbf{c} \cup \mathbf{a} \cup \mathbf{0}' = \mathbf{c} \cup \mathbf{0}'$. The result now follows from Theorem 1.7. ▯

All the classes of the generalized **a**-high/low hierarchy are non-trivial extensions of the corresponding classes of the **a**-high/low hierarchy. We prove such a result for $\mathbf{a} = \mathbf{0}$, leaving the relativization of the proof to the reader.

1.9 Theorem. *For all* $n > 0$, $\mathbf{GL_{n+1}} - \mathbf{GL_n} \neq \mathbf{L_{n+1}} - \mathbf{L_n}$; *for all* $n \geqslant 0$, $\mathbf{GH_{n+1}} - \mathbf{GH_n} \neq \mathbf{H_{n+1}} - \mathbf{H_n}$; and $\mathbf{GI} \neq \mathbf{I}$.

Proof. By Theorem III.4.2, there is a degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{0}^{(2)}$. Since $\mathbf{a} \cup \mathbf{0}' > \mathbf{0}'$, $\mathbf{a} \nleqslant \mathbf{0}'$. Hence $\mathbf{a} \in \mathbf{GL_1} - \mathbf{L_1}$, and no element of $\mathbf{D}[\mathbf{a}, \mathbf{a}']$ is in any class of the high/low hierarchy. Fix $\mathbf{C} \in \{\mathbf{L_{n+1}} - \mathbf{L_n} : n > 0\} \cup \{\mathbf{H_{n+1}} - \mathbf{H_n} : n \geqslant 0\} \cup \{\mathbf{I}\}$, and let $\mathbf{GC}$ be the corresponding generalized class. By Theorem 1.4, there is a degree $\mathbf{c} \in \mathbf{C}(\mathbf{a})$. Hence by Theorem 1.8, $\mathbf{c} \in \mathbf{GC} - \mathbf{C}$. ▯

Degrees in the generalized high/low hierarchy are high or low in the sense that their nth jumps achieve the highest or lowest possible values. In the case of the high/low hierarchy, high and low were also descriptive of the location of the degree within the poset $\mathscr{D}[\mathbf{0}, \mathbf{0}']$. This is not the case for the generalized high/low hierarchy. The next proposition shows that some of the properties proved in Proposition 1.3 for the high/low hierarchy remain true for the generalized high/low hierarchy. After that, we prove a theorem which shows that all properties mentioned in Proposition 1.3 which relate the hierarchy to the ordering fail for the generalized high/low hierarchy.

1.10 Proposition. *Fix* $\mathbf{a} \in \mathbf{D}$. *Then*:

(i) $\forall m, n \in N(m < n \rightarrow \mathbf{GL_m}(\mathbf{a}) \subseteq \mathbf{GL_n}(\mathbf{a})\ \&\ \mathbf{GH_m}(\mathbf{a}) \subseteq \mathbf{GH_n}(\mathbf{a}))$.

(ii) $\forall m, n \in N(\mathbf{GL_m}(\mathbf{a}) \cap \mathbf{GH_n}(\mathbf{a}) = \mathbf{GL_m}(\mathbf{a}) \cap \mathbf{I}(\mathbf{a}) = \mathbf{GH_n}(\mathbf{a}) \cap \mathbf{I}(\mathbf{a}) = \emptyset)$.

(iii) $\mathbf{GH_0}(\mathbf{a}) = \mathbf{D}[\mathbf{a}', \infty)$.

We leave the proof of Proposition 1.10 to the reader.

1.11 Theorem. *Let* $\mathbf{a} \in \mathbf{D}$ *be given such that* $\mathbf{a} \notin \mathbf{GH_0}$, *and fix* $\mathbf{C} \in \{\mathbf{GH_{n+1}} - \mathbf{GH_n}, \mathbf{GL_{n+1}} - \mathbf{GL_n} : n \geqslant 0\} \cup \{\mathbf{GI}\}$. *Then there is a* $\mathbf{b} \in \mathbf{D}$ *such that* $\mathbf{b} \geqslant \mathbf{a}$ *and* $\mathbf{b} \in \mathbf{C}$.

Proof. By Theorem 1.8, it suffices to find a degree $\mathbf{b} \geqslant \mathbf{a}$ such that $\mathbf{b} \in \mathbf{GL_1}$. If $\mathbf{a} \in \mathbf{GL_1}$, let $\mathbf{b} = \mathbf{a}$. If $\mathbf{a} \notin \mathbf{GL_1}$, then since $\mathbf{a} \ngeqslant \mathbf{0}'$, it must be the case that $\mathbf{a} < \mathbf{a} \cup \mathbf{0}' < \mathbf{a}'$. By the relativization of the Join Theorem for $\mathbf{0}'$ (Theorem III.5.8) to $\mathbf{D}[\mathbf{a}, \mathbf{a}']$, there is a $\mathbf{b} \in \mathbf{D}(\mathbf{a}, \mathbf{a}')$ such that $\mathbf{b} \cup (\mathbf{a} \cup \mathbf{0}') = \mathbf{b}' = \mathbf{a}'$. Since $\mathbf{b} \geqslant \mathbf{a}$, $\mathbf{b} \cup (\mathbf{a} \cup \mathbf{0}') = \mathbf{b} \cup \mathbf{0}'$. Hence $\mathbf{b} \in \mathbf{GL_1}$. ▯

1.12 Remarks. The high/low hierarchy was introduced by Soare [1974] and independently by Cooper in a preprint of [1974]. This introduction followed work by many people in which various concepts of Recursion Theory were related to properties of degrees in the high/low hierarchy. The generalized high/low hierarchy was introduced by Jockusch and Posner [1978]. Clauses (i) and (ii) of Theorem 1.4 follow from the Sacks Jump Inversion Theorem, and Theorem 1.4(iii) was proved independently by Lachlan [1965] and Martin [1966]. Theorem 1.11 is a corollary of the relativization of the Join Theorem for $\mathbf{0}'$ (Posner and Robinson [1981]).

1.13–1.17 Exercises

***1.13** Verify Proposition 1.3.

1.14 State and prove a version of Theorem 1.9 for the generalized $\mathbf{a}$-high/low hierarchy.

***1.15** Verify Proposition 1.10.

1.16 Prove the following facts for $\mathbf{a}, \mathbf{b} \in \mathbf{D}$.
(i) $\mathbf{a} \in \mathbf{GH_n(b)}$ implies $\mathbf{a} \in \mathbf{GH_n}$.
(ii) $\mathbf{a} \in \mathbf{L_n(b)}$ and $\mathbf{b} \in \mathbf{GL_n}$ implies $\mathbf{a} \in \mathbf{GL_n}$.
(iii) $\mathbf{a} \in \mathbf{I(b)}$ and $\mathbf{b} < \mathbf{0}'$ and $\mathbf{b} \notin \mathbf{I}$ implies $\mathbf{a} \in \mathbf{I}$.

1.17 Show that the following statements are false for $\mathbf{a}, \mathbf{b} \in \mathbf{D}$.
(i) $\mathbf{a} \in \mathbf{I(b)}$ and $\mathbf{b} \in \mathbf{I}$ implies $\mathbf{a} \in \mathbf{I}$.
(ii) $\mathbf{a} \in \mathbf{L_n(b)}$ and $\mathbf{b} \in \mathbf{GH_m}$ implies $\mathbf{a} \in \mathbf{GL_n}$.

2. $\mathbf{GL_1}$ *and 1-Generic Degrees*

If $\mathbf{a} \in \mathbf{L_1}$, then $\mathbf{a}$ is close to $\mathbf{0}$ in the ordering of $\mathbf{D}$, so we might expect $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ and $\mathscr{D}[\mathbf{a}, \mathbf{0}']$ to be similar. In fact, the structure theorems proved for $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ relativize to similar results for $\mathscr{D}[\mathbf{a}, \mathbf{0}']$. Since $\mathbf{a}$ is close to $\mathbf{0}$ in the ordering of $\mathbf{D}$, we might also expect $\mathscr{D}[\mathbf{0}, \mathbf{a}]$ to be relatively simple. Although this is not always the case, it is true that if $\mathscr{D}[\mathbf{0}, \mathbf{a}]$ is relatively simple (in a sense to be made precise in the next section) then $\mathbf{a} \in \mathbf{GL_2}$. Thus there is little that one can say about $\mathscr{D}[\mathbf{0}, \mathbf{a}]$ which is true for all $\mathbf{a} \in \mathbf{L_1}$. There is a nice subset of $\mathbf{GL_1}$, the set of 1-generic degrees, for which $\mathscr{D}[\mathbf{0}, \mathbf{a}]$ is relatively rich in structure. Furthermore, 1-generic degrees share many properties. This set of degrees is a useful set, and is studied in this section. We have already come across these degrees; they are the degrees of sets A which force their jump.

2.1 Definition. A set $A \subseteq N$ is *1-generic* if for every recursively enumerable set $S \subseteq \mathscr{S}_2$, either

$$\text{(i)} \qquad \exists \sigma \subset A (\sigma \in S)$$

or

$$\text{(ii)} \qquad \exists \sigma \subset A \, \forall \tau \in \mathscr{S}_2 (\sigma \subseteq \tau \to \tau \notin S).$$

A degree $\mathbf{d}$ is *1-generic* if $\mathbf{d}$ is the degree of a 1-generic set.

2.2 Lemma. *Let $A \subseteq N$ be given. Then A is 1-generic if and only if A forces its jump.*

Proof. Let $A \subseteq N$ be a 1-generic set. Let $S = \{\sigma \in \mathscr{S}_2 : \Phi_e^\sigma(e)\downarrow\}$. Then S is recursively enumerable, so 2.1(i) and 2.1(ii) imply III.3.8(i) and III.3.8(ii) respectively.

Conversely, let A force its jump. Let $S \subseteq \mathscr{S}_2$ be recursively enumerable. Define the partial B-recursive function f_B (uniformly in B) by

$$f_B(x) = \begin{cases} 0 & \text{if } \exists \sigma \in S (\sigma \subset B) \\ \uparrow & \text{otherwise} \end{cases}$$

for all $x \in N$. Then by the Enumeration Theorem, there is an $e \in N$ such that for all $B \subseteq N$, $f_B = \Phi_e^B$. Fix such an e. Letting $B = A$, we now note that III.3.8(i) and III.3.8(ii) imply 2.1(i) and 2.1(ii) respectively. ∎

We now show that all 1-generic degrees lie in $\mathbf{GL_1}$.

2.3 Lemma. *Let* $\mathbf{a} \in \mathbf{D}$ *be given such that* $\mathbf{a}$ *is 1-generic. Then* $\mathbf{a} \in \mathbf{GL_1}$. *Hence if* $\mathbf{a} \leqslant \mathbf{0}'$, *then* $\mathbf{a} \in \mathbf{L_1}$.

Proof. Since $\mathbf{a}$ is 1-generic, there is a 1-generic set A of degree $\mathbf{a}$. By Lemma 2.2, A forces its jump. The result now follows from Lemma III.3.9. ▯

The next lemma is a useful result about 1-generic sets. Its proof gives an easy example of how to use the hypothesis of 1-genericity. We first give the following definition.

2.4 Definition. We say that $A \subseteq N$ is *immune* if A is infinite but has no infinite recursive subset. (Since every infinite recursively enumerable set has an infinite recursive subset, we see that A is immune if A is infinite but has no infinite recursively enumerable subset.)

2.5 Lemma. *Let* $A \subseteq N$ *be given such that* A *is 1-generic. Then* A *is immune.*

Proof. Let R be any infinite recursive subset of N, and let f be the characteristic function of R. Let $S = \{\sigma \in \mathcal{S}_2 : \exists x(\sigma(x)\downarrow \,\&\, x \in R \,\&\, \sigma(x) \neq f(x))\}$. Note that S is recursive. Fix a 1-generic set A, and fix $\sigma \subset A$ satisfying 2.1(i) or 2.1(ii) for S. σ cannot satisfy 2.1(ii) for S, since if $\sigma \notin S$, we can define $\tau \in S$, $\tau \supseteq \sigma$, by choosing $y \in R$ such that $y > \mathrm{lh}(\sigma)$ and $\tau \in \mathcal{S}_2$ such that $\tau \supseteq \sigma$ and $\tau(y) \neq f(y)$. Hence $\sigma \in S$. But then R is not a subset of A. ▯

The next theorem shows that if $\mathbf{a}$ is a 1-generic degree, then $\mathcal{D}[\mathbf{0}, \mathbf{a}]$ is relatively rich in structure, since $\mathbf{D}(\mathbf{0}, \mathbf{a})$ must contain an infinite set of independent degrees.

2.6 Theorem. Let $\mathbf{a} \in \mathbf{D}$ be given such that $\mathbf{a}$ is 1-generic. Then there is a set $\{\mathbf{a_i} : i \in N\}$ of independent degrees such that for all $i \in N$, $\mathbf{a_i} \in \mathbf{D}(\mathbf{0}, \mathbf{a})$.

Proof. Fix a 1-generic set A of degree $\mathbf{a}$, and for each $i \in N$, let $A^{[i]}$ have degree $\mathbf{a_i}$. In order to prove that $\{\mathbf{a_i} : i \in N\}$ is a set of independent degrees, it suffices to show that for all $i, e \in N$ and all finite $F \subseteq N$ such that $i \notin F$, $\Phi_e^{A^{[F]}} \neq A^{[i]}$.

Fix $i, e \in N$ and a finite set $F \subseteq N$ such that $i \notin F$. Let

$$S = \{\sigma \in \mathcal{S}_2 : \exists x \in N \, \exists \tau \subseteq \sigma^{[F]}(\Phi_e^{\tau}(x)\downarrow \neq \sigma^{[i]}(x)\downarrow)\}.$$

Then S is a recursively enumerable set. Fix $\sigma \subset A$ satisfying 2.1(i) or 2.1(ii). If σ satisfies 2.1(i), then $\Phi_e^{A^{[F]}} \neq A^{[i]}$. Suppose that σ satisfies 2.1(ii). Fix $x \in N$ such that $\sigma^{[i]}(x)\uparrow$. There can be no $\xi \supset \sigma$ such that for some $\tau \subseteq \xi^{[F]}$, $\Phi_e^{\tau}(x)\downarrow$, else since $i \notin F$, we can find such a ξ with $\xi^{[i]}(x) = 0$ and another ξ with exactly the same τ such that $\xi^{[i]}(x) = 1$, so there will be some $\xi \supseteq \sigma$ such that $\xi \in S$. Hence $\Phi_e^{A^{[F]}}(x)\uparrow \neq A^{[i]}(x)\downarrow$. ▯

The following corollaries are now proved exactly as in Chap. II.3.

2.7 Corollary. *Let* $\mathcal{U}$ *be a finite poset, and let* $\mathbf{a}$ *be a 1-generic degree. Then* $\mathcal{U} \hookrightarrow \mathcal{D}[\mathbf{0}, \mathbf{a}]$.

2.8 Corollary. *Let* $\mathbf{a}$ *be a 1-generic degree. Then* $\exists_1 \cap \mathrm{Th}(\mathcal{D}[\mathbf{0}, \mathbf{a}])$ *is decidable.*

The next theorem shows that 1-generic degrees are always recursively enumerable relative to some smaller degree. Thus we can relativize results about recursively enumerable degrees to obtain similar results about 1-generic degrees. For example, the proofs that $\mathscr{D}[\mathbf{0},\mathbf{a}]$ is not densely ordered and that $\mathrm{Th}(\mathscr{D}[\mathbf{0},\mathbf{a}])$ is undecidable for $\mathbf{a}$ recursively enumerable will now yield similar results for $\mathbf{a}$ 1-generic. Other structural results about $\mathscr{D}[\mathbf{0},\mathbf{a}]$ can also be obtained in this manner, but frequently, a direct proof using the 1-genericity of $\mathbf{a}$ is simpler than a proof using the relative recursive enumerability of $\mathbf{a}$.

2.9 Theorem. *Let* $\mathbf{a} \in \mathbf{D}$ *be given such that* $\mathbf{a}$ *is 1-generic. Then there is a degree* $\mathbf{b} < \mathbf{a}$ *such that* $\mathbf{a}$ *is recursively enumerable in* $\mathbf{b}$.

Proof. Let $A \subseteq N$ be a 1-generic set of degree $\mathbf{a}$. We recursively identify N^2 with N, and so will treat sets $C \subseteq N$ as if they are also subsets of N^2, and strings $\sigma \in \mathscr{S}_2$ as if they are also partial functions $N^2 \to \{0,1\}$. We also write $k = \langle i, x\rangle$ under this identification.

We will define $B = \Psi(A) \subseteq N^2$ of degree $\mathbf{b}$ to satisfy the lemma. We set aside column 0 of N^2 to detach it from the coding which will have to be done. We will satisfy the following condition:

$$(1) \qquad i \in A \Leftrightarrow B^{[i+1]} \neq \emptyset.$$

Thus $i \in A \Leftrightarrow \exists x(\langle i+1, x\rangle \in B)$, so A will be recursively enumerable in B.

In order to make B recursive in A, we must let A determine the elements which are to be placed into B. We do this according to the following rule:

$$(2) \qquad \langle i, x\rangle \in B \Leftrightarrow i > 0 \,\&\, i-1 \in A \,\&\, \langle i, x\rangle \notin A.$$

Thus A and B will partition column i of N^2 whenever $i - 1 \in A$. Note that, by Lemma 2.5, A is immune, so for every i, there are infinitely many x such that $\langle i, x\rangle \notin A$. Hence (1) will hold.

The most difficult problem will be to show that $A \not\leq_T B$. The 1-genericity of A is used to verify this fact. We note that for any set C, we can define a set $D = \Psi(C)$ as in (2) with C and D in place of A and B respectively, where Ψ is a recursive functional. Hence by the Enumeration Theorem, there is an $n \in N$ such that for all $C \subseteq N$, $\Psi(C) = \Phi_n^C$. If we assume that $A = \Phi_e^B$, then we must be able to force $A = \Phi_e^B$ to be true by specifying some finite $\sigma \subset A$. The specification of σ would then determine the value of $\Phi_e^D(x)$ for all x and all D such that $D = \Psi(C)$ and $\sigma \subset C$. We thus pick a free z in column 0 of N^2 (which is not used for coding in (2)), and obtain a contradiction by showing that there are τ, $v \supset \sigma$ such that $\tau(z) \neq v(z)$, $\Psi(\tau) = \Psi(v)$, and $\Phi_e^{\Psi(\tau)}(z)\downarrow$. We fix z so that $\tau(z) = 0$ and $v(z) = 1$. Setting $v(z) = 1$ forces us to do some coding on column $z + 1$ of N^2. But if we place $k = \langle z+1, x\rangle \in D$, we must be allowed to do so by not having $k \in C$. Since we have the freedom to define $v(k)$, we can arrange the definition of v so that $k \in B \Leftrightarrow k \in D$.

Formally, we proceed as follows. For each $\sigma \in \mathscr{S}_2$ and $\langle i, x\rangle < \mathrm{lh}(\sigma)$, we define $\Psi(\sigma) \in \mathscr{S}_2$ of length $\mathrm{lh}(\sigma)$ as follows:

$$\Psi(\sigma)(\langle i, x\rangle) = \begin{cases} 1 & \text{if } \; i > 0 \,\&\, \sigma(i-1) = 1 \,\&\, \sigma(\langle i, x\rangle) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(We assume that the correspondence of N^2 with N has the property that for all $i,j,k \in N$, if $\langle i,j\rangle = k$, then $i,j \leqslant k$. Hence $\Psi(\sigma)$ is well-defined for each $\sigma \in \mathcal{S}_2$.) For each $C \subseteq N$, define the set

$$\Psi(C) = \{\langle i,x\rangle : i > 0 \,\&\, i-1 \in C \,\&\, \langle i,x\rangle \notin C\}.$$

Note that for all σ, $\tau \in \mathcal{S}_2$, $C \subseteq N$ and $i \in N$;

(3) $\quad \sigma \subseteq \tau \subset C \Rightarrow \Psi(\sigma) \subseteq \Psi(\tau) \subset \Psi(C)$.

(4) $\quad i \notin C \Rightarrow \Psi(C)^{[i+1]} = \emptyset$.

(5) $\quad i \in C \Rightarrow C^{[i+1]}$ and $\Psi(C)^{[i+1]}$ partition N.

We have already noted that A is recursively enumerable in B and $B \leqslant_T A$. We now show that $A \nleqslant_T \Psi(A) = B$.

Suppose that $A \leqslant_T B$ for the sake of obtaining a contradiction. Fix $e \in N$ such that $A = \Phi_e^B$. Let $S = \{\sigma \in \mathcal{S}_2 : \exists x(\Phi_e^{\Psi(\sigma)}(x)\downarrow \neq \sigma(x)\downarrow)\}$. Since Ψ is a recursive functional, S is recursively enumerable. There can be no $\sigma \subset A$ such that $\sigma \in S$, else $\Phi_e^B \neq A$. Hence by Definition 2.1, there is a $\sigma \subset A$ such that for all $\tau \supseteq \sigma$, $\tau \notin S$. Fix such a σ. Fix the least x such that $\sigma(\langle 0,x\rangle)\uparrow$ and $x \notin A$, and let the correspondence match $\langle 0,x\rangle$ with i. Note that such an x must exist since, by Lemma 2.5, A is immune. Fix $\tau, \beta \in \mathcal{S}_2$ such that $\tau \subset A$, $\Psi(\tau) = \beta$, and $\Phi_e^\beta(i)\downarrow$. Such τ and β must exist since Φ_e^B is total. We define $\nu \in \mathcal{S}_2$ such that $\sigma \subseteq \nu$, $\Psi(\nu) = \beta$, and $\nu(i) \neq \tau(i)$ by induction on $\{z : z < \mathrm{lh}(\tau)\}$. Note that once we prove that such a ν exists, then since $\nu \supseteq \sigma$ and $\nu \in S$, we will have contradicted the choice of σ, and so will have completed the proof of the theorem.

We assume that at the beginning of step z of the induction, we will have defined $\nu_z \subset \nu$ of length z, and if we let $\tau_z = \tau \upharpoonright z$, then the following conditions will hold:

(6) $\quad \{u : \tau_z(u) = 1\} \subseteq \{u : \nu_z(u) = 1\}$.

(7) $\quad \Psi(\tau_z) = \Psi(\nu_z)$.

We begin by defining $\nu(z) = \tau(z)$ for all $z < i$. Clearly (6) and (7) hold. Consider step z of the induction, and assume that (6) and (7) hold. Let z correspond to $\langle j,y\rangle$. We proceed by cases:

Case 1. $j = 0$. Define $\nu_{z+1} \supset \nu_z$ by $\nu_{z+1}(z) = 1$. Since $j = 0$, $\Psi(\tau_{z+1}) = \Psi(\tau_z) = \Psi(\nu_z) = \Psi(\nu_{z+1})$ by (7). (6) clearly holds with $z+1$ in place of z. Note also that $\Phi_e^\beta(i)\downarrow \neq \nu_{i+1}(i) = 1$ since $\tau(i) = 0$.

Case 2. $j > 0$ and $\nu(j-1) = 0$. Define $\nu_{z+1} \supset \nu_z$ by $\nu_{z+1}(z) = 0$. By (3) and (7), $\Psi(\nu_{z+1})(u) = \Psi(\tau_{z+1})(u)$ for all $u < z$. Since $j - 1 < z$ and $\nu(j-1) = 0$, it follows from (6) that $\tau(j-1) = 0$. Hence by (3) and (4),

$$\tau_{z+1}(z) = \Psi(\tau_{z+1})(z) = 0 = \Psi(\nu_{z+1})(z) = \nu_{z+1}(z).$$

Thus (6) and (7) hold for $z+1$ in place of z.

Case 3. $j > 0$ and $\nu(j-1) = 1$. Define $\nu_{z+1} \supset \nu_z$ by $\nu_{z+1}(z) = 1 - \beta(z)$. If $\tau(j-1) = 1$, then by (5), $\nu(z) = 1 - \beta(z) = \tau(z)$, so $\Psi(\nu_{z+1})(z) = \beta(z) = \Psi(\tau_{z+1})(z)$. If $\tau(j-1) = 0$, then by (4), $\beta(z) = 0$, so $\nu(z) = 1$. Hence by (5), $\Psi(\nu_{z+1})(z) = 0 = \Psi(\tau_{z+1})(z)$. Thus we see that in either case, (6) and (7) hold for $z + 1$ in place of z.

This completes the induction, and hence the proof of the theorem. ▯

We have just proved some results about 1-generic degrees. These degrees, and in fact the n-generic and generic degrees have been studied by Jockusch [1980], and we refer the reader to that paper for more information about connections between genericity and degrees. As Jockusch notes, some of his proofs, including the proof of Theorem 2.9 are based on ideas and proofs of Martin, who took a topological approach to the degrees in terms of Baire category (see Yates [1976].)

The original presentation of forcing and genericity for arithmetic appears in Feferman [1965]. In that presentation, forcing is defined by induction on the logical complexity of sentences, and one then builds generic sets which force every sentence or its negation. Hinman [1969] first considered n-genericity, or forcing for restricted classes of sentences. The equivalence of the original approach with that of Jockusch was proved by Posner [1977]. We have followed Jockusch's approach in this section.

We now list some (but not all) of the results which appear in Jockusch [1980]. Proofs of some of these results are left as exercises for the reader. We first note that Definition 2.1 is easily modified by changing the complexity of the class of sets from which S comes. Thus we say that A is *n-generic* (*generic*) if for every Σ_n^0 (arithmetical) set S of strings, either 2.1(i) or 2.1(ii) holds. A degree is *n-generic* (*generic*) if it contains an n-generic (generic) set.

2.10 Further Results

(i) *Generic sets exist. Furthermore, for all $n \geqslant 1$, there is an n-generic set A of degree* $\mathbf{a} \leqslant \mathbf{0^{(n)}}$.

(ii) *If* $\mathbf{a}$ *is n-generic, then* $\mathbf{a^{(n)}} = \mathbf{a} \cup \mathbf{0^{(n)}}$.

(iii) *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are n-generic, then* $\exists_n \cap \mathrm{Th}(\mathscr{D}[\mathbf{0}, \mathbf{a}]) = \exists_n \cap \mathrm{Th}(\mathscr{D}[\mathbf{0}, \mathbf{b}])$.

(iv) *If* $\mathbf{a}$ *is 1-generic, then* $\mathscr{D}[\mathbf{0}, \mathbf{a}]$ *is not a lattice.*

(v) *If $n \geqslant 2$ and* $\mathbf{a}$ *is n-generic (generic) and* $\mathbf{b} < \mathbf{a}$, *then there is an n-generic (generic) degree* $\mathbf{c}$ *such that* $\mathbf{c} < \mathbf{b}$.

(vi) *If* $\mathbf{a}$ *is 2-generic, then* $\mathbf{a}$ *does not bound a minimal degree (i.e., a degree* $\mathbf{b}$ *such that* $\mathbf{D(0, b)} = \emptyset$).

(vii) *If* $\mathbf{a}$ *is 2-generic, then there is a degree* $\mathbf{b} < \mathbf{a}$ *such that* $\mathbf{b} \in \mathbf{GL_2} - \mathbf{GL_1}$. (*Hence if* $\mathbf{a}$ *is 2-generic, then there is a degree* $\mathbf{b} < \mathbf{a}$ *such that* $\mathbf{b}$ *is not 1-generic.*)

(viii) *If A is 2-generic, then there is a set $B \leqslant_T A$ such that $A \in \Sigma_2^B - \Pi_2^B$.*

(ix) *If* $\mathbf{a}$ *is 3-generic, then there is a* $\mathbf{b} < \mathbf{a}$ *such that* $\mathbf{b} \in \mathbf{GL_3} - \mathbf{GL_2}$. *Hence there is a degree in* $\mathbf{GL_3} - \mathbf{GL_2}$ *which does not bound a minimal degree.*

2.11–2.15 Exercises

2.11 Prove that generic sets exist. Show that for all $n \geqslant 1$, there is an n-generic degree $\mathbf{a}$ for which $\mathbf{a} \leqslant \mathbf{0^{(n)}}$.

2.12 Show that if $\mathbf{a}$ is n-generic, then $\mathbf{a^{(n)}} = \mathbf{a} \cup \mathbf{0^{(n)}}$.

2.13 Show that if **a** and **b** are n-generic, then

$$\exists_n \cap \mathrm{Th}(\mathscr{D}[\mathbf{0},\mathbf{a}]) = \exists_n \cap \mathrm{Th}(\mathscr{D}[\mathbf{0},\mathbf{b}]).$$

***2.14** Let **a** be a 1-generic degree. Show that $\mathscr{D}[\mathbf{0},\mathbf{a}]$ is not a lattice. (*Hint*: Fix a 1-generic set A of degree **a**, and view A as a subset of N^2. It suffices to find an exact pair $\langle \mathbf{b},\mathbf{c}\rangle$ for the ideal **I** generated by $\{A^{[3i]}: i \in N\}$. We define B and C having degrees **b** and **c** respectively. Let $B^{[i]} = A^{[3i]}$ for all $i \in N$. Define $F \subseteq N^2$ by

$$F^{[i]} = \{j \in A^{[3i+1]}: \forall k \leqslant j(k \in A^{[3i+2]})\}$$

and define $C^{[i]} = (B^{[i]} \triangle F^{[i]})$. Now use the 1-genericity of A to show that **b** and **c** form an exact pair for **I**.)

2.15 Show that if **a** is 2-generic, then there is a degree $\mathbf{b} < \mathbf{a}$ such that $\mathbf{b} \in \mathbf{GL_2} - \mathbf{GL_1}$. (*Hint*: First note that the notion of n-genericity can be relativized to any set C. The proof of Theorem 2.9 relativizes to show that if A is 1-generic over C, then there is a set $B \leqslant_T A$ such that A is recursively enumerable in B and $A \nleqslant_T B \oplus C$. Note that by Post's Theorem, 2-genericity is the same as 1-genericity over $\emptyset'$. Choose $C = \emptyset'$ and **b** as the degree of B in the relativized version of Theorem 2.9, and use Exercise 2.12.)

3. $\mathbf{GL_2}$ *and Its Complement*

The degrees in $\mathbf{GL_2}$ are, in a sense, closer to **0** than the degrees in any other class of the generalized high/low hierarchy except for $\mathbf{GL_1}$. As we saw in the previous section, $\mathbf{D}[\mathbf{0},\mathbf{a}]$ for $\mathbf{a} \in \mathbf{L_2}$ can still be fairly complicated. However, we will show in the next chapter that there are some degrees $\mathbf{a} \in \mathbf{GL_2}$ such that $\mathbf{D}[\mathbf{0},\mathbf{a}]$ is finite; in fact, such degrees can be found in $\mathbf{GL_2} - \mathbf{GL_1}$. In this section, we prove the converse to that result, namely, if $\mathbf{D}[\mathbf{0},\mathbf{a}]$ is finite, then $\mathbf{a} \in \mathbf{GL_2}$. Thus we are really proving a theorem about $\overline{\mathbf{GL_2}} = \mathbf{D} - \mathbf{GL_2}$. For we show that if $\mathbf{a} \in \overline{\mathbf{GL_2}}$, then there is a 1-generic degree $\mathbf{b} < \mathbf{a}$. We then apply results for 1-generic degrees which were proved in Sect. 2.

$\mathbf{GL_2}$ is closely connected with relativized $\mathbf{H_1}$. For

$$\mathbf{a} \in \mathbf{GL_2} \Leftrightarrow \mathbf{a}^{(2)} = (\mathbf{a} \cup \mathbf{0}')' \Leftrightarrow \mathbf{a} \cup \mathbf{0}' \in \mathbf{H_1}(\mathbf{a}).$$

Our investigation of $\overline{\mathbf{GL_2}}$ relies on a domination property for degrees in $\mathbf{H_1}(\mathbf{a})$. This domination property is also a useful tool for studying degrees in $\mathbf{H_1}$, and so is also used in the next section. Before proving the domination lemma, we need the following definition and lemma.

3.1 Definition. Let $f: N \to N$ be given. Define $\mathrm{Tot}(f) = \{e \in N: \Phi_e^f \text{ is total}\}$.

3.2 Lemma. *For all* $f: N \to N$, $\mathrm{Tot}(f) \in \Pi_2^f$ *and* $\mathrm{Tot}(f)$ *has degree* $\mathbf{f}^{(2)}$.

Proof. We note that

$$e \in \mathrm{Tot}(f) \Leftrightarrow \forall x \exists \sigma(\sigma \subset f \,\&\, \Phi_e^{\sigma}(x)\downarrow).$$

Hence $\mathrm{Tot}(f) \in \Pi_2^f \subseteq \Delta_3^f$. By Post's Theorem, $\mathrm{Tot}(f) \leqslant_T f^{(2)}$.

In order to show that $f^{(2)} \leqslant_T \mathrm{Tot}(f)$, we construct, uniformly in f, a set $B \subseteq N^2$ such that B is recursively enumerable in f and for all $n \in N$,

(1) $\quad n \in f^{(2)} \Rightarrow B^{[n]}$ is finite;

(2) $\quad n \notin f^{(2)} \Rightarrow B^{[n]} = N$.

By the Enumeration Theorem, there will be a recursive function g such that for all $n \in N$ and $f: N \to N$, $B^{[n]} = \mathrm{dom}(\Phi_{g(n)}^f)$. Thus

$$n \in f^{(2)} \Rightarrow g(n) \in \mathrm{Tot}(f)$$

so $f^{(2)} \leqslant_T \mathrm{Tot}(f)$.

It remains only to construct B. Let $\{\sigma_i : i \in N\}$ be a one-one recursive correspondence of $\mathscr{S}_2$ with N. f' is recursively enumerable in f, so is the range of a function h recursive in f. Let $f'_s = \{y \in N : \exists x \leqslant s(h(x) = y)\}$. Note that

(3) $\quad \sigma \subset f' \Leftrightarrow \exists s \forall t \geqslant s(\sigma \subset f'_t) \Leftrightarrow \forall s \exists t \geqslant s(\sigma \subset f'_t)$

and

(4) $\quad n \in f^{(2)} \Leftrightarrow \exists i(\sigma_i \subset f' \,\&\, \Phi_n^{\sigma_i}(n)\downarrow)$.

Let $B^0 = \emptyset$. We place $\langle n, i\rangle \in B^{s+1}$ if $\langle n, i\rangle \in B^s$ or, at stage s, it is determined that for all $j \leqslant i$, σ_j does not seem to be a candidate which will witness that $n \in f'$, i.e., $i \leqslant s$ and either $\sigma_j \not\subseteq f'_s$ or $\Phi_n^{\sigma_j}(n)\uparrow$. Let $B = \cup\{B^s : s \in N\}$. Note that B is recursively enumerable in f, uniformly in f. If $n \in f'$, then by (3) and (4) there will be some $i \in N$ such that σ_i prevents $\langle n, k\rangle$ from entering B for all sufficiently large $k \in N$, so (1) holds. And if $n \notin f'$, then it follows from (3) and (4) that $\langle n, k\rangle$ will enter B for all $k \in N$, so (2) holds. ▯

3.3. Domination Lemma for $\mathbf{H}_1(\mathbf{a})$. *Let* $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ *be given such that* $\mathbf{a} \leqslant \mathbf{b}$. *Then there is a function of degree* $\leqslant \mathbf{b}$ *which dominates every function of degree* $\leqslant \mathbf{a}$ *if and only if* $\mathbf{b}' \geqslant \mathbf{a}^{(2)}$.

Proof. First assume that $\mathbf{b} \geqslant \mathbf{a}$ and that f is a function of degree $\leqslant \mathbf{b}$ which dominates every function of degree $\leqslant \mathbf{a}$. Let $A, B \subseteq N$ be given having degrees $\mathbf{a}$ and $\mathbf{b}$ respectively, and let $\{\sigma_i : i \in N\}$ be a one-one recursive enumeration of $\mathscr{S}_2$. Note that if $g: N \to N$ is recursive in A, then for some $k \in N$ $\{x : g(x) \leqslant f(x) + k\} = N$. Furthermore, if Φ_e^A is total, then the function $h_e: N \to N$ defined by $h_e(x) = \mu i[\sigma_i \subset A \,\&\, \Phi_e^{\sigma_i}(x)\downarrow]$ is recursive in A. Hence for all $e \in N$

$$e \in \mathrm{Tot}(A) \Leftrightarrow \exists k \forall x \exists i \leqslant f(x) + k(\sigma_i \subset A \,\&\, \Phi_e^{\sigma_i}(x)\downarrow).$$

Since $\mathbf{a}, \mathbf{f} \leqslant \mathbf{b}$, $\mathrm{Tot}(A) \in \Sigma_2^{f \oplus A} \subseteq \Sigma_2^B$. By Lemma 3.2, $\mathrm{Tot}(A) \in \Pi_2^A \subseteq \Pi_2^B$; hence $\mathrm{Tot}(A) \in \Delta_2^B$, i.e., $\mathrm{Tot}(A) \leqslant_T B'$. Again by Lemma 3.2, $\mathrm{Tot}(A) \equiv_T A^{(2)}$, so $\mathbf{a}^{(2)} \leqslant \mathbf{b}'$.

Conversely, assume that $\mathbf{b} \geqslant \mathbf{a}$ and that $\mathbf{b}' \geqslant \mathbf{a}^{(2)}$. Fix sets A and B having degrees $\mathbf{a}$ and $\mathbf{b}$ respectively. Then $\mathrm{Tot}(A) \leqslant_T B'$. By the Limit Lemma, there is a function $g: N^2 \to N$ recursive in B such that $\mathrm{Tot}(A) = \lim_s g$. We define $f: N \to N$ dominating every function of degree $\leqslant \mathbf{a}$. Fix $x \in N$. Let $f(x) = \max(\{y_i^x : i \leqslant x\})$ where y_i^x is defined as follows: Fix the least $s \geqslant x$ such that either $g(s,i) = 0$, or $\sigma_s \subset A$ and $\Phi_i^{\sigma_s}(x)\downarrow$. (Note that such an s must exist since if $\Phi_i^A(x)\uparrow$, then $i \notin \mathrm{Tot}(A)$ so $\lim_s g(s,x) = 0$.) Let

$$y_i^x = \begin{cases} \Phi_i^{\sigma_s}(x) & \text{if } \Phi_i^{\sigma_s}(x)\downarrow \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f \leqslant_T B$. Furthermore, if $i \in \mathrm{Tot}(A)$ then $\lim_s g(s,i) = 1$ so for all sufficiently large x, $y_i^x = \Phi_i^A(x)$; hence f dominates Φ_i^A. ▯

For $\mathbf{a}, \mathbf{b} \in \mathbf{D}$, if $\mathbf{a} \leqslant \mathbf{b} \leqslant \mathbf{a}'$ then $\mathbf{b}' \geqslant \mathbf{a}^{(2)}$ if and only if $\mathbf{b} \in \mathbf{H}_1(\mathbf{a})$. Furthermore, $\mathbf{a} \leqslant \mathbf{a} \cup \mathbf{0}' \leqslant \mathbf{a}'$, and we have already noted that $\mathbf{a} \in \mathbf{GL}_2$ if and only if $\mathbf{a} \cup \mathbf{0}' \in \mathbf{H}_1(\mathbf{a})$. We thus obtain the following corollary of the Domination Lemma for $\mathbf{H}_1(\mathbf{a})$.

3.4 Corollary. *Let $\mathbf{a} \in \mathbf{D}$ be given. Then there is a function of degree $\leqslant \mathbf{a} \cup \mathbf{0}'$ which dominates every function of degree $\leqslant \mathbf{a}$ if and only if $\mathbf{a} \in \mathbf{GL}_2$.*

We use the above characterization of $\mathbf{GL}_2$ to show that every degree in $\overline{\mathbf{GL}_2}$ bounds a 1-generic degree.

3.5 Theorem. *Let $\mathbf{a} \in \overline{\mathbf{GL}_2}$ be given. Then there is a degree $\mathbf{d} < \mathbf{a}$ such that $\mathbf{d}$ is 1-generic.*

Proof. Fix $\mathbf{a} \in \overline{\mathbf{GL}_2}$. By Proposition 1.10(i), Lemma 2.2, and Lemma 2.3, it suffices to construct a set D of degree $\mathbf{d} \leqslant \mathbf{a}$ such that D forces its jump. Let

$$P = \{\langle \sigma, e\rangle \in \mathscr{S}_2 \times N : \Phi_e^\sigma(e)\downarrow\},$$

and let

$$Q = \{\langle \sigma, e\rangle \in \mathscr{S}_2 \times N : \forall \tau \supseteq \sigma(\Phi_e^\tau(e)\uparrow)\}.$$

D will force its jump if for all $e \in N$ there is a $\sigma \in \mathscr{S}_2$ such that $\langle \sigma, e\rangle \in P \cup Q$. We construct $D = \cup\{\delta_s : s \in N\}$ through the use of an oracle of degree $\mathbf{a}$. It will thus suffice to have D satisfy the following requirements for each $e \in N$:

$$R_e : \exists \sigma \subset D(\langle \sigma, e\rangle \in P \cup Q).$$

Define the partial recursive function $\psi: \mathscr{S}_2 \times N \to N$ as follows: Given $\langle \sigma, e\rangle \in \mathscr{S}_2 \times N$, let $\psi(\sigma, e)$ be the length of the shortest $\tau \in \mathscr{S}_2$ such that $\tau \supseteq \sigma$ and $\langle \tau, e\rangle \in P$ if such a τ exists. $\psi(\sigma, e)$ is undefined otherwise. Define $f: N \to N$ by

$$f(s) = \max(\{0\} \cup \{y : \exists e \leqslant s \exists \sigma \in \mathscr{S}_2(\mathrm{lh}(\sigma) \leqslant s \,\&\, \psi(\sigma, e)\downarrow = y)\}).$$

Since ψ is partial recursive, f has degree $\leqslant \mathbf{0}'$. Thus since $\mathbf{a} \notin \mathbf{GL}_2$, by Corollary 3.4, there is a function $g: N \to N$ of degree $\leqslant \mathbf{a}$ such that $\{x : g(x) \geqslant f(x)\}$ is infinite. Fix such a g. Without loss of generality, we may assume that g is increasing, i.e., for all

$x, y \in N$, if $x < y$ then $g(x) < g(y)$. Fix a recursive one-one correspondence $\{\sigma_i : i \in N\}$ of N with $\mathscr{S}_2$.

The construction of D is similar to the construction given in Proof III.5.6; we use a slowdown procedure to take advantage of the fact that f does not dominate g, and so appoint targets for requirements. We use a priority argument construction in order to guarantee that all requirements are satisfied.

Given $\langle \sigma, e \rangle \in \mathscr{S}_2 \times N$, we define the *e-target* for σ at stage s as follows: Search for $\tau \in \mathscr{S}_2$ of shortest length such that $\tau \supset \sigma$, $\mathrm{lh}(\tau) \leqslant g(s)$, and $\langle \tau, e \rangle \in P$; the first such τ in the above ordering of $\mathscr{S}_2$ is the e-target for σ at stage s. If no such τ exists, then σ has no e-target at stage s. Note that we can determine whether or not σ has an e-target at stage s, and find this e-target if it exists, using g as an oracle; hence this determination is made using an oracle of degree $\leqslant \mathbf{a}$.

The construction proceeds as follows. Set $\delta_0 = \emptyset$ and $i_0 = 0$.

Stage $s + 1$. Fix the least $e \in N$ such that R_e is not yet satisfied and δ_s has an e-target $\tau \supset \delta_s$ at stage $s + 1$. If no such e exists, let $\delta_{s+1} = \delta_s * 0$ and $i_{s+1} = s + 1$. Otherwise, let $i_{s+1} = e$ and let δ_{s+1} be the unique $\xi \in \mathscr{S}_2$ such that $\delta_s \subset \xi \subseteq \tau$ and $\mathrm{lh}(\xi) = \mathrm{lh}(\delta_s) + 1$. If $\delta_{s+1} = \tau$, then R_e *becomes satisfied at stage* $s + 1$.

This completes the construction. We note that if $i_{s+1} = e$, then either R_e becomes satisfied at stage $s + 1$, or δ_{s+1} has an e-target at stage $s + 2$ which is not longer than its e-target at stage $s + 1$, and so $i_{s+2} \leqslant i_{s+1}$. Furthermore, if, in the latter case, $i_t \geqslant i_{s+1}$ for all $t \geqslant s + 1$, then R_e becomes satisfied at some stage $t > s$. It follows from an induction proof and the fact that if R_e becomes satisfied at stage t then $i_s \neq e$ for all $s > t$, that $\liminf_s i_s = \infty$.

We complete the proof of the theorem by verifying that D forces its jump. Fix $e \in N$. If R_e becomes satisfied during the construction, then $\langle \delta_s, e \rangle \in P$ for some $s \in N$. Suppose that R_e does not become satisfied during the construction. Since $\liminf_s i_s = \infty$, there is an $s \in N$ such that for all $t \geqslant s$, $i_t > e$. Fix $t \geqslant s$ such that $g(t) > f(t)$. Then δ_{t-1} cannot have an e-target at stage t, so by the definition of f, $\langle \delta_s, e \rangle \in Q$. (Note that $\mathrm{lh}(\delta_s) = s$ for all $s \in N$.) It is easily checked that D has degree $\leqslant \mathbf{a}$. ▯

The function f used in the proof of Theorem 3.5 has the property that for all $\mathbf{a} \in \mathbf{D}$, if there is no $\mathbf{b} \leqslant \mathbf{a}$ such that $\mathbf{b}$ is 1-generic, then f dominates every function of degree $\leqslant \mathbf{a}$. The facts we proved about 1-generic degrees in the previous section can now be used to obtain the following corollaries.

3.6 Corollary. *Let* $\mathbf{a} \in \mathbf{D}$ *be given such that* $\mathbf{D[0, a]}$ *is finite. Then* $\mathbf{a} \in \mathbf{GL_2}$.

Proof. Immediate from Theorem 3.5 and Theorem 2.6. ▯

If we relax the condition of Corollary 3.6 that $\mathbf{D[0, a]}$ is finite and only require that $\mathscr{D}[\mathbf{0}, \mathbf{a}]$ have a finite maximal chain, then we still can conclude that $\mathbf{a} \in \mathbf{GL_2}$ if we also assume that $\mathbf{a} \leqslant \mathbf{0}'$. We do not know whether the same is true if $\mathbf{a} \nleqslant \mathbf{0}'$.

3.7 Corollary. *Let* $\mathbf{a} \leqslant \mathbf{0}'$ *be given such that* $\mathscr{D}[\mathbf{0}, \mathbf{a}]$ *has a finite maximal chain. Then* $\mathbf{a} \in \mathbf{L_2}$.

Proof. Let $\mathbf{0} = \mathbf{a_0} < \mathbf{a_1} < \cdots < \mathbf{a_n} = \mathbf{a}$ be a finite maximal chain of $\mathscr{D}[\mathbf{0}, \mathbf{a}]$. Theorem 3.5 relativizes to show that $\mathbf{a_{i+1}} \in \mathbf{GL_2(a_i)}$ for all $i < n$. By Theorem 1.8,

$\mathbf{a}_{i+1} \in \mathbf{L}_2(\mathbf{a}_i)$ for all $i < n$. Hence $\mathbf{0}^{(2)} = \mathbf{a}_0^{(2)} = \mathbf{a}_1^{(2)} = \cdots = \mathbf{a}_n^{(2)}$. Thus $\mathbf{a}_n = \mathbf{a} \in \mathbf{L}_2$. ▯

Theorem 3.5 yields a great deal of information about all degrees in $\overline{\mathbf{GL}_2}$.

3.8 Corollary. *Let* $\mathbf{a} \in \overline{\mathbf{GL}_2}$ *be given. Then* $\mathscr{D}[\mathbf{0}, \mathbf{a}]$ *has an infinite set of independent degrees.*

Proof. Immediate from Theorem 3.5 and Theorem 2.6. ▯

3.9 Corollary. *Let* $\mathbf{a} \in \overline{\mathbf{GL}_2}$ *be given, and let* $\mathscr{U}$ *be a finite poset. Then* $\mathscr{U} \hookrightarrow \mathscr{D}[\mathbf{0}, \mathbf{a}]$.

Proof. Immediate from Theorem 3.5 and Corollary 2.7. ▯

3.10 Corollary. *Let* $\mathbf{a} \in \overline{\mathbf{GL}_2}$ *be given. Then* $\exists_1 \cap \mathrm{Th}(\mathscr{D}[\mathbf{0}, \mathbf{a}])$ *is decidable.*

Proof. Immediate from Theorem 3.5 and the proof of Corollary 2.8. ▯

3.11 Corollary. *Let* $\mathbf{a} \in \overline{\mathbf{GL}_2}$ *be given. Then* $\mathscr{D}[\mathbf{0}, \mathbf{a}]$ *is not a lattice.*

Proof. Immediate from Theorem 3.5 and Exercise 2.14. ▯

3.12 Corollary. *Let* $\mathbf{a} \in \mathbf{D}$ *be given. Then there is a degree* $\mathbf{d} \leqslant \mathbf{a}$ *such that* $\mathbf{d} \in \mathbf{GL}_2$.

Proof. If $\mathbf{a} \in \mathbf{GL}_2$, choose $\mathbf{d} = \mathbf{a}$. Otherwise, let $\mathbf{d}$ be a 1-generic degree such that $\mathbf{d} < \mathbf{a}$. Such a $\mathbf{d}$ exists by Theorem 3.5, and by Lemma 2.3 and Proposition 1.10(i), $\mathbf{d} \in \mathbf{GL}_1 \subseteq \mathbf{GL}_2$. ▯

3.13 Remarks. The Domination Theorem for $\mathbf{H}_1(\mathbf{a})$ was proved by Martin [1966a]. The remaining results of this section and the results mentioned in the exercises below were proved by Jockusch and Posner [1978], with the exception of Exercise 3.19 which was proved by Posner and Robinson [1981], and Exercise 3.18.

3.14–3.19 Exercises

***3.14** Let $\mathbf{a} \in \overline{\mathbf{GL}_2(\mathbf{e})}$ and $\mathbf{c} \geqslant \mathbf{a} \cup \mathbf{e}'$ be given such that $\mathbf{c}$ is recursively enumerable in $\mathbf{a}$. Construct degrees $\mathbf{b}, \mathbf{d} \leqslant \mathbf{a}$ such that $\mathbf{b}' = \mathbf{d}' = \mathbf{c}$ and $\mathbf{b} \cap \mathbf{d} = \mathbf{e}$. (*Hint*: Combine the proof of the Shoenfield Jump Inversion Theorem with the proof of Theorem 3.5 and the use of e-splittings.)

3.15 Let $\mathbf{a} \in \overline{\mathbf{GL}_2}$ and $\mathbf{c} > \mathbf{a}$ be given. Construct a degree $\mathbf{b} \in \mathbf{GL}_1$ such that $\mathbf{b} \mid \mathbf{a}$ and $\mathbf{b} \cup \mathbf{a} = \mathbf{c}$. (*Hint*: Combine the proof of the Join Theorem for $\mathbf{0}'$ with the proof of Theorem 3.5.)

***3.16** Let $\mathbf{a} \in \overline{\mathbf{GL}_2}$ be given. Construct 1-generic degrees $\mathbf{b}$ and $\mathbf{c}$ such that $\mathbf{b} \cup \mathbf{c} = \mathbf{a}$. (*Hint*: Modify the proof of Theorem 3.5 to construct two 1-generic sets C and D and code in A of degree $\mathbf{a}$ on $\{x : C(x) \neq D(x)\}$.)

3.17 Let $\mathbf{a} \in \overline{\mathbf{GL}_2}$ be given. Construct a 1-generic degree $\mathbf{d} < \mathbf{a}$ such that for every degree $\mathbf{c} > \mathbf{a}$ there is a 1-generic degree $\mathbf{b}$ such that $\mathbf{b} \ngeqslant \mathbf{a}$ and $\mathbf{b} \cup \mathbf{d} = \mathbf{c}$. (*Hint*: Combine the idea of the proof of Theorem 3.5 with the hint to Exercise 3.15. Revise the construction in such a way so as to build a set D of degree $\mathbf{d}$ simultaneously with the construction of all possible sets B of degree $\mathbf{b}$. Allow for all possible choices of C of degree $\mathbf{c}$ by using a tree construction.)

*3.18 Let $\mathbf{a} \in \overline{\mathbf{GL}_2}$ and $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_n \leqslant \mathbf{a}$ be given such that $\mathbf{c}_i \neq \mathbf{0}$ for all $i \leqslant n$. Construct a 1-generic degree $\mathbf{d}$ such that $\mathbf{d} < \mathbf{a}$ and for all $i \leqslant n$, $\mathbf{d} \ngeqslant \mathbf{c}_i$.

3.19 Let $\mathbf{a} \in \mathbf{L}_2$ be given. Construct a degree $\mathbf{b} < \mathbf{0}'$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$ and $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$. (*Hint*: The basic construction follows the construction given in the Join Theorem for $\mathbf{0}'$, but additional requirements must be satisfied. Let A be given of degree $\mathbf{a}$. Construct B having degree $\mathbf{b}$. A typical new requirement has the form: If $\Phi_e^A = \Phi_n^B$ then Φ_n^B is recursive. Given $\beta_s \subset B$, try to satisfy this requirement by finding an n-splitting of β_s. If none exists, then Φ_n^B is recursive. If one exists, then we can satisfy the requirement if we know that $\Phi_e^A(x)\downarrow$ for the x on which the strings just found n-split. Unfortunately, this question cannot be answered by an oracle of degree $\mathbf{0}'$ unless $\mathbf{a} \in \mathbf{L}_1$. However, $\mathrm{Tot}(A) \in \Delta_1^{\emptyset'}$, so by the Limit Lemma, we have an approximation to $\mathrm{Tot}(A)$ which is recursive in $\emptyset'$. Use this approximation in a way similar to its use in the Domination Lemma for $\mathbf{H}_1(\mathbf{a})$ to complete the construction.)

4. $\mathbf{GH_1}$

If $\mathbf{d} \in \mathbf{GH}_1$, and especially if $\mathbf{d} \in \mathbf{H}_1$, then $\mathscr{D}[\mathbf{0}, \mathbf{d}]$ seems to closely resemble $\mathscr{D}[\mathbf{0}, \mathbf{0}']$. Most of the results which we have proved for $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ have already had their counterparts proved for $\mathscr{D}[\mathbf{0}, \mathbf{d}]$, $\mathbf{d} \in \mathbf{GH}_1$ in Sect. 3. The two major exceptions are the Join Theorem for $\mathbf{0}'$ which we prove in this section, and the *Maximal Chain Property*: All maximal chains are infinite. It is not known whether $\mathscr{D}[\mathbf{0}, \mathbf{d}]$ has the maximal chain property for all $\mathbf{d} \in \mathbf{GH}_1$, but by Corollary 3.7, if $\mathbf{d} \in \mathbf{H}_1$ then $\mathscr{D}[\mathbf{0}, \mathbf{d}]$ has the Maximal Chain Property.

The Join Theorem for $\mathbf{GH}_1$ follows from the Join Theorem for $\mathbf{H}_1$ and other results which we have already proved. The proof of the Join Theorem for $\mathbf{H}_1$ involves an application of the Recursion Theorem. We now discuss the way in which the Recursion Theorem is used.

4.1 Remark. The Recursion Theorem will be used in the following way in the proof of the Join Theorem for $\mathbf{H}_1$. Let $\mathbf{d} \in \mathbf{GH}_1$ be given, and fix a set D of degree $\mathbf{d}$. Starting with Φ_e^D, we recursively construct, uniformly in e, a set D_e^* which is recursive in D. Hence there is a recursive function f such that $D_e^* = \Phi_{f(e)}^D$ for all $e \in N$. By the Recursion Theorem, there is an $e \in N$ such that $\Phi_e^D = \Phi_{f(e)}^D$. We fix such an e, and can then assume that the set Φ_e^D which we are using during the construction is the same as the set $\Phi_{f(e)}^D = D_e^*$ which we are constructing.

Given Φ_e^D as in Remark 4.1, we will need a certain function defined from Φ_e^D in a uniform way. This function is described in the following remark.

4.2 Remark. Let $\mathbf{d} \in \mathbf{GH}_1$ be given, and fix a set D of degree $\mathbf{d}$. Note that for all $\sigma \in \mathscr{S}_2$,

$$\sigma \subset \Phi_e^D \Leftrightarrow \exists \tau \in \mathscr{S}_2 \, \forall x < \mathrm{lh}(\sigma)(\tau \subset D \,\&\, \sigma(x) = \Phi_e^\tau(x)\downarrow),$$

so the relation $\sigma \subset \Phi_e^D$ is in Σ_1^D. Hence the function $Q_e: N \to N$ ($Q_e(n)$ determines whether the divergence of $\Phi_n^A(n)$ is forced where $A = \Phi_e^D$) defined by

$$Q_e(n) = \begin{cases} 1 & \text{if } \exists\sigma \in \mathcal{S}_2(\sigma \subset \Phi_e^D \,\&\, \forall\tau \in \mathcal{S}_2(\tau \supseteq \sigma \to \Phi_n^\tau(n)\uparrow)) \\ 0 & \text{otherwise} \end{cases}$$

is the characteristic function of a set which is in $\Sigma_1^{D\oplus\emptyset'}$. Since $\mathbf{d} \in \mathbf{GH_1}$, $Q \in \Delta_2^D$ hence by the Limit Lemma and the Enumeration Theorem, there is a function $q_e: N^2 \to N$ such that q_e is recursive in D with an index which can be found uniformly recursively from e, and for all $x \in N$, $Q_e(x) = \lim_s q_e(s, x)$.

We now prove the Join Theorem for $\mathbf{H_1}$.

4.3 Join Theorem for $\mathbf{H_1}$. *Let* $\mathbf{d} \in \mathbf{H_1}$ *and* $\mathbf{b} \in \mathbf{D(0, d]}$ *be given. Then there is a degree* $\mathbf{a} \in \mathbf{D(0, d)}$ *such that* $\mathbf{a}' = \mathbf{0}'$ *and* $\mathbf{a} \cup \mathbf{b} = \mathbf{d}$.

Proof. We modify the proof of the Join Theorem for $\mathbf{0}'$. Thus instead of being able to use Q, we use the approximation q_e to Q_e when we are trying to decide how to force the jump on n. q_e, in the limit, tells us whether we will ever succeed in forcing n out of the jump of A. While q_e predicts failure, we try to find an extension of α_s which forces n into the jump. When q_e predicts success, we abandon this attempt and assume that we have already succeeded in forcing n out of the jump of A. If we make infinitely many attempts for n, and if q_e is the correct predictor, then the construction will succeed. We will make sure that q_e is the correct predictor by defining A on all arguments at one stage if q_e makes a prediction which is incorrect in the limit and which tells us that we can force n into A'. The Recursion Theorem will then guarantee that for some $e \in N$, q_e is the correct predictor for the set A being constructed. The reader should be familiar with the proof of the Join Theorem for $\mathbf{0}'$ before continuing with this proof.

Let $\mathbf{d} \in \mathbf{H}_1$ and $b \in \mathbf{D(0, d]}$ be given. Fix a set B of degree $\mathbf{b}$, and a set D of degree $\mathbf{d}$. By the Upward Domination Lemma, we can fix a function g of degree $\leqslant \mathbf{b}$ which is not dominated by any recursive function. For each $m \in N$, let 0_m be the string of length m such that $0_m(x) = 0$ for all $x < m$, and let $\lambda_m = 0_m * 1$. Let $P = \{\langle\sigma, e\rangle \in \mathcal{S}_2 \times N: \Phi_e^\sigma(e)\downarrow\}$. P is recursively enumerable, so we can fix a recursive enumeration $\{\langle\sigma_i, e_i\rangle: i \in N\}$ of P, and let $P^s = \{\langle\sigma_i, e_i\rangle: i \leqslant s\}$ for all $s \in N$.

Let $e \in N$ be given. Starting with Φ_e^D, we construct a set $\Phi_{k(e)}^D$ where k is a recursive function. By Remark 4.1, we may assume that for some $e \in N$, $\Phi_e^D = \Phi_{k(e)}^D$. Fix such an e, and let Q and q be the Q_e and q_e of Remark 4.2 respectively.

We construct a set $A = \Phi_{k(e)}^D = D_e^*$ directly (instead of using the Bounding Principle for Forcing and Coding) in order to be able to apply Remark 4.1. We will have $A = \bigcup\{\alpha_s: s \in N\}$, where α_s is defined at stage s of the construction. Let $\{\langle j_s, m_s\rangle: s \in N\}$ be a one-one recursive enumeration of N^2.

We begin the construction at stage 0, setting $\alpha_0 = \emptyset$. The construction then proceeds as follows:

Stage $s + 1$. Let $\langle j_s, m_s\rangle = \langle j, m\rangle$. We simultaneously try to force j into A' and follow the prediction made by q, stopping the attempt to force j into A' when q predicts that this attempt will fail. Since q makes correct predictions only in the limit, at each attempt for j we follow only sufficiently late predictions made by q. Thus we define

$$i(s) = \mu r[q(s + r, j) = 1 \text{ or } \exists\sigma \supseteq \alpha_s * 0_r(\langle\sigma, j\rangle \in P^{g(r)})].$$

While we search for $i(s)$ checking out $r = 0, 1, \ldots$ in order, as we determine that $i(s) > r$, we specify that $\alpha_{s+1} \supseteq \alpha_s * 0_r$. Thus if $i(s)$ is undefined, then the construction *terminates* at this point with A defined by

$$A(x) = \begin{cases} \alpha_s(x) & \text{if } x < \mathrm{lh}(\alpha_s) \\ 0 & \text{otherwise.} \end{cases}$$

Thus we may assume that $i(s)$ has been found. Suppose first that $\exists\sigma \supseteq \alpha_s * 0_{i(s)}(\langle\sigma, j\rangle \in P^{g(i(s))})$. Fix the least $i \in N$ such that for such a σ, $\langle\sigma_i, j_i\rangle$ appears in $P^{g(i(s))}$, and define $\alpha_{s+1} = \sigma * 1 * D(s)$. Otherwise, we let $\alpha_{s+1} = \alpha_s * \lambda_{i(s)} * D(s)$.

The reader should, at this point, convince himself that Remark 4.1 is applicable to this construction. We now show that $i(s)$ is always defined. Under the assumption that this is not the case, we obtain a contradiction. Let the construction terminate at stage $s + 1$ with $j_s = j$. Let $0_\infty = \cup\{0_m : m \in N\}$. Then $A = \alpha_s * 0_\infty$ but $Q(j) = 0$. By the definition of Q, every $\sigma \subset A$ can be extended to some $\tau \in \mathscr{S}_2$ such that $\Phi_j^\tau(j)\downarrow$. Define $h: N \to N$ by $h(x) = \mu t[\exists\tau \in \mathscr{S}_2(\langle\sigma, j\rangle \in P^t \,\&\, \sigma \supseteq \alpha_s * 0_x)]$. h is a total recursive function, hence by the Upward Domination Lemma, there is an $x \in N$ such that $g(x) \geqslant h(x)$. Thus $i(s) \leqslant x$ yielding the desired contradiction.

We next show that A forces its jump. Fix $j \in N$. Fix $s \in N$ sufficiently large so that for all $t \geqslant s$, $q(t, j) = q(s, j)$, and fix $t \geqslant s$ such that $\langle j_t, m_t\rangle = \langle j, m\rangle$ for some $m \in N$. Since $i(t)$ is defined, either $\langle\alpha_{t+1}, j\rangle \in P$ and so $\Phi_j^A(j)\downarrow$, or $q(r, j) = 1$ for some $r \geqslant t$, in which case, by the choice of s, $q(r, j) = q(s, j) = Q(j) = 1$ and so $\exists\sigma \subset A\, \forall\tau \geqslant \sigma(\Phi_j^\tau(j)\uparrow)$. Thus A forces its jump, so by Lemma III.3.9, $\mathbf{a}' = \mathbf{0}'$.

Finally, we show that $D \equiv_T A \oplus B$. Since $g \leqslant_T D$, $A \oplus B \leqslant_T D$. To see that $D \leqslant_T A \oplus B$, we assume by induction that we have found α_s, and wish to compute $D(s)$ through the use of A and B oracles. Let $\langle j_s, m_s\rangle = \langle j, m\rangle$. Since $\{\lambda_k : k \in N\}$ is a set of pairwise incompatible strings, we can use the A oracle to find the unique $k \in N$ such that $\alpha_s * \lambda_k \subset A$. Such a k will exist as, in all cases, $\alpha_{s+1} \neq \alpha_s * 0_m$ for all m. Fix this k. We now use the A and B oracles to determine whether there are $\sigma \subset A$ and $i \leqslant k$ such that $\sigma \supseteq \alpha_s * 0_i$ and $\langle\sigma, j\rangle \in P^{g(i)}$. If the answer is yes, fix the first such $\langle\sigma, j\rangle$ found and let $\tau = \sigma * 1$; if the answer is no, let $\tau = \alpha_s * \lambda_k$. In either case, the A oracle now gives us the unique $n \in \{0, 1\}$ such that $\tau * n \subset A$; $\alpha_{s+1} = \tau * n$ and $s \in D$ if and only if $n = 1$. ▯

We note that the only place in the proof of Theorem 4.3 where we used the fact that $\mathbf{a} \in \mathbf{H_1}$ rather than $\mathbf{a} \in \mathbf{GH_1}$ was to obtain the function g provided by the Upward Domination Lemma. Hence the following corollary to Theorem 4.3 is immediate.

4.4 Corollary. *Let* $\mathbf{d} \in \mathbf{GH_1}$ *and* $\mathbf{b} \in \mathbf{D}(\mathbf{0}, \mathbf{d}]$ *be given. Assume that there is a function* g *of degree* $\leqslant \mathbf{b}$ *which is not dominated by any recursive function. Then there is a degree* $\mathbf{a} \in \mathbf{D}(\mathbf{0}, \mathbf{d})$ *such that* $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}'$ *and* $\mathbf{a} \cup \mathbf{b} = \mathbf{d}$.

Corollary 4.4 has the following relativization:

4.5 Corollary. *Let* $\mathbf{c} \in \mathbf{D}$, $\mathbf{d} \in \mathbf{GH_1}(\mathbf{c})$, *and* $\mathbf{b} \in \mathbf{D}(\mathbf{c}, \mathbf{d}]$ *be given. Assume that there is a function* g *of degree* $\leqslant \mathbf{b}$ *which is not dominated by any function of degree* $\leqslant \mathbf{c}$. *Then there is a degree* $\mathbf{a} \in \mathbf{D}(\mathbf{c}, \mathbf{d})$ *such that* $\mathbf{a}' = \mathbf{a} \cup \mathbf{c}'$ *and* $\mathbf{a} \cup \mathbf{b} = \mathbf{d}$.

The proof we give for the Join Theorem for $\mathbf{GH_1}$ relies on a version of the Join Theorem for $\mathbf{H_1}$ in which the degree $\mathbf{a}$ constructed does not lie above a prespecified degree $\mathbf{c} > \mathbf{0}$. We state this theorem, and indicate how to expand the proof of the Join Theorem for $\mathbf{H_1}$ to obtain a proof.

4.6 Theorem. *Let* $\mathbf{d} \in \mathbf{H_1}$ *and* $\mathbf{b}, \mathbf{c} \in \mathbf{D}(\mathbf{0}, \mathbf{d}]$ *be given. Then there is a degree* $\mathbf{a} \in \mathbf{D}(\mathbf{0}, \mathbf{d})$ *such that* $\mathbf{a}' = \mathbf{0}'$, $\mathbf{c} \not\leqslant \mathbf{a}$, *and* $\mathbf{a} \cup \mathbf{b} = \mathbf{d}$.

Sketch of Proof. We proceed as in the proof of the Join Theorem for $\mathbf{H_1}$, acting at stage $2s$ as we previously acted at stage s. We add requirements $\{\Phi_e^A \neq C : e \in N\}$ where C is a set of degree $\mathbf{c}$. The requirement $\Phi_e^A \neq C$ is attacked at those stages $2s+1$ of the construction where $\langle j_s, m_s \rangle = \langle e, m \rangle$ for some $m \in N$. Counterparts S and T_e of P and Q_e are defined as follows: $S = \{\langle \sigma, \tau, n \rangle \in \mathscr{S}_2^2 \times N : \sigma$ and τ are an n-splitting$\}$ and $T_e = \{\langle \sigma, n \rangle \in \mathscr{S}_2 \times N : \sigma \subset \Phi_e^D \,\&\, \forall \tau, \rho \supset \sigma$ (τ and ρ are not an n-splitting of σ)$\}$. S is recursively enumerable, so has a recursive approximation $S = \bigcup\{S^s : s \in N\}$, and $T_e \in \Sigma_1^{D \oplus \emptyset'} \subseteq \Delta_2^D$. If there is some $\sigma \subset A$ such that $\sigma \in T_e$ and Φ_n^A is total, then Φ_n^A must be recursive.

We use the Recursion Theorem to get $T = T_e$ for $A = \Phi_e^D$. At stage $2k+1$, while attacking $\Phi_n^A \neq C$, we simultaneously run the approximation to T and search for an n-splitting of $\alpha_{2k} * 0_r \in S^{g(r)}$. If an n-splitting is found first, take the half of the n-splitting, σ, which will force $\Phi_n^A \neq C$, and let $\alpha_{2k+1} = \sigma * 1 * D(2k)$. The reader should now be able to complete the proof along the lines of the proof of Theorem 4.3. ∎

There are corollaries to Theorem 4.6 which correspond to Corollary 4.4 and Corollary 4.5. We state the counterpart to Corollary 4.5.

4.7 Corollary. *Let* $\mathbf{a} \in \mathbf{D}$, $\mathbf{d} \in \mathbf{GH_1}(\mathbf{a})$, $\mathbf{e} \in \mathbf{D}(\mathbf{a}, \mathbf{d}]$ *and* $\mathbf{c} \in \mathbf{D}(\mathbf{0}, \mathbf{d}]$ *be given such that* $\mathbf{c} \not\leqslant \mathbf{a}$. *Assume that there is a function* g *of degree* $\leqslant \mathbf{e}$ *which is not dominated by any function of degree* $\leqslant \mathbf{a}$. *Then there is a degree* $\mathbf{b} \in \mathbf{D}(\mathbf{a}, \mathbf{d})$ *such that* $\mathbf{b}' = \mathbf{b} \cup \mathbf{a}'$, $\mathbf{c} \not\leqslant \mathbf{b}$, *and* $\mathbf{e} \cup \mathbf{b} = \mathbf{d}$.

There is one more result which will be used in the proof of the Join Theorem for $\mathbf{GH_1}$.

4.8 Theorem. *Let* $\mathbf{d} \in \mathbf{GH_1}$ *and* $\mathbf{c} \in \mathbf{D}(\mathbf{0}, \mathbf{d}]$ *be given. Then there is a degree* $\mathbf{b} \in \mathbf{D}(\mathbf{0}, \mathbf{d})$ *such that* $\mathbf{c} \not\leqslant \mathbf{b}$, $\mathbf{b}' = \mathbf{d} \cup \mathbf{0}'$ (*hence* $\mathbf{d} \in \mathbf{GH_1}(\mathbf{b})$), *and* $\mathbf{d}$ *is recursively enumerable in* $\mathbf{b}$.

Proof. Since $\mathbf{GH_1} \subseteq \overline{\mathbf{GL_2}}$, it follows from Exercise 3.18 that there is a 1-generic degree $\mathbf{e} < \mathbf{d}$ such that $\mathbf{c} \not\leqslant \mathbf{e}$. By Theorem 2.9, there is a degree $\mathbf{a} < \mathbf{e}$ such that $\mathbf{e}$ is recursively enumerable in $\mathbf{a}$. Since $\mathbf{d} \in \mathbf{GH_1}$, $\mathbf{0} \leqslant \mathbf{a} < \mathbf{e}$ and $\mathbf{e}$ is 1-generic,

$$\mathbf{d}' = (\mathbf{d} \cup \mathbf{0}')' \leqslant (\mathbf{d} \cup \mathbf{a}')' \leqslant (\mathbf{d} \cup \mathbf{e}')' = (\mathbf{d} \cup \mathbf{e} \cup \mathbf{0}')' = (\mathbf{d} \cup \mathbf{0}')' = \mathbf{d}'.$$

Hence $\mathbf{d} \in \mathbf{GH_1}(\mathbf{a})$. Since $\mathbf{a} < \mathbf{e}$ and $\mathbf{e}$ is recursively enumerable in $\mathbf{a}$, by the relativization of the Upward Domination Lemma, there is a function of degree $\leqslant \mathbf{e}$ which is not dominated by any function of degree $\leqslant \mathbf{a}$. Hence by Corollary 4.7, there is a degree $\mathbf{b} \in \mathbf{D}(\mathbf{a}, \mathbf{d})$ such that $\mathbf{b}' = \mathbf{b} \cup \mathbf{a}'$, $\mathbf{c} \not\leqslant \mathbf{b}$, and $\mathbf{b} \cup \mathbf{e} = \mathbf{d}$. Since $\mathbf{e}$ is recursively enumerable in $\mathbf{a} < \mathbf{b}$, both $\mathbf{e}$ and $\mathbf{b}$ are recursively enumerable in $\mathbf{b}$, so $\mathbf{d} = \mathbf{b} \cup \mathbf{e}$ is recursively enumerable in $\mathbf{b}$. Hence $\mathbf{b} < \mathbf{d} \leqslant \mathbf{b}'$, so $\mathbf{d} \cup \mathbf{0}' \leqslant \mathbf{b}'$.

Furthermore,

$$\mathbf{b}' = \mathbf{b} \cup \mathbf{a}' \leqslant \mathbf{b} \cup \mathbf{e}' = \mathbf{b} \cup \mathbf{e} \cup \mathbf{0}' = \mathbf{d} \cup \mathbf{0}'.$$

Hence $\mathbf{b}' = \mathbf{d} \cup \mathbf{0}'$ and $\mathbf{b}^{(2)} = (\mathbf{d} \cup \mathbf{0}')'$, so $\mathbf{d} \in \mathbf{H}_1(\mathbf{b}) \subseteq \mathbf{GH}_1(\mathbf{b})$. ▯

We are now ready to prove the Join Theorem for $\mathbf{GH}_1$.

4.9 Join Theorem for $\mathbf{GH}_1$. *Let* $\mathbf{d} \in \mathbf{GH}_1$ *and* $\mathbf{c} \in \mathbf{D}(\mathbf{0}, \mathbf{d}]$ *be given. Then there is a degree* $\mathbf{a} \in \mathbf{D}(\mathbf{0}, \mathbf{d})$ *such that* $\mathbf{a}' = \mathbf{d} \cup \mathbf{0}'$ *and* $\mathbf{a} \cup \mathbf{c} = \mathbf{d}$.

Proof. By Theorem 4.8, fix a degree $\mathbf{b} \in \mathbf{D}(\mathbf{0}, \mathbf{d})$ such that $\mathbf{c} \nleqslant \mathbf{b}$, $\mathbf{b}' = \mathbf{d} \cup \mathbf{0}'$ and $\mathbf{d} \in \mathbf{GH}_1(\mathbf{b})$. Let $\mathbf{e} = \mathbf{b} \cup \mathbf{c}$. Note that $\mathbf{b} < \mathbf{e} \leqslant \mathbf{d} \leqslant \mathbf{b}'$. Since $\mathbf{d} \in \mathbf{H}_1(\mathbf{b})$, we can apply the relativization of the Join Theorem for $\mathbf{H}_1$ to obtain a degree $\mathbf{a} \in \mathbf{D}(\mathbf{b}, \mathbf{d})$ such that $\mathbf{a} \cup \mathbf{e} = \mathbf{d}$ and $\mathbf{a}' = \mathbf{b}'$. Now $\mathbf{d} = \mathbf{a} \cup \mathbf{e} = \mathbf{a} \cup \mathbf{b} \cup \mathbf{c} = \mathbf{a} \cup \mathbf{c}$. Furthermore, $\mathbf{a}' = \mathbf{b}' = \mathbf{d} \cup \mathbf{0}'$. ▯

We do not know if we can require that $\mathbf{a} \in \mathbf{GL}_1$ in the Join Theorem for $\mathbf{GH}_1$.

In the next section, we will apply some of the results proved in this chapter to show that certain classes of degrees are *automorphism bases* for structures of degrees.

4.10 Remarks. The type of application of the Recursion Theorem made in this section originated with Jockusch [1977]. The results of this section are due to Posner [1977], with the exception of Exercise 4.17 which was proved by Shore [1981].

4.11–4.17 Exercises

4.11 Let $\mathbf{d} \in \mathbf{H}_1$ and $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbf{D}(\mathbf{0}, \mathbf{d}]$ be given. Construct a degree $\mathbf{a} \in \mathbf{D}(\mathbf{0}, \mathbf{d})$ such that $\mathbf{a}' = \mathbf{0}'$ and $\mathbf{a} \cup \mathbf{b}_i = \mathbf{d}$ for all $i \leqslant n$. (*Hint*: See Exercise III.5.20.)

4.12 Let $\mathbf{d} \in \mathbf{GH}_1$ and $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbf{D}(\mathbf{0}, \mathbf{d}]$ be given. Construct a degree $\mathbf{a} \in \mathbf{D}(\mathbf{0}, \mathbf{d})$ such that $\mathbf{a}' = \mathbf{d} \cup \mathbf{0}'$ and $\mathbf{a} \cup \mathbf{c}_i = \mathbf{d}$ for all $i \leqslant n$.

4.13 Let $\mathbf{d} \in \mathbf{GH}_1$ be given. Show that all non-trivial antichains of $\mathbf{D}[\mathbf{0}, \mathbf{d}]$ are infinite.

4.14 Let $\mathbf{d} \in \mathbf{GH}_1$ be given. Let $\mathbf{B} \subseteq \mathbf{D}(\mathbf{0}, \mathbf{d}]$ be finite but non-empty, and let $\mathbf{C} \subseteq \mathbf{D}[\mathbf{0}, \mathbf{d}]$ be uniformly of degree $\leqslant \mathbf{d}$. Construct a degree $\mathbf{a} \in \mathbf{D}(\mathbf{0}, \mathbf{d})$ such that $\mathbf{a}' = \mathbf{d} \cup \mathbf{0}'$, $\mathbf{a} \cup \mathbf{b} = \mathbf{d}$ for all $\mathbf{b} \in \mathbf{B}$, and $\mathbf{c} \nleqslant \mathbf{a}$ for all $\mathbf{c} \in \mathbf{C} - \{\mathbf{0}\}$.

4.15 Let $\mathbf{d} \in \mathbf{GH}_1$ and $\mathbf{b} \in \mathbf{D}(\mathbf{0}, \mathbf{d})$ be given such that $\mathbf{b}^{(2)} \leqslant \mathbf{d}'$. Construct a degree $\mathbf{a}$ such that $\mathbf{b} \cup \mathbf{a} = \mathbf{d}$ and $\mathbf{b} \cap \mathbf{a} = \mathbf{0}$. (*Hint*: $\mathrm{Tot}(B)$ is uniformly of degree $\leqslant \mathbf{d}$ if B has degree $\mathbf{b}$. Apply Exercise 4.14.)

***4.16** Let $n \geqslant 1$ and $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ be given such that $\mathbf{a} \geqslant \mathbf{b}^{(n)}$. Construct a finite set of degrees $\{\mathbf{c}_i : i \leqslant k\}$ such that $\cup\{\mathbf{c}_i : i \leqslant k\} = \mathbf{a}$ and for all $j \leqslant k$, $\mathbf{b} \leqslant \mathbf{c}_j$ and $\mathbf{c}_j^{(n)} \leqslant \mathbf{a}$.

4.17 Let A, $X \subseteq N$ be given such that $X \in \Sigma_4^A$ and for all $x \in X$, Φ_x^A is total. Let A have degree $\mathbf{a}$ and for all $x \in X$, let $\mathbf{a}_x$ be the degree of Φ_x^A. Let $\mathbf{I} = \{\mathbf{d} \in \mathbf{D} : \exists F \subseteq N$ (F is finite & $\mathbf{d} \leqslant \cup\{\mathbf{a}_i : i \in F\}$)$\}$. Let $\mathbf{e} \in \mathbf{GH}_1(\mathbf{a})$ be given, and suppose that $\mathbf{I}$ is an ideal of $\mathscr{D}[\mathbf{0}, \mathbf{e}]$. Construct an exact pair $\langle \mathbf{c}_0, \mathbf{c}_1 \rangle$ for $\mathbf{I}$ such that $\mathbf{c}_0, \mathbf{c}_1 \leqslant \mathbf{e}$. (*Hint*: Use the approximation to Tot(A) which has degree $\mathbf{e}$ to modify the proof of Theorem III.8.6.)

5. Automorphism Bases

In this section, we begin our study of automorphisms of various structures of degrees. We show that almost all classes of the high/low and generalized high/low hierarchies are automorphism bases for the appropriate structures of degrees.

5.1 Definition. Let $\mathscr{A}$ be any algebraic structure. An *automorphism* of $\mathscr{A}$ is an isomorphism of $\mathscr{A}$ with $\mathscr{A}$.

5.2 Example. Let $\mathscr{U} = \langle U, \leqslant \rangle$ be a poset, and let $\mathscr{U}^* = \langle U, \leqslant, \vee \rangle$ be a usl. Define $\mathrm{Id}: U \to U$ by $\mathrm{Id}(x) = x$ for all $x \in U$. Then Id is a poset automorphism of $\mathscr{U}$ and a usl automorphism of $\mathscr{U}^*$. Id is called the *identity automorphism.*

5.3 Example. Let $\mathscr{U} = \langle U, \leqslant \rangle$ be a poset and let $f: U \to U$ be a poset automorphism of $\mathscr{U}$. Define $f^{-1}: U \to U$ by $f^{-1}(x) = y$ if and only if $f(y) = x$. Then f^{-1} is also a poset automorphism of $\mathscr{U}$, and is referred to as the *inverse* of f. (A similar example can be given for usls.)

Example 5.2 and Example 5.3 describe poset automorphisms which are also usl automorphisms if the poset on which the automorphism is defined happens to be a usl. The next proposition shows that this is always the case.

5.4 Proposition. *Let $\langle U, \leqslant, \vee \rangle$ be a usl and let $f: U \to U$ be a poset automorphism of $\langle U, \leqslant \rangle$. Then f is also an automorphism of the usl $\langle U, \leqslant, \vee \rangle$.*

Proof. It suffices to show that for all $a, b \in U$, $f(a \vee b) = f(a) \vee f(b)$. Fix $a, b \in U$. Let $d = f(a) \vee f(b)$. Then there is a $c \in U$ such that $f(c) = d$. Since $f(a) \leqslant d$ and $f(b) \leqslant d$, it must be the case that $a \leqslant c$ and $b \leqslant c$. Hence $a \vee b \leqslant c$. Since f is a poset automorphism, $f(a \vee b) \leqslant f(c) = d$. But $a \leqslant a \vee b$ and $b \leqslant a \vee b$ so $f(a) \leqslant f(a \vee b)$ and $f(b) \leqslant f(a \vee b)$. Hence $d = f(a) \vee f(b) \leqslant f(a \vee b) \leqslant d$, so $f(a \vee b) = f(a) \vee f(b)$. ▯

The only known automorphism of either $\mathscr{D}$ or $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ is the identity automorphism, but additional automorphisms have not been ruled out. The results we give about automorphism bases show that automorphisms are difficult to construct for structures of degrees; for all automorphisms of such structures are uniquely determined by their values on relatively small sets of degrees.

5.5 Definition. A set $B \subseteq U$ is an *automorphism base* for the poset $\mathscr{U} = \langle U, \leqslant \rangle$ if every $f: B \to B$ has at most one extension $\hat{f}: U \to U$ such that $\hat{f}$ is an automorphism of $\mathscr{U}$.

We note that by Proposition 5.4, the study of poset automorphisms and usl automorphisms is identical for posets which are also usls. Thus we restrict our study of automorphisms to poset automorphisms.

The next lemma gives a useful characterization of automorphism bases.

5.6 Lemma. *Let $\mathscr{U} = \langle U, \leqslant \rangle$ be a poset and let $B \subseteq U$. Then the following are equivalent*:

(i) *B is an automorphism base for $\mathscr{U}$.*

(ii) *If f is an automorphism of $\mathscr{U}$ such that $f(x) = x$ for all $x \in B$, then $f = \mathrm{Id}$.*

Proof. (i) $\Rightarrow$ (ii) is immediate from Example 5.2.

(ii) $\Rightarrow$ (i): Suppose that $f, g: U \to U$ are automorphisms of $\mathscr{U}$ such that $f(x) = g(x)$ for all $x \in B$. By Example 5.3, fg^{-1} is an automorphism of $\mathscr{U}$. It is easily verified that $fg^{-1}(x) = x$ for all $x \in B$. Hence by (ii), $fg^{-1} = \mathrm{Id}$, and so $f = g$. ∎

We begin with a study of automorphism bases for $\mathscr{D}[\mathbf{0}, \mathbf{0}']$. One technique for showing that a set B is an automorphism base is to show that B *generates* the set of degrees in the following sense.

5.7 Definition. Let $\mathbf{B} \subseteq \mathbf{P} \subseteq \mathbf{D}$ be given. We say that $\mathbf{B}$ *generates* $\mathbf{P}$ if $\mathbf{P}$ is a subset of the smallest set $\mathbf{E} \subseteq \mathbf{D}$ satisfying the following conditions:

(i) $\mathbf{B} \subseteq \mathbf{E}$.

(ii) $\forall \mathbf{a}, \mathbf{b} \in \mathbf{E}(\mathbf{a} \cup \mathbf{b} \in \mathbf{E})$.

(iii) $\forall \mathbf{a}, \mathbf{b} \in \mathbf{E}$(if $\mathbf{a} \cap \mathbf{b}$ exists then $\mathbf{a} \cap \mathbf{b} \in \mathbf{E}$).

5.8 Remark. It follows from the proof of Proposition 5.4 and the dual proof for $\cap$ that if $\mathbf{B}$ generates $\mathbf{P}$ then $\mathbf{B}$ is an automorphism base for $\langle \mathbf{P}, \leqslant \rangle$.

It will be shown that for any $\mathbf{a} \leqslant \mathbf{0}'$, $\mathbf{J(a)} = \{\mathbf{b} \leqslant \mathbf{0}' : \mathbf{b}' = \mathbf{a}'\}$ generates $\mathbf{D}[\mathbf{0}, \mathbf{0}']$ and hence is an automorphism base for $\mathscr{D}[\mathbf{0}, \mathbf{0}']$. We begin by showing that certain classes in the high/low hierarchy generate $\mathbf{D}[\mathbf{0}, \mathbf{0}']$. We recall Exercise 3.14 which will be a key tool in identifying generating sets.

(1) Let $\mathbf{e} \in \mathbf{D}$, $\mathbf{a} \in \overline{\mathbf{GL}_2}(\mathbf{e})$ and $\mathbf{c} \geqslant \mathbf{a} \cup \mathbf{e}'$ be given such that $\mathbf{c}$ is recursively enumerable in $\mathbf{a}$. Then there are degrees $\mathbf{b}, \mathbf{d} \leqslant \mathbf{a}$ such that $\mathbf{b}' = \mathbf{d}' = \mathbf{c}$ and $\mathbf{b} \cap \mathbf{d} = \mathbf{e}$.

5.9 Lemma. $\overline{\mathbf{L}_2}$ *generates* $\mathbf{D}[\mathbf{0}, \mathbf{0}']$.

Proof. It suffices to show that $\mathbf{H}_1$ generates $\mathbf{L}_2$. Fix $\mathbf{e} \in \mathbf{L}_2$. Applying (1) with $\mathbf{a} = \mathbf{0}'$ and $\mathbf{c} = \mathbf{0}^{(2)}$, we obtain $\mathbf{b}, \mathbf{d} \in \mathbf{H}_1$ such that $\mathbf{b} \cap \mathbf{d} = \mathbf{e}$. ∎

5.10 Lemma. $\mathbf{L}_1$ *generates* $\mathbf{D}[\mathbf{0}, \mathbf{0}']$.

Proof. By Lemma 5.9 it suffices to show that $\overline{\mathbf{L}_2}$ generates $\mathbf{L}_1$. Given $\mathbf{d} \in \mathbf{H}_1$ we apply Exercise 3.16 to obtain 1-generic degrees $\mathbf{a}$ and $\mathbf{c}$ such that $\mathbf{a} \cup \mathbf{c} = \mathbf{d}$. Since $\mathbf{a}, \mathbf{c} \leqslant \mathbf{0}'$, $\mathbf{a}, \mathbf{c} \in \mathbf{L}_1$ by Lemma 2.3. ∎

We can now show the all the jump classes $\mathbf{J(a)}$ generate $\mathbf{D}[\mathbf{0}, \mathbf{0}']$.

5.11 Theorem. *For all* $\mathbf{p} \leqslant \mathbf{0}'$, $\mathbf{J(p)} = \{\mathbf{b} \leqslant \mathbf{0}' : \mathbf{b}' = \mathbf{p}'\}$ *generates* $\mathbf{D}[\mathbf{0}, \mathbf{0}']$.

Proof. Fix $\mathbf{p} \leqslant \mathbf{0}'$. By Lemma 5.10, it suffices to show that $\mathbf{J(p)}$ generates $\mathbf{L}_1$. Fix $\mathbf{e} \in \mathbf{L}_1$, and apply (1) with $\mathbf{a} = \mathbf{0}'$ and $\mathbf{c} = \mathbf{p}'$ to obtain $\mathbf{b}, \mathbf{d} \leqslant \mathbf{0}'$ such that $\mathbf{b}, \mathbf{d} \in \mathbf{J(p)}$ and $\mathbf{b} \cap \mathbf{d} = \mathbf{e}$. ∎

5.12 Corollary. *Let* $\mathbf{C} \subseteq \{\mathbf{I}\} \cup \{\mathbf{L}_{n+1} - \mathbf{L}_n : n \in N\} \cup \{\mathbf{H}_{n+1} - \mathbf{H}_n : n \in N\}$. *Then* $\mathbf{C}$ *is an automorphism base for* $\mathbf{D}[\mathbf{0}, \mathbf{0}']$; *in fact*, $\mathbf{C}$ *generates* $\mathbf{D}[\mathbf{0}, \mathbf{0}']$.

Proof. By Theorem 1.4, no such class $\mathbf{C}$ is empty, hence $\mathbf{J(a)} \subseteq \mathbf{C}$ for some $\mathbf{a} \leqslant \mathbf{0}'$. Apply Theorem 5.11. ▯

We now show that almost all classes of the generalized high/low hierarchy generate $\mathbf{D}$ and hence form automorphism bases for $\mathscr{D}$.

5.13 Lemma. $\overline{\mathbf{GL}_2}$ *generates* $\mathbf{D}$.

Proof. Let $\mathbf{e} \in \mathbf{D}$ be given. Apply (1) with $\mathbf{a} = \mathbf{e}'$ and $\mathbf{c} = \mathbf{e}^{(2)}$ to obtain degrees $\mathbf{b}, \mathbf{d} \in \mathbf{H}_1(\mathbf{e})$ such that $\mathbf{b} \cap \mathbf{d} = \mathbf{e}$. We show that $\mathbf{b} \in \overline{\mathbf{GL}_2}$. By symmetry, it will follow that $\mathbf{d} \in \overline{\mathbf{GL}_2}$. Since $\mathbf{e} \leqslant \mathbf{b} \leqslant \mathbf{e}'$, $\mathbf{b} \cup \mathbf{0}' \leqslant \mathbf{e}'$. Hence $(\mathbf{b} \cup \mathbf{0}')' \leqslant \mathbf{e}^{(2)} < \mathbf{e}^{(3)} = \mathbf{b}^{(2)}$. ▯

5.14 Lemma. $\mathbf{GL}_1$ *generates* $\mathbf{D}$.

Proof. Immediate from Lemma 5.13 and Exercise 3.16. ▯

5.15 Theorem. *Let* $\mathbf{C} \in \{\mathbf{GI}\} \cup \{\mathbf{GL}_{n+1} - \mathbf{GL}_n : n \in N\} \cup \{\mathbf{GH}_{n+1} - \mathbf{GH}_n : n \in N\}$ *be given. Then* $\mathbf{C}$ *is an automorphism base for* $\mathscr{D}$. *In fact,* $\mathbf{C}$ *generates* $\mathbf{D}$.

Proof. Fix $\mathbf{C}$. It suffices to show that $\mathbf{C}$ generates $\mathbf{GL}_1$. Fix $\mathbf{e} \in \mathbf{GL}_1$. Fix the symbol $\mathbf{X}$ such that $\mathbf{C}$ has the form $\mathbf{GX}$. By the relativization of the proof of Theorem 5.11, there are degrees $\mathbf{b}, \mathbf{d} \in \mathbf{X(e)}$ such that $\mathbf{b} \cap \mathbf{d} = \mathbf{e}$. By Theorem 1.8, $\mathbf{X(e)} = \mathbf{GX} \cap \mathbf{D[e, e']}$, so $\mathbf{b}, \mathbf{d} \in \mathbf{GX} = \mathbf{C}$. ▯

Other automorphism bases for classes of degrees are discussed in the exercises below and in later chapters. We list some additional automorphism bases below. These results are due to Jockusch and Posner [1981] with the exception of (viii) which was proved by Posner. Along with the results listed, Jockusch and Posner show that if a set of degrees is large in the appropriate sense of category or measure, then it is an automorphism base.

5.16 Further Results. *The following sets generate* $\mathbf{D[0, 0']}$ *hence are automorphism bases for* $\mathscr{D}[\mathbf{0}, \mathbf{0}']$.

(i) $\{\mathbf{b} \leqslant \mathbf{0}' : \mathbf{b}$ *is 1-generic*$\}$.

(ii) $\{\mathbf{b} \leqslant \mathbf{0}' : \mathbf{b} \ngeqslant \mathbf{a}\}$ *for any* $\mathbf{a} \in \mathbf{D(0, 0']}$.

(iii) $\{\mathbf{b} \leqslant \mathbf{0}' : \mathbf{b} \cap \mathbf{a} = \mathbf{0}\}$ *for any* $\mathbf{a} \in \mathbf{L}_2$. (*Posner shows this for* $\mathbf{a} < \mathbf{0}'$.)

(iv) $\{\mathbf{b} \leqslant \mathbf{0}' : \mathbf{b} \cup \mathbf{a} = \mathbf{0}'\}$ *for all* $\mathbf{a} < \mathbf{0}'$.

(v) $\{\mathbf{b} \leqslant \mathbf{0}' : \mathbf{b} \geqslant \mathbf{c}$ *or* $\mathbf{b} \geqslant \mathbf{d}\}$ *for some choice of* $\mathbf{c}, \mathbf{d} \in \mathbf{D(0, 0')}$.

(vi) $\{\mathbf{b} \leqslant \mathbf{0}' : \mathbf{b} \in \mathbf{L}_1 \,\&\, \mathbf{b} \cup \mathbf{a} = \mathbf{0}'\}$ *for all* $\mathbf{a} < \mathbf{0}'$.

(vii) $\{\mathbf{b} \leqslant \mathbf{0}' : \mathbf{b} \cup \mathbf{a} = \mathbf{0}' \,\&\, \mathbf{b} \cap \mathbf{a} = \mathbf{0}\}$ *for all* $\mathbf{a} \in \mathbf{L}_2 - \{\mathbf{0}\}$.

(viii) $\{\mathbf{b} \leqslant \mathbf{0}' : \mathbf{b}$ *is minimal*$\}$.

The following sets are automorphism bases for $\mathscr{D}$.

(ix) $\{\mathbf{a} \in \mathbf{D} : \mathbf{a}^{(n)} = \mathbf{a} \cup \mathbf{0}^{(n)}\}$ *for all* $n \geqslant 1$.

(x) $\{\mathbf{a} \in \mathbf{D} : \forall \mathbf{b} \leqslant \mathbf{a}(\mathbf{b}$ *is not minimal*$)\}$.

(xi) $\{\mathbf{a} \in \mathbf{D} : \mathbf{a} \geqslant \mathbf{b}\}$ *for some choice of* $\mathbf{b} > \mathbf{0}$.

(xii) $\{\mathbf{a} \in \mathbf{D} : \mathbf{a}$ *is minimal*$\}$.

5.17 Remarks. The study of automorphism bases for structures of degrees began with results of Lerman [1977] who studied bases for the recursively enumerable degrees. Automorphism bases for the lattice of recursively enumerable sets had previously been studied by Shore [1977], following up on a suggestion of Nerode. The results proved in this section come from Jockusch and Posner [1981].

5.18–5.21 Exercises.

5.18 Show that $\{\mathbf{d} \leqslant \mathbf{0}' : \mathbf{d}$ is 1-generic$\}$ generates $\mathbf{D}[\mathbf{0}, \mathbf{0}']$.

5.19 Show that for every $\mathbf{a} \in \mathbf{D}(\mathbf{0}, \mathbf{0}']$, $\{\mathbf{b} \leqslant \mathbf{0}' : \mathbf{b} \not\geqslant \mathbf{a}\}$ generates $\mathbf{D}[\mathbf{0}, \mathbf{0}']$.

5.20 Show that for all $\mathbf{a} \in \mathbf{L}_2$, $\{\mathbf{b} \leqslant \mathbf{0}' : \mathbf{b} \cap \mathbf{a} = \mathbf{0}\}$ generates $\mathbf{D}[\mathbf{0}, \mathbf{0}']$.

5.21 Show that for all $\mathbf{a} < \mathbf{0}'$, $\{\mathbf{b} \leqslant \mathbf{0}' : \mathbf{b} \cup \mathbf{a} = \mathbf{0}'\}$ generates $\mathbf{D}[\mathbf{0}, \mathbf{0}']$.