Introduction

Degree theory, as it is studied today, traces its development back to the fundamental papers of Post [1944] and Kleene and Post [1954]. These papers introduced algebraic structures which arise naturally from the classification of sets of natural numbers in terms of the amount of additional oracular information needed to compute these sets. Thus we say that *A is computable from B* if there is a computer program which identifies the elements of A, using a computer which has access to an oracle containing complete information about the elements of B.

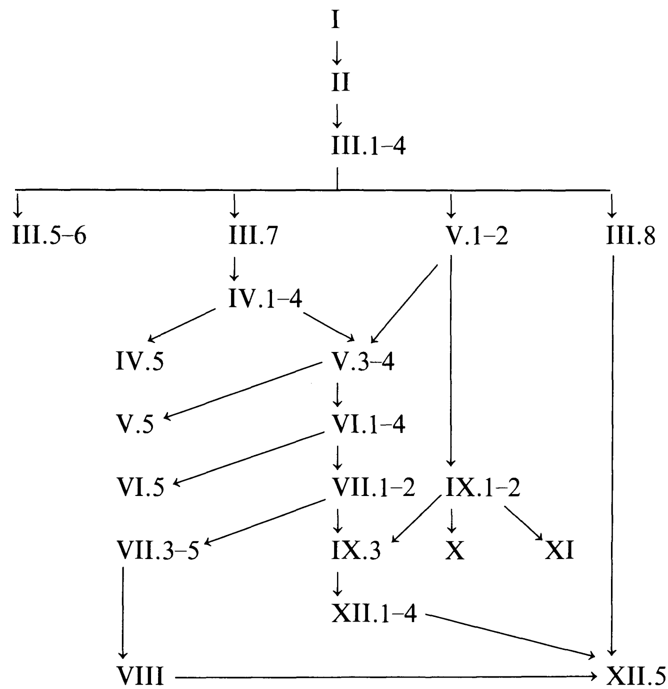
The idea of comparing sets in terms of the amount of information needed to compute them has been extended to notions of computability or constructibility which are relevant to other areas of Mathematical Logic such as Set Theory, Descriptive Set Theory, and Computational Complexity as well as Recursion Theory. However, the most widely studied notion of degree is still that of *degree of unsolvability* or *Turing degree*. The interest in this area lies as much in the fascinating combinatorial proofs which seem to be needed to obtain the results as in the attempt to unravel the mysteries of the structure. An attempt is made, in this book, to present a study of the degrees which emphasizes the methods of proof as well as the results. We also try to give the reader a feeling for the usefulness of local structure theory in determining global properties of the degrees, properties which deal with questions about homogeneity, automorphisms, decidability and definability.

This book has been designed for use by two groups of people. The main intended audience is the student who has already taken a graduate level course in Recursion Theory. An attempt has been made, however, to make the book accessible to the reader with some background in Mathematical Logic and a good feeling for computability. Chapter 1 has been devoted to a summary of basic facts about computability which are used in the book. The reader who is intuitively comfortable with these results should be able to master the book. The second intended use for the book is as a reference to enable the reader to easily locate facts about the degrees. Thus the reader is directed to further results which are related to those in a given section whenever the treatment of a topic within a section and its exercises is not complete.

The material which this book covers deals only with part of Classical Recursion Theory. A major omission is the study of the lattice of recursively enumerable sets, and the study of the recursively enumerable degrees is only cursory. These areas are normally covered in a first course in Recursion Theory, and the books of Soare [1984], Shoenfield [1971] and Rogers [1967] are recommended as sources for this material.

The book contains more material than can be covered in a one semester course. If time is short, it is advisable to sample material in some of the sections rather than cover whole sections. Sample courses for one semester would contain a core consisting of Chaps. I–V and Chap. IX, with the remaining time spent either on Chaps. VI–VIII (perhaps skipping some of the structure results, and either assuming them for the purposes of the applications of Chap. VIII, or using the exercises at the end of Chap. VI to replace the structure results of Chap. VIII in those applications), or on Chaps. X and XI. Chapter XII is best left to the reader to puzzle through on his own. The material in the appendices may be covered immediately before the section where it is used, but it is recommended that this material be left to the reader.

The following chart describes the major dependencies of one section on another within the book.



Some proofs are left unfinished, to be worked out by the reader. This is done either to avoid repeating a proof which is similar to one already presented, or when straightforward details remain to be worked out. Hints are provided for the more difficult exercises, along with references to the original papers where these results appeared. Exercises which are used later in the text have been starred.

Although an attempt has been made to be accurate in the attribution of results, it is inevitable that some omissions and perhaps errors occur. We apologize in advance for those unintentional errors.

Theorems, definitions, etc. are numbered and later referred to by chapter, section, and number within the section. Thus VI.1.2 is the numbered paragraph in

Sect. 1 of Chap. VI with number 1.2. If the reference to this paragraph is within Chap. VI, we refer to the paragraph as 1.2, dropping the VI. There are two appendices, A and B, and a reference to A.1.2 is a reference to paragraph 1.2 of Appendix A.

Definitions and Notation. The following definitions and notation will be used without further comment within the book.

Sets will be determined by listing their elements as $\{a_0, a_1, \ldots\}$ or by specification as the set of all x satisfying property P, denoted by $\{x: P(x)\}$. If A and B are sets, then we write $x \in A$ for *x is an element of A* and $A \subseteq B$ for *A is a subset of B*. We use $A \subset B$ to denote $A \subseteq B$ but $A \neq B$ (placing / through a relation symbol denotes that the relation fails to hold for the specified elements). $A \cup B$ is the *union* of A and B, i.e., the set of all elements which appear either in A or in B, and $A \cap B$ denotes the *intersection* of A and B, i.e., the set of all elements which appear in both A and B. The *difference* of A and B is denoted by $A - B$ and consists of those elements which lie in A but not in B. The *symmetric difference* of A and B is denoted by $A \triangle B = (A - B) \cup (B - A)$. We will denote the *maximum* or greatest element of the partially ordered set $\langle A, \leqslant \rangle$ by $\max(A)$, and the *minimum* or least element of this set by $\min(A)$ if such maximum and/or minimum elements exist.

Let A, B and C be sets. The *cartesian product* of A and B, $A \times B$, is the set of all ordered pairs $\langle x, y \rangle$ such that $x \in A$ and $y \in B$. The cartesian product operation can be iterated, so that $A \times B \times C$ is used to denote $(A \times B) \times C$. We use A^k to denote the cartesian product of k copies of A (which is the same as the set of all k-tuples of elements of A) and $A^{<\omega}$ to denote the set of all finite sequences of elements of A. If $\bar{x} = \langle x_1, \ldots, x_k \rangle$ is a k-tuple, we use $\bar{x}^{[k]}$ to denote x_k, the kth coordinate of $\bar{x}$. Given $S \subseteq A \times B$ and $i \in A$, we use $S^{[i]}$ to denote $\{x \in B: \langle i, x \rangle \in S\}$.

We use $\emptyset$ to denote the empty set, and N to denote the set of natural numbers $\{0, 1, \ldots\}$. Given $A, B \in N$, we denote the *direct sum* of A and B by $A \oplus B = \{2x: x \in A\} \cup \{2x + 1: x \in B\}$. For any set A, $|A|$ will denote the cardinality of A. The infinite cardinal numbers are $\aleph_0, \aleph_1, \ldots$ in order, and $2^{\aleph_0}$ is the cardinality of the continuum.

A *partial function* φ from A to B (written $\varphi: A \to B$) is a subset of the set of ordered pairs $\{\langle x, y \rangle : x \in A \,\&\, y \in B\}$ such that for each $x \in A$ there is at most one $y \in B$ such that $\langle x, y \rangle \in \varphi$. We write $\varphi(x)\downarrow$ ($\varphi(x)$ *converges*) for $\langle x, y \rangle \in \varphi$, and $\varphi(x)\uparrow$ ($\varphi(x)$ *diverges*) if $x \in A$ and for all $y \in B$ $\langle x, y \rangle \notin \varphi$. We will sometimes denote the function φ with the notation $x \mapsto \varphi(x)$. The *domain* of φ is denoted by $\mathrm{dom}(\varphi) = \{x \in A: \varphi(x)\downarrow\}$ and B is called the range of φ, denoted by $\mathrm{rng}(\varphi)$. If $\mathrm{dom}(\varphi) = A$, we call φ a *total function*. The word total, however, will frequently be dropped. Thus unless otherwise specified, a function will always be total. In general, we use the lower case Roman letters $f, g, h, \ldots$ to denote functions with domain N and lower case Greek letters $\varphi, \psi, \theta, \ldots$ to denote partial functions with domain $\subseteq N$. The corresponding upper case letters are reserved for *functionals*, i.e., maps taking functions into functions. A set S is identified with its *characteristic function* χ_S where $\chi_S(x) = 1$ if $x \in A$ and $\chi_S(x) = 0$ otherwise. If φ is a partial function and $B \subseteq \mathrm{dom}(\varphi)$, then $\varphi \restriction B$ is the *restriction* of φ to B, i.e., the function with domain B which agrees with φ on B. By the previous definition, the restriction notation applies to sets as well as to functions.

Given $f: N \to N$, we write $\lim_s f(s) = y$ if $\{s: f(s) \neq y\}$ is finite, and $\lim_s f(s) = \infty$ if, for every $y \in N$, $\{s: f(s) = y\}$ is finite. We write $\limsup_s f(s) = y$ if $\{s: f(s) = y\}$ is infinite and $\{s: f(s) > y\}$ is finite; and $\limsup_s f(s) = \infty$ if, for every $y \in N$, $\{s: f(s) \geqslant y\}$ is infinite. We write $\liminf_s f(s) = y$ if $\{s: f(s) = y\}$ is infinite and $\{s: f(s) < y\}$ is finite; and $\liminf_s f(s) = \infty$ if, for every $y \in N$, $\{s: f(s) \leqslant y\}$ is finite. If $\{\alpha_s : s \in N\}$ is a sequence of finite sequences of integers, then we write $\lim_s \alpha_s$ for the partial function θ such that for all $x \in N$, $\theta(x)\downarrow$ if and only if $\lim_s \alpha_s(x)\downarrow$, in which case $\theta(x) = \lim_s \alpha_s(x)$. Given two sequences of integers α and β, we say that α *lexicographically precedes* β if either $\alpha \subset \beta$ or $\alpha(x) < \beta(x)$ for the least x such that $\alpha(x) \neq \beta(x)$. We write $\limsup_s \alpha_s = \theta$ if θ is a sequence of integers and for all $x \in N$, $\{s: \alpha_s \upharpoonright x = \theta \upharpoonright x\}$ is infinite and $\{s: \theta$ lexicographically precedes $\alpha_s\}$ is finite. We write $\liminf_s \alpha_s = \theta$ for θ as in the preceding sentence if $\{s: \alpha_s \upharpoonright x = \theta \upharpoonright x\}$ is infinite for each $x \in N$, and $\{s: \alpha_s \nsubseteq \theta$ and α_s lexicographically precedes $\theta\}$ is finite.

We use Church's lambda notation to define new functions from old ones. If $f(x_1, \ldots, x_n, y_1, \ldots, y_k)$ is a function of $n + k$ variables, then $\lambda x_1 \cdots x_n f(x_1, \ldots, x_n, y_1, \ldots, y_k)$ denotes the function g of n variables defined by $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, y_1, \ldots, y_k)$.

If φ and ψ are partial functions, then we write $\varphi \subseteq \psi$ (ψ *extends* φ) if $\mathrm{dom}(\varphi) \subseteq \mathrm{dom}(\psi)$ and for all $x \in \mathrm{dom}(\varphi)$, $\psi(x)\downarrow = \varphi(x)$. We say that φ and ψ are incomparable and write $\varphi \mid \psi$ if neither $\varphi \subseteq \psi$ or $\psi \subseteq \varphi$.

We write A^B for the set of all functions from B into A. Since sets are identified with their characteristic functions, 2^S then denotes the *power set* of S, i.e., the set of all subsets of S.

Standard interval notation will be used for sets A partially ordered by $\leqslant$. Thus $[a, b]$ will denote $\{x \in A: a \leqslant x \leqslant b\}$, (a, b) will denote $\{x: a < x < b\}$, (a, ∞) will denote $\{x: x \geqslant a\}$, $(-\infty, b]$ will denote $\{x: x \leqslant b\}$, etc. Structures will be denoted by $\mathscr{A} = \langle A, R_0, \ldots, R_n, f_0, \ldots, f_k, c_0, \ldots, c_m \rangle$ where A is the *universe* of $\mathscr{A}$, $R_0, \ldots, R_n$ are relations on cartesian products of A, $f_0, \ldots, f_k$ are functions from cartesian products of A into A, and $c_0, \ldots, c_m$ are designated elements of A. The partially ordered set above is thus denoted by $\mathscr{A} = \langle A, \leqslant \rangle$.

We will use the logical symbols & to denote *and*, $\vee$ to denote *or*, $\neg$ to denote *not*, $\to$ to denote *implies*, and $\leftrightarrow$ to denote *if and only if*. $\exists$ will denote the *existential quantifier* and $\forall$ will denote the *universal quantifier*. We will write $\bigwedge_{i=0}^{n} \sigma_i$ for $\sigma_0 \,\&\, \sigma_1 \,\&\, \cdots \,\&\, \sigma_n$ and $\bigvee_{i=0}^{n} \sigma_i$ to denote $\sigma_0 \vee \sigma_1 \vee \cdots \vee \sigma_n$. When we are not using a formal language, we will use $\Rightarrow$ and $\Leftrightarrow$ in place of $\to$ and $\leftrightarrow$ respectively.

$\blacksquare$ will denote the end of a proof.