# *Chapter II*
# Embeddings and Extensions of Embeddings in the Degrees

We define the degrees of unsolvability in this chapter, and show that these degrees from an uppersemilattice. Much of the rest of this book will be devoted to studying this uppersemilattice. The study begins in this chapter, with sections on embedding theorems and on extensions of embeddings into the degrees. We also prove the decidability of a certain natural class of sentences about the degrees.

## *1. Uppersemilattice Structure for the Degrees*

We are now ready to define the degrees of unsolvability, and to show that Turing reducibility induces a partial ordering on these degrees which gives rise to an uppersemilattice. In Section 4 we will prove that the degrees do not form a lattice.

We begin with some algebraic definitions.

**1.1. Definition.** A *partially ordered set* (*poset*) $\langle P, \leqslant \rangle$ is a set $P$ together with a binary relation $\leqslant \subseteq P^2$ having the following properties:

(i) *Reflexivity*: $\forall x \in P(x \leqslant x)$.

(ii) *Antisymmetry*: $\forall x, y \in P(x \leqslant y \,\&\, y \leqslant x \rightarrow x = y)$.

(iii) *Transitivity*: $\forall x, y, z \in P(x \leqslant y \,\&\, y \leqslant z \rightarrow x \leqslant z)$.

**1.2 Definition.** An *uppersemilattice* (*usl*) is a triple $\langle P, \leqslant, \vee \rangle$ such that $\langle P, \leqslant \rangle$ is a poset, and $\vee: P^2 \rightarrow P$ (write $x \vee y = z$ for $\vee(x, y) = z$) satisfies:

(i) $$\forall x, y \in P(x \leqslant x \vee y \,\&\, y \leqslant x \vee y)$$

and

(ii) $$\forall x, y, u \in P(x \leqslant u \,\&\, y \leqslant u \rightarrow x \vee y \leqslant u).$$

Thus a usl is a poset in which every pair of elements has a least upper bound.

Clause (ii) of Definition 1.1 prevents the use of $\leqslant_T$ to directly transform $N^N$ into a poset. This obstruction is circumvented by using certain equivalence classes of $N^N$, *the degrees*, as the domain of the poset. The equivalence relation used is the following.

**1.3 Definition.** For $f, g \in N^N$, define $f \equiv_T g$ if $f \leqslant_T g$ and $g \leqslant_T f$.

We leave the proof of the fact that $\equiv_T$ is an equivalence relation to the reader (Exercises 1.11 and 1.12). $\equiv_T$ partitions $N^N$ into equivalence classes which are now defined.

**1.4 Definition.** Let $f \in N^N$ be given. The *degree* (*of unsolvability*) *of* $f$, $\mathbf{f}$, is $\{g \in N^N : g \equiv_T f\}$.

**1.5 Notation.** $\{\mathbf{f}: f \in N^N\}$ will henceforth be denoted by $\mathbf{D}$.

**1.6 Remark.** Since $|N^N| = 2^{\aleph_0}$ and for each $\mathbf{d} \in \mathbf{D}$, $|\mathbf{d}| = \aleph_0$, a simple computation in cardinal arithmetic shows that $|\mathbf{D}| = 2^{\aleph_0}$.

The next two definitions indicate the natural way in which usl structure is induced on $\mathbf{D}$.

**1.7 Definition.** Let $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ be given. We say that $\mathbf{a} \leqslant \mathbf{b}$ if

$$\forall f, g \in N^N (f \in \mathbf{a} \,\&\, g \in \mathbf{b} \rightarrow f \leqslant_T g).$$

We leave it to the reader (Exercise 1.13) to show that

$$\mathbf{a} \leqslant \mathbf{b} \Leftrightarrow \exists f, g \in N^N (f \in \mathbf{a} \,\&\, g \in \mathbf{b} \,\&\, f \leqslant_T g).$$

**1.8 Definition.** Let $\mathbf{a}, \mathbf{b} \in \mathbf{D}$, $f \in \mathbf{a}$ and $g \in \mathbf{b}$ be given. Define $\mathbf{a} \cup \mathbf{b}$ to be the degree of the function $f \oplus g \in N^N$ defined by

$$f \oplus g(x) = \begin{cases} f(x/2) & \text{if } x \text{ is even,} \\ g(x - 1/2) & \text{if } x \text{ is odd.} \end{cases}$$

Let $\mathscr{D} = \langle \mathbf{D}, \leqslant \rangle$ and $\mathscr{D}\mathscr{U} = \langle \mathbf{D}, \leqslant, \cup \rangle$. We leave it to the reader (Exercise 1.14) to verify that $\mathscr{D}$ is a poset and that $\mathscr{D}\mathscr{U}$ is a usl. Note that $\mathbf{D}$ has a smallest element, namely, the degree of the recursive functions (Exercise 1.16).

**1.9 Notation.** We will write $\mathbf{a} = \mathbf{b}$ for $\mathbf{a} \leqslant \mathbf{b}$ and $\mathbf{b} \leqslant \mathbf{a}$. $<, \geqslant, >, \neq$, etc. will have the obvious meaning. $\mathbf{0}$ will denote the smallest degree. $\cup\{\mathbf{a}_i : 1 \leqslant i \leqslant n\}$ will denote $\mathbf{a}_1 \cup \cdots \cup \mathbf{a}_n$, and $\cap\{\mathbf{a}_i : 1 \leqslant i \leqslant n\}$ will denote the greatest element $\mathbf{d} \in \mathbf{D}$ such that $\mathbf{d} \leqslant \mathbf{a}_i$ for $i = 1, 2, \ldots, n$ if such an element exists, and will be undefined otherwise.

The study of relative recursion, or equivalently, computation from oracles leads naturally to the study of the degrees. Questions about information contained in functions which can be computed from an $f$ oracle are best formulated in terms of the structure of the degrees below $f$. Hence the study of $\mathscr{D}$ will shed light on relative recursion.

Several algebraic and logical problems arise naturally in the study of $\mathscr{D}$. We would like to have a classification of the usls which can be embedded into $\mathscr{D}$, and to develop structure theory for $\mathscr{D}$. We would like to have answers to certain questions about the elementary theory of $\mathscr{D}$, e.g., "is the theory decidable?", and "how complicated is this theory?". Some of these questions have been answered, while a complete answer to the others still remains to be found. (Note that for the questions mentioned above, $\mathscr{D}$ and $\mathscr{D}\mathscr{U}$ are interchangeable.) These, and other questions will be studied in this book, a study which begins in the next section.

**1.10 Remark.** $\mathscr{D}$ was first defined and studied by Kleene and Post [1954]. This paper has an interesting history. Kleene received a letter from Post with some of the definitions and theorems, and suggested that Post publish those results. Post was reluctant to do so, feeling that some of the most important initial questions about the degrees had not yet been answered. Some of these questions were later answered by Kleene, who added his results to Post's and had the paper published. This was done while Post was terminally ill, and we do not know whether or not Post ever read the paper.

**1.11–1.17 Exercises**

***1.11** Show that $\leqslant_T$ is transitive.

***1.12** Show that $\equiv_T$ is an equivalence relation.

***1.13** Show that $\mathbf{a} \leqslant \mathbf{b} \Leftrightarrow \exists f, g \in N^N (f \in \mathbf{a} \,\&\, g \in \mathbf{b} \,\&\, f \leqslant_T g)$.

***1.14** Show that $\mathscr{D}\mathscr{U}$ is a usl.

***1.15** Show that every degree contains a set (i.e., a characteristic function).

***1.16** Show that for all degrees $\mathbf{a}, \mathbf{0} \leqslant \mathbf{a}$.

***1.17** Given $\{f_i\colon N \to N\colon i = 0, 1, \ldots, n-1\}$, define $\oplus_{i=0}^{n-1} f_i\colon N \to N$ by $(\oplus_{i=0}^{n-1} f_i)(nx + b) = f_b(x)$ where $0 \leqslant b < n$. Show that $\oplus_{i=0}^{n-1} f_i$ and $((\cdots((f_0 \oplus f_1) \oplus f_2) \oplus \cdots) \oplus f_{n-1})$ have the same degree.

## *2. Incomparable Degrees*

Embeddings into the degrees are considered in this section. Many constructions of classes of degrees with various properties can be carried out through the use of the method of forcing. We describe forcing in this section, and use it to construct incomparable degrees.

Rather than begin immediately with the abstract notion of forcing, we first give a classical proof of the existence of incomparable degrees. We next describe the relationship between this proof and the forcing proof. Forcing is then introduced, and is used to prove the same theorem.

**2.1 Definition.** Let $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ be given. Then $\mathbf{a}$ and $\mathbf{b}$ are *incomparable* (write $\mathbf{a} \mid \mathbf{b}$) if $\mathbf{a} \nleqslant \mathbf{b}$ and $\mathbf{b} \nleqslant \mathbf{a}$.

**2.2 Theorem.** *There exist* $\mathbf{a}_0, \mathbf{a}_1 \in \mathbf{D}$ *such that* $\mathbf{a}_0 \mid \mathbf{a}_1$.

*Proof.* We construct sets $A_0, A_1 \subseteq N$ such that $A_0 \nleqslant_T A_1, A_1 \nleqslant_T A_0$ and set $\mathbf{a}_i = \mathbf{A}_i$ for $i = 0, 1$. By the Enumeration Theorem, it suffices to satisfy the *requirements*

(1) $\quad P_{e,i}\colon \Phi_e^{A_i} \neq A_{1-i}$

for all $e \in N$ and $i \in \{0, 1\}$, where we say that $\Phi_e^{A_i} \neq A_{1-i}$ is *satisfied* if

(2) $\exists x \in N(\Phi_e^{A_i}(x)\downarrow \neq A_{1-i}(x)$ or $\Phi_e^{A_i}(x)\uparrow)$.

For $i = 0, 1$, we will define $A_i$ as the union of a finite sequence of elements of $\mathcal{S}_2$ (recall that $\mathcal{S}_2$ is the set of finite sequences of 0s and 1s) $\alpha_i^0 \subset \alpha_i^1 \subset \cdots$. (Since each string is a partial function and each set is identified with its characteristic function, there is no ambiguity in this definition of $A_i$.) We say that the requirement $\Phi_e^{A_i} \neq A_{1-i}$ is *satisfied by* $\langle \beta_0, \beta_1 \rangle$ if $\beta_0, \beta_1 \in \mathcal{S}_2$ and for some $x \in N$, either

(3) $\Phi_e^{\beta_i}(x)\downarrow \neq \beta_{1-i}(x)\downarrow$

or

(4) for all $\beta \in \mathcal{S}_2$ such that $\beta \supseteq \beta_i$, $\Phi_e^{\beta}(x)\uparrow$.

We first prove the following lemma.

**2.3 Lemma.** *Fix a requirement $P_{e,i}$, and let $\alpha_0, \alpha_1 \in \mathcal{S}_2$ be given. Then there are $\beta_0, \beta_1 \in \mathcal{S}_2$ such that $\beta_0 \supset \alpha_0$, $\beta_1 \supset \alpha_1$, and $P_{e,i}$ is satisfied by $\langle \beta_0, \beta_1 \rangle$.*

*Proof.* Fix $P_{e,i}$: $\Phi_e^{A_i} \neq A_{1-i}$. Fix $\alpha_0, \alpha_1 \in \mathcal{S}_2$ and let $x = \mathrm{lh}(\alpha_{1-i})$.

*Case 1.* $\Phi_e^{\beta}(x)\downarrow$ for some $\beta \in \mathcal{S}_2$ such that $\beta \supset \alpha_i$. Let $\beta_i$ be such a $\beta$. Define $\beta_{1-i}$ of length $x + 1$ by

$$\beta_{1-i}(y) = \begin{cases} \alpha_{1-i}(y) & \text{if } \; y < x, \\ 0 & \text{if } \; y = x \;\&\; \Phi_e^{\beta_i}(x)\downarrow \neq 0, \\ 1 & \text{if } \; y = x \;\&\; \Phi_e^{\beta_i}(x)\downarrow = 0. \end{cases}$$

It follows immediately from (3) that $P_{e,i}$ is satisfied by $\langle \beta_0, \beta_1 \rangle$.

*Case 2.* Otherwise. Then $\Phi_e^{\beta}(x)\uparrow$ for all $\beta \in \mathcal{S}_2$ such that $\beta \supseteq \alpha_i$. Fix $\beta_j \supset \alpha_j$ arbitrarily for $j = 0, 1$. It follows immediately from (4) that $P_{e,i}$ is satisfied by $\langle \beta_0, \beta_1 \rangle$. ▯

To prove the theorem, we now let $\{R_i : i \in N\}$ be a list of all requirements in $\{P_{e,i}: e \in N \;\&\; i \leqslant 1\}$. Set $\alpha_0^0 = \alpha_1^0 = \emptyset$. Given $\alpha_0^s, \alpha_1^s \in \mathcal{S}_2$, choose $\alpha_0^{s+1} \supset \alpha_0^s$ and $\alpha_1^{s+1} \supset \alpha_1^s$ as in Lemma 2.3 so that $R_s$ is satisfied by $\langle \alpha_0^{s+1}, \alpha_1^{s+1} \rangle$. Let $A_j = \bigcup\{\alpha_j^s : s \in N\}$ for $j = 0, 1$. Then $A_0, A_1 \subseteq N$, and every $R_i$ is satisfied. ▯

**2.4 Corollary.** *The degrees are not linearly ordered by* $\leqslant$.

Many constructions of classes of degrees with given properties, such as the construction of incomparable degrees, conform to the following pattern.

*Step 1.* Reduce the statement of the theorem to an equivalent infinite set of requirements on subsets of $N$. (For incomparable degrees, this is done in (1).)

*Step 2.* Define *satisfaction of requirements.* (For incomparable degrees, this is done in the first paragraph of the proof of Theorem 2.2.)

*Step 3.* Show how requirements can be satisfied while leaving infinitely much of the sets being constructed unspecified. (For incomparable degrees, this is done in the second paragraph of the proof of Theorem 2.2.)

*Step 4*. Show that any requirement can be satisfied by specifying a *little bit more* of the sets being constructed than has been specified at a given point in the construction. (For incomparable degrees, this is done in Lemma 2.3.)

*Step 5*. Show how to satisfy all requirements by applying Step 4 inductively. (For incomparable degrees, this is done after the proof of Lemma 2.3.)

The sequence of steps just described can be recast in the language of forcing. One can then prove general theorems about forcing, eliminating much of the repetition from proof to proof. In particular, Step 5 can be carried out in the context of forcing, enabling us to avoid repeating the inductive step in each proof. Forcing does, however, tend to obscure the intuition behind the constructions. The reader should be able to reconstruct this intuition by analyzing any forcing proof in terms of the above sequence of five steps.

Many of the theorems which will be proved using forcing were first proved before the invention of forcing and do not use the full power of forcing. We do not feel it advisable to introduce forcing in complete abstraction, i.e., to relate forcing to satisfaction in a very general setting. Rather than introduce a formal language, state all requirements as formulas of this language, and then define forcing syntactically for this language, we will define forcing only for those requirements which are needed to prove a given theorem. Occasional comments will be made to enable the reader already familiar with forcing to relate our approach to forcing in set theory.

We will begin our treatment of forcing with definitions of *notion of forcing*, *dense set* and *$\mathscr{C}$-generic set* where $\mathscr{C}$ is a collection of dense sets. If $G$ is a $\mathscr{C}$-generic set, then we will be able to recover the subsets of $N$ which we wanted to construct from $A_G = \cup G$. We note the relationship between the forcing approach and the steps previously outlined. Step 1 remains unchanged, and the change in Step 2 is just a change in terminology. Step 3 becomes the *Satisfaction Lemma* and Step 4 becomes the *Density Lemma*. Step 5 becomes the *Existence Theorem for $\mathscr{C}$-generic Sets*.

Forcing conditions for a set are meant to specify information about what the set looks like, e.g., whether or not certain numbers are in the set. Thus we write $q \leqslant p$ for *$q$ refines $p$*, saying that $q$ contains more information, hence less freedom, than $p$. Hence for $\sigma, \tau \in \mathscr{S}_2$, $\sigma \leqslant \tau$ will mean $\sigma \supseteq \tau$; for $\sigma$ specifies more of the final set $A$ than does $\tau$.

**2.5 Definition.** A *notion of forcing* is a partially ordered set $\langle F, \leqslant_F \rangle$ with a greatest element $1_F$. The elements of $F$ are called *conditions*. For $p, q \in F$, we say that *$p$ refines $q$* if $p \leqslant_F q$, *$p$ is compatible with $q$* if there is an $r \in F$ such that $r \leqslant_F p$ and $r \leqslant_F q$, and *$p$ is incompatible with $q$* if $p$ is not compatible with $q$. We write $p \mid q$ for $p$ is incompatible with $q$.

Consider the example where we take, as our set $F$ of forcing conditions, the set of all partial functions $\psi \colon N \to \{0, 1\}$, and order the conditions by extension, i.e., $\psi \leqslant_F \theta$ if and only if $\psi \supseteq \theta$. Let $\psi_0$ be the partial function with domain $\{0\}$ such that $\psi_0(0) = 0$, let $\psi_1$ be the partial function with domain $\{1\}$ such that $\psi_1(1) = 1$, let $\psi_2$ be the partial function with domain $\{0, 1\}$ such that $\psi_2(0) = \psi_2(1) = 0$, and let $\psi_3$ be the partial function with domain $\{0, 1\}$ such that $\psi_3(0) = 0$ and $\psi_3(1) = 1$. Then $\psi_0, \psi_1$, and $\psi_3$ are pairwise compatible since they have the common extension $\psi_3$. $\psi_0$ and $\psi_2$ have the common extension $\psi_2$, so they are compatible. But $\psi_2$ is incompatible with both $\psi_1$ and $\psi_3$ since $\psi_2(1) = 0 \neq 1 = \psi_1(1) = \psi_3(1)$, so $\psi_2$ cannot have a common extension with $\psi_1$ or $\psi_3$.

**2.6 Definition.** Let $\langle F, \leqslant_F \rangle$ be a notion of forcing. $E \subseteq F$ is *dense* if every condition in $F$ has a refinement in $E$.

Again consider the example where we take as our notion of forcing the set of finite partial functions $\psi: N \to \{0, 1\}$. An example of a dense set is $\{\psi: |\mathrm{dom}(\psi)|$ is even$\}$. For every finite partial function has an extension to one whose domain has even cardinality.

We next define the notion of $\mathscr{C}$-generic set. Such sets are used to naturally define subsets of $N$ satisfying a specified set of requirements. Each requirement will give rise to a dense set, the set of all conditions which *force* the requirement. If we let $\mathscr{C}$ be the collection of all such dense sets, then a $\mathscr{C}$-generic set is just a set of conditions with certain closure properties whose intersection with every $C \in \mathscr{C}$ is non-empty. Thus given a $\mathscr{C}$-generic set $G$ and a requirement $R$, we will have a condition $p \in G$ which forces the requirement $R$ to be satisfied.

**2.7 Definition.** Let $\langle F, \leqslant_F \rangle$ be a notion of forcing, let $G \subseteq F$ be given, and let $\mathscr{C}$ be a set of dense subsets of $F$. Then $G$ is said to be *$\mathscr{C}$-generic* if:

(i) $1_F \in G$.

(ii) $\forall p \in G \forall q \leqslant_F p (q \in G)$.

(iii) $\forall p, q \in G \exists r \in G (r \leqslant_F p \,\&\, r \leqslant_F q)$.

(iv) $\forall C \in \mathscr{C} (G \cap C \neq \emptyset)$.

Having defined $\mathscr{C}$-generic sets, we show that they exist.

**2.8 Existence Theorem for $\mathscr{C}$-generic Sets.** *Let $\langle F, \leqslant_F \rangle$ be a notion of forcing and let $p \in F$ be given. Let $\mathscr{C}$ be a countable set of dense subsets of $F$. Then there exists a $\mathscr{C}$-generic set $G$ such that $p \in G$.*

*Proof.* Let $\mathscr{C} = \{C_i: i \in N\}$. Let $q_0 = p$ and let $q_{s+1}$ be any refinement of $q_s$ in $C_s$. Let $G = \{r \in F: \exists s (q_s \leqslant_F r)\}$. It is easily verified that $G$ is $\mathscr{C}$-generic. ▯

It is useful to isolate requirements which make $\bigcup G$ total on $\bigcup\{\mathrm{dom}(p): p \in F\}$ when $F$ consists of partial functions. Such requirements are needed in all forcing constructions of this chapter.

**2.9 Existence Theorem for Total $\mathscr{C}$-generic Sets.** *Let $\langle F, \leqslant_F \rangle$ be a notion of forcing such that each $p \in F$ is a partial function. Assume that $X = \bigcup\{\mathrm{dom}(p): p \in F\}$ is countable, and that for all $p \in F$ and $x \in X$ there is a $q \leqslant_F p$ such that $q(x)\downarrow$. Let $\mathscr{C}$ be a countable set of dense subsets of $F$, and let $p \in P$ be given. Then there is a $\mathscr{C}$-generic set $G$ such that $p \in G$ and for all $x \in X$ there is a $q \in G$ such that $q(x)\downarrow$.*

*Proof.* Let $\mathscr{C} = \{C_i: i \in N\}$ and let $X = \{x_i: i \in N\}$. Let $q_0 = p$, let $q_{2s+1}$ be any refinement of $q_{2s}$ in $C_s$, and let $q_{2s+2}$ be any $r \leqslant_F q_{2s+1}$ such that $r(x_s)\downarrow$. Then $G = \{r \in F: \exists s (q_s \leqslant_F r)\}$ is the desired $\mathscr{C}$-generic set. ▯

Before introducing any specific notions of forcing, we give some notational conventions, and then state the lemmas which will have to be proved for each forcing construction. Below, $A_G$ will be a collection of sets naturally defined in terms of a $\mathscr{C}$-generic set $G$.

**2.10 Notation.** Let $R$ be a requirement. We write $A_G \models R$ if $R$ is satisfied by $A_G$ and $p \Vdash R$ if the condition $p$ forces $R$.

**2.11 Density Lemma.** *For each requirement* $R$, $C_R = \{p \in F: p \Vdash R\}$ *is a dense set.*

**2.12 Satisfaction Lemma.** *If* $G$ *is* $\mathscr{C}$*-generic and* $C_R \in \mathscr{C}$, *then* $A_G \models R$.

**2.13 Remark.** (This side remark is meant only for the reader who is familiar with forcing in set theory.) We indicate how our approach to forcing is an adaptation, to a simpler setting, of the set-theoretical approach. The reader should refer to Shoenfield [1971a] for a corresponding set-theoretical approach.

Although it is possible to do so, we do not fix a language and then treat forcing syntactically. Rather, our requirements are those *sentences* of the would-be language which we want to be satisfied (i.e., to hold in our *model* $A_G$), and we define *satisfaction* on an ad hoc basis to force the sets constructed to have the desired properties. (The ad hoc definition of satisfaction would coincide with the appropriate syntactical definition.) Once forcing is defined, we prove the Satisfaction Lemma relating forcing to satisfaction. The Satisfaction Lemma corresponds to Shoenfield's Truth Lemma, but is much easier to prove because of the ad hoc nature of our definition of satisfaction.

Since we are only interested in satisfying certain requirements, we need only make sure that our generic set meets the dense sets corresponding to those requirements, rather than every dense set definable in our base model. (This idea is also present in uses of Martin's Axiom (see Martin and Solovay [1970]).) Thus Shoenfield's Definability Lemma is replaced by our Density Lemma, each having, as its purpose, the proof that every appropriate dense set is met by every generic set. In our approach, it must be shown that the appropriate set is dense, while in the set-theoretical approach, the density of this set is shown independently of the particular notion of forcing, but the definability of the set is a problem.

Because of the special nature of our requirements, once a requirement is forced by a condition $p$, it is forced by all conditions $q$ refining $p$. If we examine the proof of Theorem 2.2, we see that this permanence property follows from the use property of the Enumeration Theorem. Hence Shoenfield's Extension Lemma becomes unnecessary, its content being absorbed into our Satisfaction Lemma.

We have already noted that we need only look at sets directly related to our generic set. Thus typically for this chapter, the desired sets will be of the form $A_G^{[i]}$, cross-sections of $A_G = \cup G$. This should be contrasted with set-theoretical forcing where we look at the model of all sets generated by $G$.

Our first application of the method of forcing is the use of finite forcing to prove Theorem 2.2. We take the notion of forcing to be $\langle \mathscr{S}_2^2, \supseteq \rangle$ i.e., the set of all ordered pairs of strings of 0s and 1s ordered coordinatewise by $\supseteq$. Thus given $\sigma_1, \tau_1, \sigma_2, \tau_2 \in \mathscr{S}_2$, we say that $\langle \sigma_1, \tau_1 \rangle \supseteq \langle \sigma_2, \tau_2 \rangle$ if $\sigma_1 \supseteq \sigma_2$ and $\tau_1 \supseteq \tau_2$. Note that the greatest element of $\mathscr{S}_2^2$ under this ordering is $\langle \emptyset, \emptyset \rangle$.

**2.14 Forcing Proof of Theorem 2.2.** Establish requirements as in (1). Satisfaction of requirements is defined as in (2). Fix a requirement $R$, say $\Phi_e^{A_i} \neq A_{1-i}$. We say that $\langle\beta_0,\beta_1\rangle \Vdash R$ if (3) or (4) holds. For each requirement $R$, let $C_R = \{\langle\beta_0,\beta_1\} \in \mathscr{S}_2^2 : \langle\beta_0,\beta_1\rangle \Vdash R\}$, and let $\mathscr{C} = \{C_R : R \text{ is a requirement}\}$. The Density Lemma is then Lemma 2.3. By the Existence Theorem for Total $\mathscr{C}$-generic Sets (Theorem 2.9), there is a $\mathscr{C}$-generic set $G$. Let $A_G = \cup G = \langle A_0, A_1\rangle \subseteq N^2$. The Satisfaction Lemma now follows from (1)–(4), the definition of $A_G$ and the Use Property of the Enumeration Theorem. (The paragraph following the proof of Lemma 2.3 is superfluous here. It merely repeats the proof of the existence of a $\mathscr{C}$-generic set.) ▯

Almost all theorems which we will prove about the degrees can be relativized. We now indicate how to relativize Theorem 2.2, but leave some of the details to the reader.

**2.15 Definition.** Let $S$ be any statement about the degrees, and let $\mathbf{d} \in \mathbf{D}$ be given. The *relativization of* $S$ *to* $\mathbf{d}$ is the assertion that $S$ is true about the degrees $\geqslant \mathbf{d}$.

**2.16 Remark.** We can easily modify 2.14 to obtain a proof of the relativization of Theorem 2.2 to any degree $\mathbf{d}$. This relativization states that for any degree $\mathbf{d}$, there are degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \geqslant \mathbf{d}$, $\mathbf{b} \geqslant \mathbf{d}$, and $\mathbf{a} \mid \mathbf{b}$. Fix a set $D \in \mathbf{d}$. The notion of forcing used to prove this relativization is $\{\langle\theta_0,\theta_1\rangle : \theta_0, \theta_1$ are partial functions with range $\subseteq \{0,1\}$, $\theta_0(2x)\downarrow = \theta_1(2x)\downarrow = D(x)$ for all $x \in N$, and $\mathrm{dom}(\theta_0)$ and $\mathrm{dom}(\theta_1)$ each has finite intersection with the odd numbers$\}$, ordered by $\supseteq$ defined coordinatewise. We leave it to the reader (Exercise 2.18) to carry out the proof. This notion of forcing is called *infinite-coinfinite pointed forcing.*

**2.17 Remarks.** Theorem 2.2 was proved by Kleene and Post [1954]. The concept of forcing is due to Cohen [1963], and the forcing we have done in this section is a simplified version of Cohen forcing. The reals constructed by Cohen also have incomparable Turing degrees. The connection between Cohen forcing and some prior constructions in Recursion Theory was made shortly after the invention of Cohen forcing by Gandy and Sacks independently. Feferman [1965] has developed the corresponding version of forcing for arithmetic, and more recently, Jockusch [1980] has studied the application of forcing to proving new theorems about the degrees. Some of Jockusch's work is discussed in Chap. IV.

**2.18–2.19 Exercises**

***2.18** Given $\mathbf{d} \in \mathbf{D}$, construct a pair of incomparable degrees above $\mathbf{d}$.

**2.19** Construct infinitely many degrees $\mathbf{d}_0, \mathbf{d}_1, \ldots$ such that for all $i, j \in N$, if $i \neq j$ then $\mathbf{d_i} \mid \mathbf{d_j}$.

## *3. Embeddings into the Degrees*

The main result of Section 2.2, Theorem 2.2, states that a certain poset can be embedded into $\mathscr{D}$. We investigate embeddings of other posets into $\mathscr{D}$ in this section. In particular, we show that any finite poset can be embedded into the degrees. This result will be used to show that a certain natural fragment of the elementary theory of $\mathscr{D}$ is decidable.

For the most part, the embeddings we will consider are poset or usl embeddings. Embeddings for such structures are now defined.

**3.1 Definition.** Let $\mathscr{U} = \langle U, \leqslant_U \rangle$ and $\mathscr{T} = \langle T, \leqslant_T \rangle$ be posets. A *poset embedding* of $\mathscr{U}$ into $\mathscr{T}$ is a one-one map $h: U \rightarrow T$ satisfying

$$\forall x, y \in U(x \leqslant_U y \leftrightarrow h(x) \leqslant_T h(y)).$$

**3.2 Definition.** Let $\mathscr{U} = \langle U, \leqslant_U, \vee_U \rangle$ and $\mathscr{T} = \langle T, \leqslant_T, \vee_T \rangle$ be usls. A *usl embedding* is a one-one map $h: U \rightarrow T$ such that $h$ is a poset embedding of $\langle U, \leqslant_U \rangle$ into $\langle T, \leqslant_T \rangle$ which satisfies

$$\forall x, y \in U(h(x \vee_U y) = h(x) \vee_T h(y)).$$

On occasion, we will talk about embeddings or isomorphisms for structures other than posets or usls. Such embeddings and isomorphisms are now defined.

**3.3 Definition.** Let

$$\mathscr{U} = \langle U, \{R_i\colon i \in I\}, \{f_j\colon j \in J\}, \{c_k\colon k \in K\} \rangle$$

and

$$\mathscr{T} = \langle T, \{Q_i\colon i \in I\}, \{g_j\colon j \in J\}, \{d_k\colon k \in K\} \rangle$$

be *similar structures*, i.e.

(i) $\forall i \in I \exists n \in N(R_i \subseteq U^n \,\&\, Q_i \subseteq T^n)$;

(ii) $\forall j \in J \exists n \in N(f_j\colon U^n \rightarrow U \,\&\, g_j\colon T^n \rightarrow T)$;

(iii) $\forall k \in K(c_k \in U \,\&\, d_k \in T)$.

A map $h: U \rightarrow T$ is said to be an *embedding* of $\mathscr{U}$ into $\mathscr{T}$ if $h$ is one-one and satisfies

(iv) $\forall i \in I \forall s_1, \ldots, s_n \in N(\langle s_1, \ldots, s_n \rangle \in R_i \leftrightarrow \langle h(s_1), \ldots, h(s_n) \rangle \in Q_i)$;

(v) $\forall j \in J \forall s_1, \ldots, s_n \in N(h(f_j(s_1, \ldots, s_n)) = g_j(h(s_1), \ldots, h(s_n)))$;

(vi) $\forall k \in K(h(c_k) = d_k)$.

If $h$ is an embedding mapping $\mathscr{U}$ onto $T$, then $h$ is said to be an *isomorphism* of $\mathscr{U}$ with $\mathscr{T}$. We write $\mathscr{U} \hookrightarrow \mathscr{T}$ if there is an embedding of $\mathscr{U}$ into $\mathscr{T}$, and $\mathscr{U} \simeq \mathscr{T}$ if $\mathscr{U}$ and $\mathscr{T}$ are isomorphic.

The embeddings of this section will be constructed from a collection of subsets of $N$ whose degrees form an independent set. This notion of independence is now defined.

**3.4 Definition.** A set of degrees $\{\mathbf{a_i}\colon i \in I\}$ is *independent* if for all finite subsets $J \subseteq I$ and all $i \in I - J$, $\mathbf{a_i} \nleqslant \cup\{\mathbf{a_j}\colon j \in J\}$.

We will construct $A \subseteq N^2$ such that $\{A^{[i]}: i \in N\}$ is a collection of sets whose degrees form an independent set. It will be convenient to have notation for finite disjoint unions of such sets.

**3.5 Definition.** Let $F \subseteq A$ and $\theta: A \times B \to C$ be given. Let $\{n_i: i < |F|\}$ be an enumeration of the elements of $F$ in order of magnitude. Define $\theta^{[F]}: |F| \times B \to C$ by $\theta^{[F]}(i, x) = \theta(n_i, x)$. If $j \in A$ and $F = A - \{j\}$, then we write $\theta^{[\hat{j}]}$ for $\theta^{[F]}$.

**3.6 Theorem.** *There is a countable set of independent degrees.*

*Proof.* We will use forcing to prove this theorem. The domain of the notion of forcing used is $F = \{\theta \subseteq N^2: \text{dom}(\theta) \text{ is finite}\}$. We let $F$ be ordered by $\supseteq$ where $\theta \supseteq \psi$ if $\theta^{[i]} \supseteq \psi^{[i]}$ for all $i \in N$.

Given a class $\mathscr{C}$ of dense sets and a $\mathscr{C}$-generic set $G$, let $A_G = \cup G = A$, and let $A_i = A_G^{[i]}$. Then $\{A_i: i \in N\}$ will be the collection of sets whose degrees form an independent set.

By the Existence Theorem for Total $\mathscr{C}$-generic Sets, it suffices to have $A$ satisfy the following requirements for all $e, i \in N$:

$$(1) \qquad R(e, i): \Phi_e^{A^{[\hat{i}]}} \neq A_i.$$

Given $\theta \in F$, the forcing definition is given by

$$(2) \qquad \theta \Vdash R(e, i) \leftrightarrow \exists x \in N \exists \sigma \in \mathscr{S}_2((\sigma \subseteq \theta^{[\hat{i}]} \,\&\, \Phi_e^\sigma(x)\downarrow \neq \theta^{[i]}(x)\downarrow)$$
$$\text{or } \exists x \in N\, \forall \sigma \in \mathscr{S}_2(\theta^{[\hat{i}]} \subseteq \sigma \to \Phi_e^\sigma(x)\uparrow)).$$

(Note that $(N \times N)^{[\hat{i}]}$ is a space, so can be identified with $N$, and hence we are permitted to make the above identification of $\theta^{[\hat{i}]}$ with a partial function of one variable.)

The Satisfaction Lemma follows easily from the Use Property of the Enumeration Theorem. It thus suffices to verify the Density Lemma, i.e., to show that for each requirement $R$ as in (1), $C_R = \{\theta \in F: \theta \Vdash R\}$ is dense. The desired sets $\{A_i: i \in N\}$ can then be recovered as above from any $\mathscr{C}$-generic set $G$, where $\mathscr{C} = \{C_R: R \text{ is a requirement}\}$.

Fix $e, i \in N$. Let $R = R(e, i)$ and let $C_R = \{\theta \in F: \theta \models R\}$. Let $\psi \in F$ be given. Fix the least $x \in N$ such that $\psi^{[i]}(x)\uparrow$. If there are $\theta \in F$ and $\sigma \in \mathscr{S}_2$ such that $\theta \supseteq \psi$, $\sigma \subseteq \theta^{[\hat{i}]}$, and $\Phi_e^\sigma(x)\downarrow$, fix such a $\theta$ for which $\theta^{[i]}(x)\downarrow$ and $\Phi_e^\sigma(x) \neq \theta^{[i]}(x)$. Such a $\theta$ will exist as $\psi^{[i]}(x)\uparrow$. It then follows from (2) that $\theta \in C_R$. If no such $\theta$ exists, then it follows from the second disjunct of (2) that $\psi \Vdash R$, so $\psi \in C_R$. In either case, $\psi$ has a refinement in $C_R$, so $C_R$ is dense. ∎

The set $\{\mathbf{a_i}: i \in N\}$ of independent degrees constructed in Theorem 3.6 has an even stronger independence property. For if we let $\hat{\mathbf{a}}_\mathbf{i}$ be the degree of $A^{[\hat{i}]}$, then we have shown that $\mathbf{a_i} \not\leq \hat{\mathbf{a}}_\mathbf{i}$. This fact is used in Exercise 3.14.

We now characterize the finite lattices which can be embedded into $\mathscr{D}$.

**3.7 Corollary.** *Let $\mathscr{U} = \langle U, \leqslant \rangle$ be a finite poset. Then $\mathscr{U} \hookrightarrow \mathscr{D}$.*

*Proof.* Let $U = \{u_i\colon i < n\}$. By Theorem 3.6, we can fix a set $\{\mathbf{a_i}\colon i < n\}$ of independent degrees. For each $i < n$, let $A_i$ be a set of degree $\mathbf{a_i}$. Let $A \subseteq [0, n) \times N$ be given such that $A^{[i]} = A_i$ for all $i < n$. For each $k < n$, let $F(k) = \{i\colon u_i \leqslant u_k\}$, $B_k = A^{[F(k)]}$, and let $B_k$ have degree $\mathbf{b_k}$. It is easily verified that

(3) $\forall j, k < n(u_j \leqslant u_k \rightarrow A_j \leqslant_T B_k)$.

Let $g\colon U \rightarrow \mathbf{D}$ be defined by $g(u_k) = \mathbf{b_k}$ for every $k < n$. To verify that $g\colon \mathscr{U} \hookrightarrow \mathscr{D}$, we must show that for all $i, j < n$, $u_j \leqslant u_k \Leftrightarrow B_j \leqslant_T B_k$.

First assume that $u_j \leqslant u_k$. Then $F(j) \subseteq F(k)$ so for all $i \in F(j)$ there are $m, r < n$ such that

$$A^{[i]} = (A^{[F(j)]})^{[m]} = (A^{[F(k)]})^{[r]}.$$

Hence $B_j \leqslant_T B_k$.

Conversely, suppose that $B_j \leqslant_T B_k$. We assume that $u_j \nleqslant u_k$ and obtain a contradiction to complete the proof of the corollary. It follows from (3) that

$$A_j \leqslant_T B_j \leqslant_T B_k.$$

Hence

$$\mathbf{a_j} \leqslant \mathbf{b_k} = \cup\{\mathbf{a_i}\colon u_i \leqslant u_k\}$$

contradicting the choice of $\{\mathbf{a_i}\colon i < n\}$ as a set of independent degrees. ▯

Corollary 3.7 will be used to show that a natural class of sentences about $\mathscr{D}$ is decidable. We first need some definitions.

**3.8 Definition.** Let $\mathscr{L}$ be the language of the pure predicate calculus with one additional binary symbol, $\leqslant$. A formula of $\mathscr{L}$ is an $\exists_0$ formula if it contains no quantifiers. For all $n \geqslant 0$, a formula of $\mathscr{L}$ is an $\forall_n$ formula if its negation is logically equivalent to an $\exists_n$ formula. And for all $n \geqslant 0$, a formula of $\mathscr{L}$ is an $\exists_{n+1}$ formula if it is of the form $\exists x_1, \ldots, x_k(A(x_1, \ldots, x_k))$ where $A(x_1, \ldots, x_k)$ is an $\forall_n$ formula.

**3.9 Definition.** Th($\mathscr{D}$), the *elementary theory of* $\mathscr{D}$, is the collection of all sentences of $\mathscr{L}$ which are true in $\mathscr{D}$.

**3.10 Definition.** A set of sentences in a language is *decidable* if that set of sentences is a recursive set. (Note that the set of all finite sequences of symbols in a countable language is a space.)

We now show that a natural class of sentences of $\mathscr{L}$ is true about $\mathscr{D}$.

**3.11. Corollary.** Th($\mathscr{D}$) $\cap\, \exists_1$ *is decidable.*

*Proof.* An $\exists_1$ sentence of $\mathscr{L}$ asserts the existence of finitely many (not necessarily distinct) elements $a_0, \ldots, a_n$ such that for any $i, j \leqslant n$, either $a_i \leqslant a_j$ is specified, or $a_i \nleqslant a_j$ is specified, or neither of these formulas in specified. Such a sentence is potentially true if there is a poset $\mathscr{U} = \langle U, \leqslant\rangle$ having at most $n + 1$ elements and an assignment of the variables of the sentence to the elements of $U$ which makes the sentence true in $\mathscr{U}$. There are only finitely many possible choices for $\mathscr{U}$ (up to isomorphism), and for each such choice, only finitely many possible assignments of variables. Furthermore, a list of all possible posets and assignments can effectively be given from the number of variables in the sentence. Hence the class of potentially true $\exists_1$ sentences of $\mathscr{L}$ is decidable. By Corollary 3.7, the potentially true $\exists_1$ sentences are exactly the $\exists_1$ sentences of $\mathscr{L}$ true in $\mathscr{D}$. ▯

**3.12 Remarks and Further Results.** Theorem 3.6 and Corollary 3.7 were first proved by Kleene and Post [1954]. In fact, they proved that any countable linearly ordered set can be embedded into $\mathscr{D}$. We leave this result to the reader to prove (Exercise 3.14). Sacks [1961a] considered embeddings of uncountable posets into $\mathscr{D}$ and proved the following theorems:

(S1) Let $\mathscr{U} = \langle U, \leqslant \rangle$ be a poset such that $|U| \leqslant \aleph_1$. Then $\mathscr{U} \hookrightarrow \mathscr{D} \Leftrightarrow$ each member of $U$ has only countably many predecessors.

(S2) Let $\mathscr{U} = \langle U, \leqslant \rangle$ be a poset such that $|U| \leqslant 2^{\aleph_0}$ and each member of $U$ has at most $\aleph_1$ successors. Then $\mathscr{U} \hookrightarrow \mathscr{D} \Leftrightarrow$ each member of $U$ has only countably many predecessors.

(S3) Let $\mathscr{U} = \langle U, \leqslant \rangle$ be a poset such that $|U| \leqslant 2^{\aleph_0}$. If each member of $U$ has only finitely many predecessors then $\mathscr{U} \hookrightarrow \mathscr{D}$.

Sacks' results completely solve the embedding problem under the assumption of the continuum hypothesis, namely, $2^{\aleph_0} = \aleph_1$. Groszek and Slaman [1983] have constructed a model of Set Theory containing a poset of cardinality the continuum each of whose elements has at most countably many predecessors, such that the poset is not embeddable into $\mathscr{D}$. Sacks' methods are similar to those discussed in the next section, which deals with extension theorems.

**3.13–3.16 Exercises**

**3.13** Prove the relativization of Theorem 3.6 to any degree **d**.

**3.14** Show that any countable poset can be embedded into $\mathscr{D}$. (*Hint*: Choose an appropriate **d** and apply Exercise 3.13.)

**3.15** (Sacks [1961a]) Show that there exists a set **S** of independent degrees such that $|\mathbf{S}| = 2^{\aleph_0}$. (*Hint*: Take as the domain of the notion of forcing the functions $T\colon \mathscr{S}_2 \to \mathscr{S}_2$ whose domain is $\{\sigma \in \mathscr{S}_2\colon \mathrm{lh}(\sigma) \leqslant i\}$ for some $i \in N$ and such that $\forall \sigma, \tau \in \mathrm{dom}(T)((\sigma \subseteq \tau \to T(\sigma) \subseteq T(\tau))\ \&\ (\sigma | \tau \to T(\sigma) | T(\tau)))$. This set is partially ordered by $\supseteq$, i.e., $S$ refines $T$ if $S$ extends $T$ as a partial function. The sets whose degrees form an independent set will be $\{A_S\colon S \subseteq N\}$, where $A_S = \bigcup\{T(\sigma)\colon \sigma \subset S\}$.

For each $e \in N$, $\sigma \in \mathscr{S}_2$, and subset $F = \{\tau_1, \ldots, \tau_k\}$ of $\{\tau \in \mathscr{S}_2\colon \mathrm{lh}(\tau) = \mathrm{lh}(\sigma)\ \&\ \tau \neq \sigma\}$, establish the requirement

$$R_{e,\sigma,F}\colon \forall A_1, \ldots, A_k, B, C(\sigma \subseteq B\ \&\ \tau_1 \subseteq A_1\ \&\ \cdots\ \&\ \tau_k \subseteq A_k\ \&$$
$$C = \oplus\{A_i\colon 1 \leqslant i \leqslant k\} \to \Phi_e^C \neq B).$$

Show that it suffices to satisfy all such requirements. Define forcing for requirements and prove that the Density and Satisfaction Lemmas are true.)

**3.16** Prove (S3). (*Hint*: Use the independent degrees from Exercise 3.15 to define the embedding.)

# *4. Extensions of Embeddings into the Degrees*

More information about the structure of $\mathscr{D}$ is extracted from the extension theorems proved in this section. In particular, we show that there is no greatest lower bound operation which would transform $\mathscr{D}\mathscr{U}$ into a lattice. Some results of this section are used in later chapters to obtain information about the decidability of classes of sentences of Th($\mathscr{D}$).

**4.1 Definition.** Let $\mathscr{L} = \langle L, \leqslant_L \rangle$ and $\mathscr{M} = \langle M, \leqslant_M \rangle$ be posets. We say that $\mathscr{L}$ is a *subposet* of $\mathscr{M}$ (write $\mathscr{L} \subseteq \mathscr{M}$) if $L \subseteq M$ and for all $a, b \in L$, $a \leqslant_L b \leftrightarrow a \leqslant_M b$.

All theorems proved in this section are extension theorems, that is, they have the following form. We start with a poset $\mathscr{M} = \langle M, \leqslant_M \rangle$, a subposet $\mathscr{L} = \langle L, \leqslant_L \rangle$ of $\mathscr{M}$, and an embedding $f$: $\mathscr{L} \hookrightarrow \mathscr{D}$. We then extend $f$ to an embedding $g$: $\mathscr{M} \hookrightarrow \mathscr{D}$ (i.e., for all $x \in L$, $g(x) = f(x)$). The theorems thus assert that the following diagram commutes:

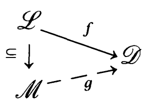


Fig. 4.1

**4.2 Definition.** Let $\mathscr{M} = \langle M, \leqslant_M \rangle$ be a poset. A *chain* of $\mathscr{M}$ is a subset $C$ of $M$ such that any two elements of $C$ are *comparable*, i.e., if $a, b \in C$, then $a \leqslant_M b$ or $b \leqslant_M a$. An *antichain* of $\mathscr{M}$ is a subset $A$ of $M$ such that any two elements of $A$ are *incomparable* (i.e., not comparable). A *maximal chain* (*antichain* resp.) $C$ of $\mathscr{M}$ is one which is not contained in a strictly larger chain (antichain resp.) of $\mathscr{M}$.

We begin by studying the sizes of maximal chains and antichains of $\mathscr{D}$.

**4.3 Theorem.** *Let* $\mathbf{C}$ *be a countable chain of* $\mathscr{D}$. *Then there is a* $\mathbf{d} \in \mathbf{D}$ *such that* $\mathbf{d} > \mathbf{c}$ *for all* $\mathbf{c} \in \mathbf{C}$.

*Proof.* Let $\mathbf{C} = \{\mathbf{c_i}: i \in N\}$ and let $C_i$ be a set of degree $\mathbf{c_i}$ for all $i \in N$. Define $B \subseteq N^2$ by letting $B^{[i]} = C_i$ for all $i \in N$. Then $C_i \leqslant_T B$ for all $i \in N$. Since every set recursive in $B$ is of the form $\Phi_e^B$ for some $e \in N$, there are only countably many such sets. As there are continuum many subsets of $N$, there must be a set $D \subseteq N$ which is not recursive in $B$. Let $\mathbf{d}$ be the degree of $B \oplus D$. (A degree $\mathbf{d}$ satisfying the conclusion of the theorem can also be obtained from Remark 2.16.) ▯

**4.4 Corollary.** *Every maximal chain of degrees has cardinality* $\aleph_1$.

*Proof.* By Theorem 4.3, every maximal chain of $\mathscr{D}$ has cardinality $\geqslant \aleph_1$. As in the proof of Theorem 4.3, we note that every degree has only countably many predecessors, hence no chain of degrees can have cardinality $> \aleph_1$. ▯

Having determined the size of maximal chains of $\mathscr{D}$, we turn our attention to maximal antichains of $\mathscr{D}$. Note that $\{\mathbf{0}\}$ is an antichain of $\mathscr{D}$ which we call the *trivial* antichain. The following theorem is the key result for extending antichains.

**4.5 Theorem.** *Let* $\mathbf{A}$ *be a countable non-trivial antichain of* $\mathscr{D}$. *Then there is a* $\mathbf{b} \in \mathbf{D}$ *such that* $\mathbf{A} \cup \{\mathbf{b}\}$ *is an antichain of* $\mathscr{D}$ *and* $\mathbf{b} \notin \mathbf{A}$.

*Proof.* Let $\mathbf{A} = \{\mathbf{a_i}: i \in I \subseteq N\}$ be a non-trivial countable antichain of $\mathscr{D}$. For each $i \in I$, fix a set $A_i$ of degree $\mathbf{a_i}$. We construct a set $B$ of degree $\mathbf{b}$ such that $\mathbf{b} \notin \mathbf{A}$ and $\mathbf{A} \cup \{\mathbf{b}\}$ is an antichain of $\mathscr{D}$. If $\mathbf{A} = \emptyset$, choose $\mathbf{b}$ to be any non-zero degree. Otherwise, it suffices to have $B$ satisfy the following requirements for all $i \in I$ and $e \in N$:

$$Q_{e,i}\colon \Phi_e^{A_i} \neq B;$$

$$R_{e,i}\colon \Phi_e^B \neq A_i.$$

We use $\langle \mathscr{S}_2, \supseteq \rangle$ as our notion of forcing. We say that $\sigma \Vdash Q_{e,i}$ if one of the following conditions holds:

$$(1) \qquad \exists x(\Phi_e^{A_i}(x)\uparrow);$$

$$(2) \qquad \exists x(\Phi_e^{A_i}(x)\downarrow \neq \sigma(x)\downarrow).$$

We say that $\sigma \Vdash R_{e,i}$ if one of the following conditions holds:

$$(3) \qquad \exists x \forall \tau \supseteq \sigma(\Phi_e^\tau(x)\uparrow);$$

$$(4) \qquad \exists x(\Phi_e^\sigma(x)\downarrow \neq A_i(x)).$$

For each requirement $R$ as above, let $C_R = \{\sigma \in \mathscr{S}_2\colon \sigma \Vdash R\}$ and let $\mathscr{C} = \{C_R\colon R = Q_{e,i}$ or $R = R_{e,i}$ for some $i \in I$ and $e \in N\}$. We first prove the Density Lemma. Fix $C_R \in \mathscr{C}$. Suppose that $R = Q_{e,i}$ for some $i \in I$ and $e \in N$. Fix $\sigma \in \mathscr{S}_2$ and the least $x$ such that $\sigma(x)\uparrow$. If $\Phi_e^{A_i}(x)\uparrow$, then by (1), $\sigma \Vdash Q_{e,i}$. Otherwise, $\Phi_e^{A_i}(x)\downarrow$, in which case we define $\tau \supset \sigma$ such that $\mathrm{lh}(\tau) = \mathrm{lh}(\sigma) + 1$ and $\Phi_e^{A_i}(x) \neq \tau(x)$. By (2), $\tau \Vdash Q_{e,i}$. Hence $C_R$ is dense. Next suppose that $R = R_{e,i}$ for some $i \in I$ and $e \in N$. Fix $\sigma \in \mathscr{S}_2$. We may assume that

$$(5) \qquad \forall x \exists \tau \supseteq \sigma(\Phi_e^\tau(x)\downarrow)$$

else by (3), it is immediate that $\sigma \Vdash R_{e,i}$. If there is a $\tau \supseteq \sigma$ such that

$$(6) \qquad \exists x(\Phi_e^\tau(x)\downarrow \neq A_i(x))$$

then by (4), $\tau \Vdash R_{e,i}$. Suppose that no $\tau$ satisfying (6) exists. We obtain a contradiction by showing that $A_i$ is recursive. To compute $A_i(x)$, search for $\tau \supseteq \sigma$ such that $\Phi_e^\tau(x)\downarrow$. Such a $\tau$ must exist by (5). But then by the assumed falsity of (6), $\Phi_e^\tau(x) = A_i(x)$. Since this procedure is recursive, we have the desired contradiction.

Since $C_R$ is dense for every requirement $R$, we may fix a $\mathscr{C}$-generic set $G$. Let $B = \cup G$. The Satisfaction Lemma is easily verified. ▯

**4.6 Corollary.** *Let* $\mathbf{A}$ *be a non-trivial maximal antichain of* $\mathscr{D}$. *Then* $|\mathbf{A}| \geqslant \aleph_1$. *(In fact,* $|\mathbf{A}| = 2^{\aleph_0}$.*)*

*Proof.* Let $\mathbf{A}$ be a non-trivial maximal antichain of $\mathscr{D}$. It is immediate from Theorem 4.5 that $\mathbf{A}$ is uncountable, so $|\mathbf{A}| \geqslant \aleph_1$. [In Chap. V.2, we show that there is a set $\mathbf{M}$ which consists of $2^{\aleph_0}$ minimal degrees. (A degree $\mathbf{d}$ is *minimal* if $\mathbf{d} \neq \mathbf{0}$ and $(\mathbf{0}, \mathbf{d}) = \emptyset$.) Since every degree has only countably many predecessors, if $|\mathbf{A}| < 2^{\aleph_0}$ then there must be a degree $\mathbf{d} \in \mathbf{M}$ such that $\mathbf{d} \nleqslant \mathbf{a}$ for all $\mathbf{a} \in \mathbf{A}$. Fix such a degree $\mathbf{d}$. Since $\mathbf{d}$ is minimal, $\mathbf{d} \mid \mathbf{a}$ for every $\mathbf{a} \in A$. Hence $\mathbf{A} \cup \{\mathbf{d}\}$ is an antichain properly extending $\mathbf{A}$, contradicting the maximality of $\mathbf{A}$.] ▯

The proof of Theorem 4.5 can be modified to show that every maximal independent set of degrees is uncountable. We leave the proof of this fact to the reader (Exercise 4.13). After proving this fact, Sacks [1961a] asked whether every maximal independent set of degrees has cardinality $2^{\aleph_0}$. Groszek and Slaman [1983] have shown that the answer is dependent on the model of Set Theory chosen.

The next theorem will be used to show that the degrees do not form a lattice. It is also used in Chap. VIII.3 to help determine the degree of Th($\mathscr{D}$). We first need a definition.

**4.7 Definition.** Let $\mathscr{U} = \langle U, \leqslant, \vee \rangle$ be a usl. An *ideal* of $\mathscr{U}$ is a subset $I \neq \emptyset$ of $U$ which satisfies:

(i) $\forall a, b \in U(a \leqslant b \,\&\, b \in I \rightarrow a \in I)$.

(ii) $\forall a, b \in I(a \vee b \in I)$.

**4.8 Exact Pair Theorem.** *Let* $\mathbf{I}$ *be a countable ideal of* $\mathscr{D}$. *Then there are* $\mathbf{a_0}, \mathbf{a_1} \in \mathbf{D}$ *such that for all* $\mathbf{c} \in \mathbf{D}$

$$\mathbf{c} \in \mathbf{I} \leftrightarrow \mathbf{c} \leqslant \mathbf{a_0} \,\&\, \mathbf{c} \leqslant \mathbf{a_1}.$$

$\{\mathbf{a_0}, \mathbf{a_1}\}$ *is called an exact pair for the ideal* $\mathbf{I}$.

*Proof.* (It will follow from the proof of Corollary 4.10 that we cannot always replace an exact pair by a single degree in the conclusion of this theorem.) Let $\mathbf{I} = \{\mathbf{c_i} : i \in N\}$ be a countable ideal of $\mathscr{D}$. For each $i \in N$, fix a set $C_i$ of degree $\mathbf{c_i}$. We construct sets $A_j \subseteq N^2$ for $j = 0, 1$ and let $\mathbf{a_j}$ be the degree of $A_j$. (Since $N^2$ is a space, we recursively identify $N^2$ with $N$ and occasionally treat $A_j$ as a subset of $N$.) It suffices to show that $A_0$ and $A_1$ satisfy the following requirements for all $e, k \in N$ and $j = 0, 1$:

$Q_{e,j}$: $C_e \leqslant_T A_j$.

$R_{e,k}$: If $\Phi_e^{A_0} = \Phi_k^{A_1}$ and both are total, then $\Phi_e^{A_0} \leqslant_T C_i$ for some $i \in N$.

The notion of forcing which we use here has domain $F$, where $F$ is the subset of $\{\langle \theta_0, \theta_1 \rangle : \theta_j : N^2 \rightarrow N\}$ which satisfies conditions (7)–(9) below for $j = 0, 1$. $F$ is ordered by $\supseteq$, which is coordinatewise extension for partial functions.

(7) $\{i: \text{dom}(\theta_j^{[i]}) \neq \emptyset\}$ is finite.

(8) $\forall i \in N(\text{dom}(\theta_j^{[i]})$ is either finite or is equal to $N)$.

(9) $\forall i \in N(\text{dom}(\theta_j^{[i]}) = N \to \{x: \theta_j^{[i]}(x) \neq C_i(x)\}$ is finite).

We say that $\langle \theta_0, \theta_1 \rangle \Vdash Q_{e,j}$ if

(10) $\{x: \theta_j^{[e]}(x)\downarrow\} = N \,\&\, \{x: \theta_j^{[e]}(x) \neq C_e(x)\}$ is finite.

(Although (9) and (10) seem to serve the same purpose, both are necessary. Without (9), it would not be possible to prove the Density Lemma for the requirements $Q_{e,j}$. And without (10), $Q_{e,j}$ might fail to be satisfied, as a generic set could be built from conditions each of which forces only finitely much of $A_j^{[i]}$, hence (9) could fail with $A$ in place of $\theta$.) We say that $\langle \theta_0, \theta_1 \rangle \Vdash R_{e,k}$ if one of the following conditions holds:

(11) $\exists x \in N(\Phi_e^{\theta_0}(x)\downarrow \neq \Phi_k^{\theta_1}(x)\downarrow)$.

(12) $\forall \langle \xi_0, \xi_1 \rangle \in F(\langle \xi_0, \xi_1 \rangle \supseteq \langle \theta_0, \theta_1 \rangle \to$ (11) fails for $\langle \xi_0, \xi_1 \rangle$ in place of $\langle \theta_0, \theta_1 \rangle)$.

For each requirement $R \in \{Q_{e,j}: e \in N \,\&\, j = 0, 1\} \cup \{R_{e,k}: e, k \in N\}$, let $C_R = \{\langle \theta_0, \theta_1 \rangle \in F: \langle \theta_0, \theta_1 \rangle \Vdash R\}$, and let $\mathscr{C}$ be the set of all such sets $C_R$. We first prove the Density Lemma. Suppose first that $R = Q_{e,j}$ for some $e \in N$ and $j = 0, 1$, and let $\langle \theta_0, \theta_1 \rangle \in F$ be given. By (9), if $\text{dom}(\theta_j^{[e]}) = N$ then $\langle \theta_0, \theta_1 \rangle \Vdash R$. Otherwise, by (8), $\text{dom}(\theta_j^{[e]})$ is finite, so we can define $\langle \xi_0, \xi_1 \rangle \supseteq \langle \theta_0, \theta_1 \rangle$ such that $\langle \xi_0, \xi_1 \rangle \in F$, $\text{dom}(\xi_j^{[e]}) = N$, and $\{x: \xi_j^{[e]}(x) \neq C_e(x)\}$ is finite. By (10), $\langle \xi_0, \xi_1 \rangle \Vdash R$, so $C_R$ is dense. Next suppose that $R = R_{e,k}$ for some $e, k \in N$, and let $\langle \theta_0, \theta_1 \rangle \in F$ be given. If there is no $\langle \xi_0, \xi_1 \rangle \in F$ such that $\langle \xi_0, \xi_1 \rangle \supseteq \langle \theta_0, \theta_1 \rangle$ and (11) holds for $\langle \xi_0, \xi_1 \rangle$ in place of $\langle \theta_0, \theta_1 \rangle$, then (12) holds. Hence there is a $\langle \xi_0, \xi_1 \rangle \supseteq \langle \theta_0, \theta_1 \rangle$ such that $\langle \xi_0, \xi_1 \rangle \Vdash R$, so again $C_R$ is seen to be dense.

By the Density Lemma, there exists a $\mathscr{C}$-generic set $G$. Let $\langle A_0, A_1 \rangle = \cup G$. We complete the proof of the theorem by verifying the Satisfaction Lemma. The satisfaction of $Q_{e,j}$ for $e \in N$ and $j = 0, 1$ follows immediately from (10). Fix $e, k \in N$, and assume that $\Phi_e^{A_0} = \Phi_k^{A_1}$ and both are total. Then by the Enumeration Theorem, there is a $\langle \theta_0, \theta_1 \rangle \in F$ which satisfies (12) such that $\langle \theta_0, \theta_1 \rangle \subseteq \langle A_0, A_1 \rangle$. Hence

(13) $\forall \sigma, \tau \in \mathscr{S}_2 \, \forall x \in N(\sigma$ compatible with $\theta_0 \,\&\, \tau$ compatible with $\theta_1 \,\&$ $\Phi_e^{\sigma}(x)\downarrow \,\&\, \Phi_k^{\tau}(x)\downarrow \to \Phi_e^{\sigma}(x) = \Phi_k^{\tau}(x) = \Phi_e^{A_0}(x))$.

Given $x \in N$, $\Phi_e^{A_0}(x)$ is computed as follows. Search for $\sigma$ compatible with $\theta_0$ such that $\Phi_e^{\sigma}(x)\downarrow$. $\sigma$ will exist since $\Phi_e^{A_0}$ is total. Then $\Phi_e^{A_0}(x) = \Phi_e^{\sigma}(x)$, and $\sigma$ can be found recursively from any oracle which can decide whether a given string $\tau$ is compatible with $\theta_0$. By (7)–(9), the $C$ oracle is such an oracle, where $C = \oplus\{C_i: \text{dom}(\theta_0^{[i]}) = N\}$. Since $\mathbf{I}$ is an ideal, $\mathbf{C} \in \mathbf{I}$, so there is an $r \in N$ such that $C \equiv_T C_r$. Hence $\Phi_e^{A_0} \leqslant_T C_r$, so $R_{e,k}$ is satisfied. ∎

**4.9 Definition.** A *lattice* $\langle L, \leqslant, \vee, \wedge\rangle$ is a usl $\langle L, \leqslant, \vee\rangle$ together with a function $\wedge$: $L^2 \to L$ (write $x \wedge y = z$ for $\wedge(x,y) = z$) satisfying:

(i) $\forall x, y \in L(x \wedge y \leqslant x \,\&\, x \wedge y \leqslant y)$;

(ii) $\forall x, y, z \in L(z \leqslant x \,\&\, z \leqslant y \to z \leqslant x \wedge y)$;

(i.e., every pair of elements of $L$ has a greatest lower bound under $\leqslant$).

**4.10 Corollary.** *$\mathscr{D}$ is not a lattice.*

*Proof.* Iterating Theorem 4.3, we get a set of degrees $\{\mathbf{d_i}: i \in N\}$ such that for all $i, j \in N, i < j \Rightarrow \mathbf{d_i} < \mathbf{d_j}$. Let $\mathbf{I} = \{\mathbf{d} \in \mathbf{D}: \exists i \in N(\mathbf{d} \leqslant \mathbf{d_i})\}$. It is easily verified that $\mathbf{I}$ is a countable ideal of $\mathscr{D}$. Choose $\mathbf{a_0}, \mathbf{a_1}$ as in Theorem 4.8 for $\mathbf{I}$. If $\mathscr{D}$ were a lattice, then $\mathbf{a_0}$ and $\mathbf{a_1}$ would have a greatest lower bound $\mathbf{d} \in \mathbf{D}$. Since $\mathbf{d} \leqslant \mathbf{a_0}$ and $\mathbf{d} \leqslant \mathbf{a_1}$, $\mathbf{d} \in \mathbf{I}$. Hence $\mathbf{d} \leqslant \mathbf{d_i}$ for some $i \in N$. But then $\mathbf{d} < \mathbf{d_{i+1}} \leqslant \mathbf{a_0}$ and $\mathbf{d} < \mathbf{d_{i+1}} \leqslant \mathbf{a_1}$ which is impossible since $\mathbf{d} = \mathbf{a_0} \cap \mathbf{a_1}$. Thus $\mathbf{a_0} \cap \mathbf{a_1}$ cannot exist. ▯

Note that the proof of Corollary 4.10 shows that no strictly increasing sequence of degrees can have a least upper bound.

The final extension theorem of this section gives a sufficient condition for determining whether, given finite posets $\mathscr{H} = \langle H, \leqslant_H\rangle \subseteq \langle M, \leqslant_M\rangle = \mathscr{M}$ and an isomorphic copy $\mathscr{T}$ of $\mathscr{H}$ which is a subposet of $\mathscr{D}$, it is always possible to extend $\mathscr{T}$ to $\mathscr{V} = \langle \mathbf{V}, \leqslant_V\rangle \subseteq \mathscr{D}$ so that the following diagram commutes.

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{\subseteq} & \mathscr{M} \\ {\scriptstyle f}\downarrow{\scriptstyle \simeq} & & {\scriptstyle g}\downarrow{\scriptstyle \simeq} \\ \mathscr{T} & \dashrightarrow_{\subseteq} & \mathscr{V} \end{array}$$

Fig. 4.2

The theorem states that such an extension can always be found if $\langle H, \leqslant_H, \vee_H\rangle$ is a usl, the embedding of $\mathscr{H}$ into $\mathscr{M}$ preserves least upper bounds, and if for all $a \in M - H$ and $b \in H$, $a \nleqslant_M b$, i.e., no new elements are placed below any old elements. It will follow from results proved in Chap. VII that without these conditions, an extension as in Fig. 4.2 will sometimes fail to exist. These two results will enable us to produce an algorithm which decides $\mathrm{Th}(\mathscr{D}) \cap \forall_2$.

**4.11 Theorem.** *Let $\langle H, \leqslant_H, \vee_H\rangle$ be a finite usl and let $\mathscr{M} = \langle M, \leqslant_M\rangle$ be a finite poset which extends $\mathscr{H} = \langle H, \leqslant_H\rangle$. Let $\mathscr{T} \subseteq \mathscr{D}$ be an isomorphic copy of $\mathscr{H}$. Assume that:*

(i) $\forall m \in M \forall p, q \in H(p \leqslant_M m \,\&\, q \leqslant_M m \to p \vee_H q \leqslant_M m)$.

(ii) $\forall a \in M - H \forall b \in H(a \nleqslant_M b)$.

*Then there is a poset $\mathscr{V} \subseteq \mathscr{D}$ for which Fig. 4.2 is a commuting diagram.*

*Proof.* Let $H = \{p_i: i < n\}$ and let $M - H = \{m_i: i < r\}$. Let $f$: $H \to T$ be an isomorphism of $\mathscr{H}$ with $\mathscr{T} = \langle \mathbf{T}, \leqslant_T\rangle$, and for all $i < n$, let $f(p_i) = \mathbf{d_i}$ and let $D_i$ be a set of degree $\mathbf{d_i}$. We will use the method of forcing to construct a set $A \subseteq [0, r) \times N$. For each $i < r$, let $i^* < n$ be determined by letting $p_{i^*}$ be the greatest element of $H$ such that $p_{i^*} \leqslant_M m_i$. And for each $i < r$, let $G_i = \{k < r: m_k \leqslant_M m_i\}$. We will extend $f$ to an isomorphism $g$ taking $M$ into $\mathbf{D}$ by defining $g(i)$ to be the degree $\mathbf{c_i}$ of the set $C_i = A^{[G_i]} \oplus D_{i^*}$. By (i), $g$ is well-defined, and the fact that $g$ is an isomorphism follows easily from (ii) once we show that the following requirements are satisfied for all $e \in N$, $i, k < r$ and $j < n$:

(14) $R^0_{e,i,k}$: $m_k \nleqslant_M m_i \Rightarrow \Phi_e^{C_i} \neq C_k$.

(15) $R^1_{e,i,j}$: $p_j \nleqslant_M m_i \Rightarrow \Phi_e^{C_i} \neq D_j$.

(16) $R^2_{e,i,j}$: $\Phi_e^{D_i} \neq C_i$.

(17) $R^3_{i,j}$: $p_j \leqslant_M m_i \Rightarrow D_j \leqslant_T C_i$.

(18) $R^4_{i,k}$: $m_k \leqslant_M m_i \Rightarrow C_k \leqslant_T C_i$.

It follows from the definition of $C_i$ that for all $i, k < r$ and $j < n$, $R^3_{i,j}$ and $R^4_{i,k}$ are satisfied.

We take as the domain $F$ of our notion of forcing $\{\theta \subseteq N^r: \forall i < r(\mathrm{dom}(\theta^{[i]})$ is finite$)\}$. These forcing conditions are ordered by $\supseteq$ defined coordinatewise. Forcing of requirements is defined as follows: We say that $\theta \Vdash R^0_{e,i,k}$ if either $m_k \leqslant_M m_i$ or one of the following conditions holds, where, for $\theta \in F$, we define $\theta^*(i)$ to be $\theta^{[G_i]} \oplus D_{i^*}$:

(19) $\exists x \in N(\Phi_e^{\theta^*(i)}(x)\downarrow \neq \theta^{[k]}(x)\downarrow)$.

(20) $\exists x \in N \forall \xi \in F(\xi \supseteq \theta \rightarrow \Phi_e^{\xi^*(i)}(x)\uparrow)$.

We say that $\theta \Vdash R^1_{e,i,j}$ if either $p_j \leqslant_M m_i$ or (20) holds or

(21) $\exists x \in N(\Phi_e^{\theta^*(i)}(x)\downarrow \neq D_j(x))$.

We say that $\theta \Vdash R^2_{e,i,j}$ if

(22) $\Phi_e^{D_j}$ total $\Rightarrow \exists x \in N(\Phi_e^{D_j}(x) \neq \theta^{[i]}(x)\downarrow)$.

For each requirement $R$ just mentioned, let $C_R = \{\theta \in F: \theta \Vdash R\}$, and let $\mathscr{C}$ be the collection of all such sets $C_R$. We will show that each $C_R \in \mathscr{C}$ is dense. Assuming that this has been shown, let $G$ be a $\mathscr{C}$-generic set and let $A = \cup G$. Note that $A^{[i]} \leqslant_T C_i$ so for any set $S$, if $A^{[i]} \nleqslant_T S$ then $C_i \nleqslant_T S$. The Satisfaction Lemma now follows easily from the Enumeration Theorem.

We complete the proof of the theorem by showing that each $C_R \in \mathscr{C}$ is dense. Let $R = R^0_{e,i,k}$ and fix $\eta \in F$ and the least $x \in N$ such that $\eta^{[k]}(x)\uparrow$. We suppose that $m_k \nleqslant_M m_i$, else $\eta \Vdash R$. If there are $\xi \supseteq \eta$ and $y \in N$ such that $\Phi_e^{\xi^*(i)}(x)\downarrow = y$, then since $k \notin G_i$, we can find $\theta \in F$ such that $\theta \supseteq \eta$ and $\theta^{[k]}(x)\downarrow \neq y$ and so satisfy (19). Otherwise, letting $\theta = \eta$, we see that (20) is satisfied. Hence $C_R$ is dense.

Next let $R = R^1_{e,i,j}$ and fix $\theta \in F$. We suppose that $p_j \nleqslant_M m_i$ else $\theta \Vdash R$. If it is not the case that

(23) $\forall x \in N \exists \xi \in F(\xi \supseteq \theta \,\&\, \Phi_e^{\xi^*(i)}(x)\downarrow)$,

then (20) is satisfied, so $\theta \,||\!\!- R$. If it is not the case that

(24) $\forall x, y \in F\, \forall \tau \in F(\tau \supseteq \theta \,\&\, \Phi_e^{\tau^*(i)}(x)\downarrow = y \rightarrow y = D_j(x))$,

then (21) will be satisfied by some $\tau \supseteq \theta$ and for such $\tau$, $\tau \,||\!\!- R$. Hence $C_R$ is dense unless both (23) and (24) hold, which we assume to be the case in order to obtain a contradiction. Under this assumption, given $x \in N$ we can compute $D_j(x)$ by finding $\sigma \in \mathscr{S}_2$ and $\xi \in F$ such that $\sigma \subseteq \xi^*(i)$ and $\Phi_e^{\sigma}(x)\downarrow$; then $\Phi_e^{\sigma}(x) = D_j(x)$. By (23) and (24), such $\sigma$ and $\xi$ exist and can be found recursively in $D_{i^*}$. Hence $D_j \leqslant_T D_{i^*}$. Since $f$ is an isomorphism, $p_j \leqslant_M p_{i^*} \leqslant_M m_i$ so $R^1_{e,i,j}$ was forced earlier by $\theta$.

Finally, let $R = R^2_{e,i,j}$. Fix $\eta \in F$ and the least $x$ such that $\eta^{[i]}(x)\uparrow$. Suppose that $\Phi_e^{D_j}$ is total. Then we can find $\theta \in F$ such that $\theta \supseteq \eta$ and $\theta^{[i]}(x)\downarrow \neq \Phi_e^{D_j}(x)$. It now follows from (22) that $C_R$ is dense. ▯

Although restrictions (i) and (ii) in Theorem 4.11 are necessary, the condition that $H$ and $M$ be finite is unnecessary. Only minor modifications are needed to prove a version of this theorem if $H$ and $M - H$ are countable if we also assume that for all $m \in M - H$, $\{p \in H: p <_M m\}$ has a greatest element. This latter theorem follows easily from results of Kleene and Post [1954]. A more complicated proof presented in Sacks [1961a] will prove the theorem without this added assumption. In fact, Sacks [1961a] proves such an extension theorem in the case where $|H| < 2^{\aleph_0}$ and $M - H$ is countable. This latter result is the key to obtaining the embedding results attributed to Sacks in the previous section.

**4.12 Remarks.** Kleene and Post [1954] introduced most of the methods used in this section, and proved Theorems 4.3, 4.5, and 4.11, and Corollary 4.10. Shoenfield [1960] proved Corollary 4.6. Theorem 4.8 was proved by Spector [1956].

**4.13–4.16 Exercises**

**4.13** Show that every maximal independent subset of $\mathscr{D}$ is uncountable.

**4.14** Let $\mathbf{d} \in \mathbf{D}$ be given, and let $\mathbf{A}$ be a countable antichain of $\mathscr{D}$ such that for all $\mathbf{a} \in \mathbf{A}$, $\mathbf{d} < \mathbf{a}$. Show that there is a $\mathbf{b} \in \mathbf{D}$ such that $\mathbf{b} > \mathbf{d}$, $\mathbf{b} \notin \mathbf{A}$, and $\mathbf{A} \cup \{\mathbf{b}\}$ is an antichain of $\mathscr{D}$.

**4.15** Prove Theorem 4.11 under the modified assumption that $H$ and $M - H$ are countable and that for all $m \in M - H$, $\{p \in H: p <_M m\}$ has a greatest element.

**4.16** Prove Theorem 4.11 under the assumption that $H$ and $M - H$ are countable. (*Hint*: Construct each $C_i$ directly, coding into $C_i$ those sets $D_j$ such that $p_j \leqslant_M m_i$ and those sets $C_k$ such that $m_k \leqslant_M m_i$.)