Chapter XVII

NDOP: Theories Without the Dimensional Order Property

In this chapter we investigate theories without the dimensional order property. As in the previous chapter, our main concerns will be with **S**-models of superstable theories and arbitrary models of ω-stable theories. We will show that for any acceptable class K, if T does not have the DOP, each K-model of T can be decomposed as a K-prime model over a skeleton which is an independent tree of 'small' K-models. If the model has cardinality λ, this tree will be isomorphic to a subset of $\lambda^{<\omega}$.

The following simple example is a good prototype for the kind of counting done in this chapter. The theory in the example is *deep* in the sense made precise in Section 2 and does not have the dimensional order property. Thus by Theorem 4.8, it has 2^λ models of power λ for each uncountable λ. We sketch here how this result can be seen directly.

Let T be the theory with a single function symbol f such that there is a unique point which is mapped to itself by f and all points have infinitely many preimages. T admits elimination of quantifiers. The models of T are best regarded as unions of components where two elements a and b are in the same component if for some m and n, $f^m(a) = f^n(b)$. There are, in fact, 2^λ possibilities for the unique component which has a base point so the others can be ignored in calculating the number of models of power λ. The component with a base point can be thought of as a subset of the tree $\lambda^{<\omega}$ which is closed under predecessor. Thus our task reduces to showing there are 2^λ such trees. To see this, we will show that each ordinal $\alpha < \lambda$ can be coded by a tree, $T_\alpha \subseteq \lambda^{<\omega}$. The T_α are constructed similarly to the coding of an arbitary countable ordinal by a subtree of $\omega^{<\omega}$. The difference is that rather than placing a single copy of T_β, for each $\beta < \alpha$, on the first level to code α, we place λ copies of each such T_β. This allows each point to have infinitely many preimages without disturbing the effect of the coding. There is a series of exercises at the end of Section 3 which explores the meaning of the notions defined in this chapter for this example.

Section 1 contains some preliminary results on the type of tree we use for a skeleton. In Section 2 we discuss representations of models and prove the decomposition theorem. In Section 3 we define the depth of a theory and use the decompostion theory to produce an upper bound on $I(\aleph_\beta, K)$ in terms of β and the depth of T. This requires that we calculate the number of (labeled) trees of each depth contained in $\lambda^{<\omega}$. In Section 4, we begin the proofs of the lower bounds on the spectrum function. We are able to complete the computation for theories with high depth and when $\aleph_\beta \geq \aleph_\omega$. The remaining cases are considered in the next chapter.

Since we only defined the dimensional order property for superstable theories, there is a de facto assumption in this section that the theory is superstable. In fact, this hypothesis is irrelevant to the discussion in Section 1 of normal trees. It becomes important when we assume the existence of **S**-prime models and in Section 2 where we show the existence of a decomposition. Shelah [Shelah 1982] discusses the dimensional order property for arbitrary stable theories.

1. Normal Trees

This section contains technical results on a generalization of an independent tree, namely a normal tree. We also introduce the notion of a stable system and prove some preliminary results about such systems. They play a small role in this book and a much larger role in Shelah's study of superstable but not ω-stable theories. We assume familiarity with the properties of independent systems which were discussed at the end of Section II.2.

We begin by establishing some notation for the trees that will form the skeletons of the decomposition. Informally, a tree is a partially ordered set which is isomorphic to a subset of $\langle \lambda^{<\omega}, \subseteq \rangle$ for some infinite λ. We give a more formal definition below, but these are the structures to remember. In the following notations for the 'cone above an element' or 'set of predecessors of an element' note that we form a union in our definition. This simplifies the notation when these concepts are applied.

1.1 Notation. i) Recall that in a discrete partial order, we denote by a^- the predecessor of a. If $b^- = a$ we say 'a precedes b' or 'b succeeds a'.

ii) We now define by induction the *cone above* a. Let $a^1_> = \{b : b^- = a\}$ and for each n, $a^{n+1}_> = \bigcup\{b^n_> : b^- = a\}$. The cone (strictly) above a is $a_> = \bigcup_{n<\omega} a^n_>$. Finally $a_\geq = a_> \cup \{a\}$. Similarly we can define $a_< = \bigcup\{b : b < a\}$. Recall that $a_\#$ denotes the set of elements incomparable to a.

iii) A partial order $(A, <)$ is a *tree* if no pair of incomparable elements has a common upper bound. Thus in a tree the set of predecessors of each element is linearly ordered by the induced ordering.

iv) A *well-founded* tree is a tree $(A, <)$ such that for each $a \in A$, $a_<$ is wellordered. The ordinal of this wellordering is called the *height* of a, and denoted $\mathrm{ht}(a)$. If each element has at most λ successors we call λ the *width* of the tree. Note that the cardinality of the tree is $\lambda^{<\omega} = \lambda$ as λ is always infinite. A node with no successor is called a *leaf.*

1.2 Exercise. Show $a_{>} = \bigcup\{b : b > a\}$

The next definition specifies the form we want the skeleton of a model to have.

1.3 Definition. (Fig. 1). $(A, <, A_{\langle\,\rangle})$ is a *normal tree* if

i) The partial order $(A, <)$ is isomorphic to a downward closed subset of the tree $(\lambda^{<\omega}, \subseteq)$ for some cardinal λ.
ii) $a < b$ implies $a \subseteq b$.
iii) $A_{\langle\,\rangle}$ denotes the element mapped to the empty node.
iv) For every $a \in A$, $a^1_{>}$ is an independent set over a.
v) For every $a \in A$ with $a^- \neq A_{\langle\,\rangle}$, $t(a; a^-) \dashv a^{--}$.

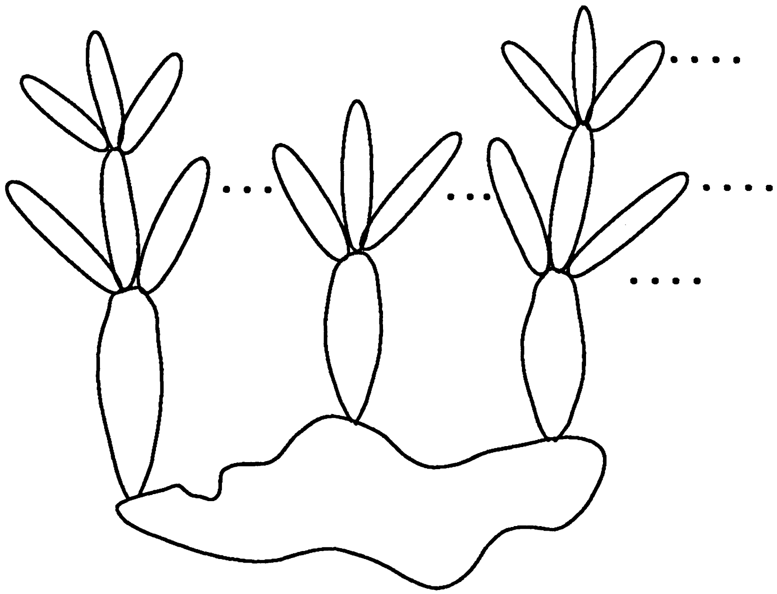

Fig. 1. A normal tree

Note that ii) implies that for each $a \in A$, $a^- = a_{<}$. Condition v) implies that the partial order $<$ on A is the same as the partial order $<'$ defined by $a <' b$ if a dominates b over $A_{\langle\,\rangle}$.

Our first task is to verify that any normal tree is independent with respect to $<$. We could, of course, have added that property to the definition but the version used reflects more clearly the way in which normal trees are constructed. The proof proceeds via two claims.

1.4 Claim 1. *For every* n, $t(a^n_{\geq}; a) \dashv a^-$. *Hence,* $t(a_>; a) \dashv a^-$.

Proof. If $n = 1$, the result follows from the triviality of orthogonality. Specifically, this is Exercise VI.2.5. Suppose we have this result for m and $n = m + 1$. Then, by induction, for each $b \in a^1_>$, we have $t(b^m_{\geq}; b) \dashv a$. We also have $t(b; a) \dashv a^-$. Thus, by the transitivity of $\dashv$, (compare Lemma VI.3.10 and Exercise VI.3.11), $t(b^m_{\geq}; a) \dashv a^-$. Since $a^1_>$ is an independent set, by Theorem VI.2.21 $\{b^m_{\geq} : b \in a^1_>\}$ is an independent set. Now as in the case $n = 1$, $t(a_>; a) \dashv a^-$.

The following remark is actually contained in the proof of Claim 1.

1.5 Exercise. Show that for any a in a normal tree, $\{b_{\geq} : b^- = a\}$ is an independent set.

1.6 Claim 2. *For every* a *in a normal tree,* $a \downarrow_{a_<} a_{\#}$.

Proof. We induct on the height of a. If $\mathrm{ht}(a) = 0$, there is nothing to prove. Suppose $a^- = b$ and we have shown by induction that $b \downarrow_{b_<} b_{\#}$. To show $a \downarrow_{a_<} a_{\#}$, note that $a_{\#} = b_{\#} \cup X$ where $X = \{c : c \geq b \wedge a \not\leq c\}$. Since A is normal, $b = a_<$ and $b^- = b_<$. By Claim 1, $t(b_>; b) \dashv b^-$; we conclude $t(b_>; b) \perp t(b_{\#}; b^-)$. Since $b_> = a_{\geq} \cup X$, we have, in particular, that $b_{\#} \downarrow_b (a_{\geq} \cup X)$. By monotonicity, this yields $(b_{\#} \cup X) \downarrow_{X \cup b} a_{\geq}$. But, the last exercise shows $(X \cup b) \downarrow_b a_{\geq}$ so by transitivity $(b_{\#} \cup X) \downarrow_b a_{\geq}$. That is, $a_{\#} \downarrow_{a_<} a_{\geq}$ which is more than is required.

1.7 Claim 3. *If* $(A, <, A_{\langle\,\rangle})$ *is a normal tree then for any* $a, b \in A$ *with neither equal to* $A_{\langle\,\rangle}$ *and* $a^- \neq b^-$, $t(a; a^-) \perp t(b; b^-)$.

Proof. As $a^- \cap b^-$ is a proper subset of $a \cap b$, $t(a; a^-) \dashv (a^- \cap b^-)$ and, since $a^- \downarrow_{a^- \cap b^-} b^-$ by Theorem VI.2.21, the result follows easily.

One of the major tasks in the remainder of this text is the reduction of problems about the class of all models to the consideration of problems about **S**-saturated models. Relative **I**-saturation (Section X.1) is an important tool for this purpose. To apply it we need Theorem 1.8.

In some of the ensuing definitions we have to deal with elements of the sets in an independent family. Recall that in Chapter XVI we adopted the convention of referring in such situations to the systems as $\bar{A}$, $\bar{B}$, etc.,the elements of the system as A, B etc. and the elements of the elements of the system as a, b, etc. We often identify a system $\bar{A}$ or an ideal J with its union and depend on the context to indicate which is meant. Sometimes for emphasis we insert the union sign.

The proof of the following theorem contains an important idea masquerading as a technical device. Let $\bar{A}$ be a partially ordered set of sets and $\bar{a} \in \cup\bar{A}$. Then $\bar{a}$ can be partitioned as $\langle \bar{a}_0, \ldots, \bar{a}_k \rangle$ for some k so that there is a set $\langle A_0, \ldots, A_k \rangle$ of elements of $\bar{A}$ such that the A_i are incomparable with respect to the partial order and $\bar{a} \cap A_i = \bar{a}_i$. The crucial point is that when $(\bar{A}, <)$ is an independent partial order, this partition breaks $\overline{a}$ down into smaller independent pieces. This reduction permits one to carry out an induction on the length of $\overline{a}$. This idea also shows up in, e.g. [Shelah 1975a] and [Baldwin & Shelah 1985].

We prove the following result in the context of an admissible class K. Thus the sequence $\overline{a}$ may be infinite (e.g. $K = \mathbf{S}_{\aleph_1}$.) However, for the major applications in this book $\lg(\overline{a})$ and so γ are finite.

1.8 Theorem. *Let $(\bar{A}, <)$ be an independent family of models from a class K and J an ideal in $\bar{A}$ such that*

for every $A \in \bar{A}$ there is a maximal element B of J which is less than A. $(*)$

Then $J \leq_I \bar{A}$.

Proof. Suppose $q(\overline{x}; \overline{b})$ is an I-formula with parameters $\overline{b} \in J$. Suppose for $\overline{a} \in \bar{A}$, $\models q(\overline{a}; \overline{b})$. Partition $\overline{a}$ as $\langle \overline{a}_i : i < \gamma \rangle$ where each $\overline{a}_i = A_i \cap \overline{a}$ for some $A_i \in \bar{A}$ and the A_i for $i < \gamma$ are incomparable with respect to the partial ordering. For each $i < \mu$, choose using $(*)$ B_i to be a maximal member of J which is contained in A_i.

Now we show by induction on $i < \gamma$ that there exists a function f from $\langle \overline{a}_i : i < \gamma \rangle$ into J such that for each i

$$\models q(f(\overline{a}_0), f(\overline{a}_1), \ldots, \overline{a}_i, \overline{a}_{i+1}, \ldots; \overline{b}).$$

Suppose we have defined $f(\overline{a}_l)$ for $l < i$. Let $J_1 = \{A \in \bar{A} : A \leq A_i\}$ and let J_2 be the ideal generated by J and the A_j for $j \neq i$. Since $(\bar{A}, <)$ is an independent partial order, $J_1 \downarrow_{J_1 \cap J_2} J_2$. By monotonicity $\overline{a}_i \downarrow_{B_i} \overline{a}'\frown\overline{b}$ where $\overline{a}'$ is the sequence containing $f(\overline{a}_l)$ for $l < i$, and the $\overline{a}_l$ for $l > i$. Since B_i is strongly I-saturated there is an $\overline{a}'_i = f(\overline{a}_i) \in B_i$ such that

$$\models q(f(\overline{a}_0), f(\overline{a}_1), \ldots, \overline{a}'_i, \overline{a}_{i+1}, \ldots; \overline{b}).$$

The condition $(*)$ imposed on I in Theorem 1.8 seems to be the abstract content (at least in this context) imposed on a partial order by Shelah's concrete representation of a *stable system* as a family of sets indexed by finite subsets of some other set. For more information see [Shelah 198?].

1.9 Exercise. Show that if $(\bar{A}, <)$ is a normal tree and J is an ideal in $\bar{A}$, then J satisfies $(*)$.

We deduce the following immediately from Corollary X.1.18.

1.10 Theorem. *Suppose $\lambda(I)$ is regular, J is an ideal in a normal tree and M_J is I-prime over J. Then $M_J \downarrow_J \bar{A}$.*

The following result which is applied in Section XVII.4 also follows easily from Corollary X.1.18. If J is an ideal in a normal tree $\bar{A}$ of I-models M_J denotes the I-prime model over J.

1.11 Corollary. *If $J_0 \subseteq J_1$ are a pair of ideals in a normal tree $\bar{A}$ then $M_{J_0} \downarrow_{J_0} J_1$ so M_{J_1} is constructible over M_{J_0}. If $\langle J_n : n < \omega \rangle$ is an increasing* *sequence of ideals and M_n is I-prime over J_n then $\bigcup M_n$ is I-prime over $\bigcup J_n$.*

Recall that the notion of parallel free amalgams was introduced in Notation XVI.1.10. Since two parallel amalgams are one of the simplest examples of an independent partial order we can immediately conclude

1.12 Corollary. *Let the* **AT**-*amalgam* $\mathcal{N} = \langle N_0, N_1, N_2 \rangle$ *be parallel to the set-amalgam* $\mathcal{A} = \langle A_0, A_1, A_2 \rangle$. *Then* $N_1 \cup N_2 \leq_{\mathbf{AT}} A_1 \cup A_2$.

1.13 Exercise. Show that if $\mathcal{A}$ is a set-amalgam, there is a parallel amalgam of **S**-models.

Using Theorem 1.8 we can construct a full amalgam of **S**-models, $\mathcal{N}'$, parallel to a given full amalgam of models, $\mathcal{N}$. We will use this result to give another proof that the dimensional order property depends on T rather than on a particular acceptable class K.

1.14 Theorem. *If* $\mathcal{N} = \langle N_0, N_1, N_2, N_3 \rangle$ *is a full* **AT**-*amalgam then there is a full* **S**-*amalgam parallel to it.*

Proof. Choose N_0', an **S**-model containing N_0 and independent from $N_1 \cup N_2$ over N_0. Extend $N_1 \cup N_0'$ to an **S**-model N_1' with $N_1' \downarrow_{N_0' \cup N_1} N_2$ and hence by transitivity with $N_1' \downarrow_{N_0'} N_2$. Then extend $N_2 \cup N_0'$ to an **S**-model N_2' with $N_2' \downarrow_{N_0' \cup N_2} N_1'$ and hence by transitivity with $N_2' \downarrow_{N_0'} N_1'$. By Theorem 1.8, $N_1 \cup N_2 <_{\mathbf{AT}} N_1' \cup N_2'$. Let $\langle a_i : i < \alpha \rangle$ be a construction of N_3 over $N_1 \cup N_2$. By Lemma X.1.18 ii), for each i,

$$N_1 \cup N_2 \cup A_i \leq_{\mathbf{AT}} N_1' \cup N_2' \cup A_i.$$

Invoking Theorem X.1.17 this implies $t(a_i; N_1' \cup N_2' \cup A_i)$ is isolated. Thus, N_3 is constructible over $N_1' \cup N_2'$. So N_3 can be embedded in any **S**-prime model over $N_1' \cup N_2'$ which completes the theorem.

Now we give a second argument that the notion of DOP does not really depend upon a class of models but is a property of a theory. For simplicity, we show only the equivalence for the two most important cases.

1.15 Theorem. *Let T be a countable ω-stable theory. Then T has the* **S**-*NDOP if and only if T has the* **AT**-*NDOP.*

Proof. It is immediate that the **AT**-NDOP implies the **S**-NDOP. Suppose T has the **S**-NDOP. If $\mathcal{N}$ is a full **AT**-amalgam we can by Theorem 1.14 extend $\mathcal{N}$ to a parallel full **S**-amalgam. Now let p be an arbitrary type in $S(N_3)$ and p' a nonforking extension to $S(N_3')$. By the **S**-NDOP, p' and hence p is not orthogonal to, say N_1'. By the choice of a parallel amalgam, $N_1' \downarrow_{N_1} N_2$. As N_3 is **AT**-prime over $N_1 \cup N_2$, we deduce from Corollary X.1.20 that $N_1' \downarrow_{N_1} N_3$. Since $p \not\perp N_1'$, Corollary VI.2.22 yields $p \not\perp N_1$. Thus T satisfies **AT**-NDOP.

1.16 Historical Notes. Shelah introduced the notion of a normal tree in [Shelah 1982]. The treatment here relies both on the account in [Makkai

1984] and [Harrington & Makkai 1985] and on [Shelah 198?]. The proof of Theorem 1.8 given here is based on that in [Shelah 198?] which is considerably simpler than that in [Shelah 1982] which was reproduced in [Makkai 1984].

2. Decompositions of Models

In this section we show that if T is a countable superstable theory without the dimensional order property then every **S**-model of T admits a decomposition into models of power the continuum. Moreover, if T is ω-stable we can require the constituent models to be countable.

The difficult extension of this result to get countable models in the decomposition of a model of an arbitrary superstable theory has been obtained by both Hart and Saffe [Hart 1986].

2.1 Definition. By a *K-representation* of a model M we mean a normal tree $(\bar{A}, <, A_{\langle\,\rangle})$ of subsets of M such that:

i) For each $A \in \bar{A}$, $A \in K$.
ii) For every $A \in \bar{A}$ except $A_{\langle\,\rangle}$, there is an $a_A \in A$ such that $t(a_A; A^-)$ is K-strongly regular and $A = A^-[a_A]$.
iii) M is K-prime over $\bigcup \bar{A}$.

We write p_A for $t(A; A^-)$ and abbreviate $(\bar{A}, <, A_{\langle\,\rangle})$ by $\bar{A}$. Note that for each A, p_A is a stationary type.

Note that each p_A is a weight one type so $\not\perp$ is an equivalence relation on the set of p_A for $A \in \bar{A}$. Moreover, by 1.7 (Claim 3), each equivalence class consists entirely of successors of a single node. There may be several equivalence classes above any particular node. We often refer to a K-representation of M as K-decomposition of M. The difference is approximately that between an internal and external direct sum.

Before showing that under favorable conditions each K-model has a K-decomposition we deduce a few properties which are implied by the existence of such a decomposition. The first extends the triviality of $\dashv$ in a theory without DOP by replacing the pair of independent sets by an independent tree.

2.2 Lemma. *Let T be a superstable theory which satisfies NDOP. If $\bar{A}$ is a K-decomposition of the model M and $p \not\dashv M$ then for some $A \in \bar{A}$, $p \not\dashv A$.*

Proof. First, we show that we can assume $\bar{A}$ is finite. Let $q \in S(M)$ with $p \not\perp q$. Since $M \in K$, there is a finite subset $B \subseteq M$ with q strongly based on B. Thus, $p \not\perp q|B$ and $p \not\dashv B$. For some finite $\bar{A}_0 \subseteq \bar{A}$, B is **I**-atomic over $A_0 = \cup\bar{A}_0$ and so we can choose M_0 which contains B, is contained in M and is K-prime over A_0. Renaming M_0 as M and $\bar{A}_0$ as $\bar{A}$, we have the required reduction.

Now, we work by induction on the finite cardinality of $\bar{A}$. If $\bar{A}$ is linearly ordered the result is evident. If not, $\bar{A} = \bar{A}_1 \cup \bar{A}_2$ where the $\bar{A}_i$ are ideals and $\bar{A}_1 \cap \bar{A}_2$ is linearly ordered and has a maximal element A. Since $\bar{A}$ is an independent tree, $\bar{A}_1 \downarrow_A \bar{A}_2$. Moreover if, for $i = 1, 2$, we choose N_i to be K-prime over $\cup \bar{A}_i$, $N_1 \downarrow_A N_2$. If M_3 is K-prime over $N_1 \cup N_2$, there is, as M is prime over $\cup \bar{A}$, an embedding of M into M_3. Thus $p \not\perp M_3$. But then by NDOP, $p \not\perp$ one of N_1, N_2, say N_1. By induction, $p \not\perp A'$ for some $A' \in \bar{A}_1$ and we finish.

From this result we can deduce that M is tied even more tightly to an $\bar{A}$ which represents it.

2.3 Definition. The K-model M is K-*minimal* over $\bar{A}$ if $\cup \bar{A} \subseteq M$ and there does not exist a K-model N with $\cup \bar{A} \subseteq N \subseteq M$ and $N \neq M$.

2.4 Corollary. *Let T be a superstable theory which satisfies NDOP. If $\bar{A}$ is a K-representation of M then M is K-minimal over $\bar{A}$.*

Proof. Let $N \prec M$ be K-prime over $\cup \bar{A}$. If N is a proper subset of M, there is a $p \in S(N)$ which is realized in $M - N$ and is K-strongly regular. By Lemma 2.2, $p \not\perp A$ for some $A \in \bar{A}$. Thus, there is a K-strongly regular $q \in S(A)$ such that the nonforking extension q^N of q to $S(N)$ satisfies $q^N \not\perp p$. So q^N is realized in $M - N$ by some $\bar{b}$. But $\bar{A} \rhd_A M$ implies $\bar{b} \downarrow_A M$ and this contradiction yields the theorem.

2.5 Exercise. Show that if T is ω-stable and $\bar{A}$ is an **S**-decomposition of the **S**-model M then, in fact, M is **AT**-minimal over $\bar{A}$.

The next two results are further technical properties of a decomposition which we will need in the next section. To simplify the statements of these results we fix some more notation. Recall that $A^+ = \{B : B^- = A\}$. In a K-representation we define A_+ as $\{a_B : B^- = A\}$. Now we describe A_+.

2.6 Corollary. *If $\bar{A}$ is a K-representation of M then for each $A \in \bar{A}$, A_+ is a maximal independent set of realizations in M of strongly regular types over A.*

Proof. Note that $A_+ \rhd_A \bar{A}$. If $t(\bar{c}; A)$ is K-strongly regular and $\bar{c} \downarrow_A A^+$ then $\bar{c} \downarrow_A \bar{A}$. But we can choose $\bar{c} \in M = A[\bar{A}]$ so this is impossible.

We want to extend the descriptions of the successors of a point in a normal tree to a description of the successors of an ideal. Thus, we write I^+ for $\{A : A^- \in I\}$ and I_+ for $\{a_A : A^- \in I\}$. As usual we systematically confuse I^+ with $\bigcup I^+$. In order to study I_+ it is important that we find a common domain for the types in I_+. The natural candidate is M_I.

2.7 Lemma. *If I is an ideal in the K representation, $\bar{A}$, of M then I_+ is a maximal independent set of realizations in M of K-strongly regular types over M_I.*

Proof. Let $\bar{a} \in M$ realize p, a K-strongly regular type over M_I and suppose $\bar{a} \downarrow_{M_I} I_+$. Since T satisfies NDOP, $p \not\perp A$ for some $A \in I$. Thus, for some regular $q \in S(A)$, $q \not\perp p$. But then, $p \not\perp q^{M_I}$ so q^{M_I} is realized by some $\bar{c} \in M_I[\bar{a}]$. Since $\bar{a} \rhd_{M_I} M_I[\bar{a}]$, $\bar{a} \downarrow_{M_I} I_+$ implies $\bar{c} \downarrow_{M_I} I_+$. By transitivity of independence, $\bar{c} \downarrow_A I_+$. But, $\bar{A} \downarrow_I M_I$ implies $A_+ \subseteq I_+$ so this contradicts the previous corollary.

This result implies that I^+ is a set of models each of which is K-prime over one of a maximal independent set of realizations in M of K-strongly regular types over M_I. Note that A^+ (I^+) and A_+ (I_+) dominate each other over A (M_I). Now we obtain the most important consequence of the NDOP.

2.8 Theorem. *If T is superstable and does not have the dimensional order property, then every K-model of T has a K-representation.*

Proof. (Fig. 2). Let $M \in K$ and suppose $|M| = \lambda$. We define a partial isomorphism, $f : \eta \mapsto M_\eta$ from $\lambda^{<\omega}$ into the set of K-submodels of M such that the image of f is the required K-representation. Let $M_{\langle\,\rangle}$ be any

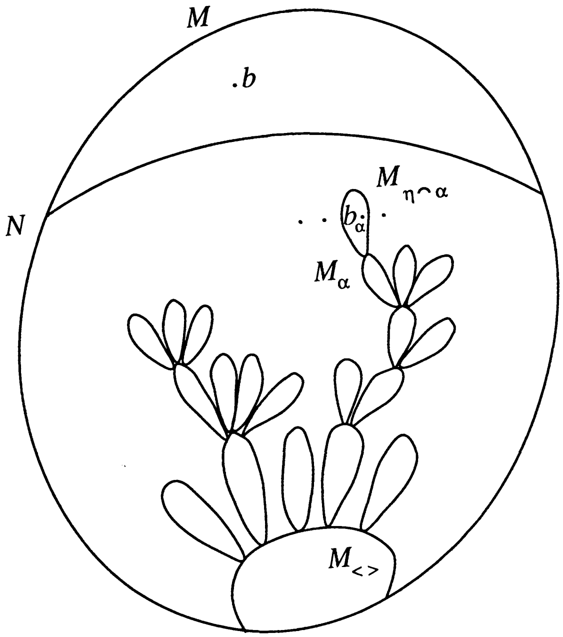

Fig. 2. Theorem XVII.2.8: The Decomposition Lemma

copy of the K-prime model of T in M. If $\eta \in \lambda^{<\omega}$ and $f(\eta)$ is defined, let $\{b_\alpha : \alpha < \mu \leq \lambda\}$ be a maximal subset of M such that:

i) $\{b_\alpha : \alpha < \mu\}$ is independent over M_η.
ii) For each α, $t(b_\alpha; M_\eta)$ is K-strongly regular.
iii) If $\eta \neq \langle\,\rangle$, for each α, $t(b_\alpha; M_\eta) \dashv M_{\eta^-}$.

Now, for $\alpha < \mu$, define $f(\eta^\frown\alpha) = M_\eta[b_\alpha]$. We must show that if $\bar{A}$ is the range of f and $<$ is $\subseteq$ then $(\bar{A}, <, M_{\langle\,\rangle})$ is a K-representation of M. Clearly conditions i)-ii) of the definition are satisfied. We must show M is K-prime over $\bar{A}$. Suppose $N \prec M$ is K-prime over $\bar{A}$. If $N \neq M$ then, as in the proof of Lemma 2.2, there is an $M_\eta \in \bar{A}$ and a $q \in S(M_\eta)$ such that some $b \in M - N$ realizes q^N. Moreover, we can choose η minimal so that $q \dashv M_\nu$ for any $\nu \subsetneq \eta$ or $\eta = \langle\,\rangle$. In particular, $b \downarrow_{M_\eta} N$. Thus if $\{b_\alpha : \alpha < \mu\} \cup \{b\}$ is the set of independent elements associated with M_η in the construction, $\{b_\alpha : \alpha < \mu\} \cup \{b\}$ is independent. If $\eta = \langle\,\rangle$, we clearly contradict the maximality of the b_α. If $\eta \neq \langle\,\rangle$, we must also check that $t(b; M_\eta) \dashv M_{\eta^-}$. But this follows by the minimal choice of η.

2.9 Historical Notes. Most of this material is from [Shelah 1982] and [Harrington & Makkai 1985]. The proof of Theorem 2.8 is from [Harrington & Makkai 1985] and greatly simplifies Shelah's construction (which applies in more general situations). Shelah's argument for the existence of a representation requires an approximation from both above and below. The simplest form of this argument which has no need for prime models appears in [Baldwin & Shelah 1985]. Construction of $A^-[a_A]$ in the general setting but without appealing to prime models is explained in [Hart 1986] and [Shelah 1982], [Shelah 198?]. Lemmas 2.6 and 2.7 which simplify the arguments in Section XVII.4 are taken from [Baldwin & Harrington].

3. Trees, Labeled Trees and Upper Bounds

The major goal of this section is to compute upper bounds on the number of models of various theories which do not have the K-dimensional order property. In the last section we saw that each K-model of such a theory could be represented by a tree. Thus estimating the number of such trees will give this upper bound. In fact, we will be able to refine our estimates by assigning another invariant to the theory, its depth. We begin by discussing the notion of depth and counting the number of trees with a given depth in a purely combinatorial context. Then we will show how to extend the definitions and apply the results in a more general model theoretic context.

3.1 Definition. Let $(A, <, A_{\langle\,\rangle})$ be a tree.

i) We define by induction the *κ-depth* of $a \in A$, denoted $\mathrm{dp}_\kappa(a)$.
For every $a \in A$, $\mathrm{dp}_\kappa(a) \geq 0$.
For any β, $\mathrm{dp}_\kappa(a) \geq \beta$ if for every $\alpha < \beta$ there exists at least κ successors, c, of b with $\mathrm{dp}_\kappa(c) \geq \alpha$.

ii) The κ-depth of $(A, <, A_{\langle\,\rangle}) = \mathrm{dp}_\kappa(A_{\langle\,\rangle})$.

We may omit the subscript κ if the value is clear from context. Note that when $\kappa = 1$ we are dealing with the usual foundation rank of the tree.

Our next step is to calculate the number of trees of width λ and depth α. Of course, the calculation of dp_κ actually also depends on κ. We deal here with the case $\kappa = 1$; other κ are treated in Lemma 4.11. The function which enumerates such trees has a curious discontinuity at depth ω.

3.2 Definition. i) Let $t(\gamma, \alpha)$ denote the following function.

- If $1 \leq \alpha < \omega$, $t(\gamma, \alpha) = \beth_{\alpha-1}(|\gamma|)$.
- If $\alpha \geq \omega$, $t(\gamma, \alpha) = \beth_{\alpha+1}(|\gamma|)$.

ii) Let $T(\gamma, \alpha) = \min(t(\gamma, \alpha), 2^{\aleph_\gamma})$.

The proof of the following lemma is a straightforward induction based on the following observation. For each α, the number of trees of depth α and width $\aleph_\beta$ is the same as the number of functions from the set of trees with depth less than α into the number of cardinals less than $\aleph_\beta$. The most delicate point is to notice that this observation forces the jump at $\alpha = \omega$. We include the proof because several variants of it are discussed later.

3.3 Lemma. *For any α and β, the number of trees $(A, <, A_{\langle\,\rangle}) \subseteq \aleph_\beta^{<\omega}$ with depth at most α is $T(\beta + \omega, \alpha)$.*

Proof. Let $g(\beta, \alpha)$ denote the number of trees $(A, <, \langle\,\rangle) \subseteq \aleph_\beta^{<\omega}$ with depth at most α. As an aid to showing $T(\beta + \omega, \alpha) = g(\beta, \alpha)$, let $T^*(\beta, \alpha)$ denote $\sum_{\gamma<\alpha} T(\beta, \gamma)$ and $g^*(\beta, \alpha)$ denote $\sum_{\gamma<\alpha} g(\beta, \gamma)$. Now for fixed β, we prove the result by induction on α.

If $\alpha = 1$, the number of trees of power at most $\aleph_\beta$ is the number of cardinals less than $\aleph_\beta$, that is, $\beta + \omega = T(\beta + \omega, 1)$. For $\alpha > 1$, each tree, A, of power at most $\aleph_\beta$ with depth at most α is determined by a function, f, from the set of trees of depth $< \alpha$ into the set of cardinals $\leq \aleph_\beta$. Namely, if B is a tree with depth $< \alpha$, $f(B) = \kappa$ if and only if there are κ elements $a \in A$ with height 1 such that $(a)_\geq \approx B$. There are $|\beta + \omega|^{g^*(\beta,\alpha)}$ such functions.

Now if $\alpha < \omega$, $g^*(\beta, \alpha) = g(\beta, \alpha - 1)$ so $g(\beta, \alpha) = |\beta + \omega|^{g(\beta,\alpha-1)} = |\beta + \omega|^{T(\beta+\omega,\alpha-1)} = T(\beta + \omega, \alpha)$. While if $\alpha \geq \omega$, $g^*(\beta, \alpha) = T^*(\beta, \alpha)$ so $g(\beta, \alpha) = |\beta + \omega|^{T^*(\beta,\alpha)} = T(\beta + \omega, \alpha + 1)$.

In order to generalize this remark to calculate an upper bound on the number of K-models of a theory without the DOP, we apply the decomposition theorem to extend the notion of depth from trees to models of such theories. We must study a slightly more complicated combinatorial object, a labeled tree.

If we decompose a model of a theory without the dimensional order property by the procedure in Theorem 2.8 we can assign a depth to each element of the model according to its position in the representing tree. In order to make the discussion uniform over the various models of T we define the depth of a type by induction. There are several minor variations of the definition of depth in the literature. The definition here agrees with that in [Shelah 1982] and [Saffe 1983] but disagrees with [Harrington & Makkai 1985] and [Lascar 1985].

3.4 Definition. Suppose T is a theory without the dimensional order property. Let M be an **S**-model and $p = t(\overline{a}; M)$ be regular and let N be **S**-prime over $M \cup \overline{a}$.

i) We define the *depth* of p by induction as follows:

- $\mathrm{dp}(p) \geq 0$ for all such p.
- If α is zero or a successor ordinal, $\mathrm{dp}(p) \geq \alpha + 1$ if there is some $q \in S(N)$, $q \dashv M$ and $\mathrm{dp}(q) \geq \alpha$.
- If β is a limit ordinal, $\mathrm{dp}(p) \geq \beta + 1$ if for every $\alpha < \beta$, $\mathrm{dp}(p) > \alpha$.

Finally, $\mathrm{dp}(p)$ is the least β such that $\mathrm{dp}(p) \not\geq \beta + 1$ or ∞ if there is no such β.

ii) If $p \in S(A)$ is stationary and regular, let M be **S**-prime over A and set $\mathrm{dp}(p) = \mathrm{dp}(p^M)$.

iii) $\mathrm{dp}(T)$ is one more than the supremum of the $\mathrm{dp}(p)$ for all regular p if this supremum exists. If the supremum exists we say T is *shallow*. If not, T is *deep*.

If q has weight one, let $\mathrm{dp}(q) = \mathrm{dp}(r)$ where r is a strongly K-regular type which is not orthogonal to q. We deduce from Theorem XIII.2.22 and Theorem 3.14 (below) that this assignment does not depend on the choice of r.

We define the *eni-depth* of a type or a theory by a similar induction but with the added condition that the type p has eni-depth ≥ 1 if and only if there is an eni type $q \in S(M[a])$ with $q \dashv M$.

Note that a theory has depth 1 just if it is bounded. Observe that neither a type nor a theory can have limit depth. When computing the depth of a theory remember the type p in clause i) of the definition of depth cannot be algebraic.

3.5 Exercise. Show that the theory of an infinite set has depth 1.

3.6 Exercise. Let $p \in S(M)$ and suppose that a realization $\overline{a}$ of p occurs in the **S**-decomposition $(\bar{A}, <)$ of a model N. Show that $\mathrm{dp}(\overline{a})$ in $\bar{A}$ is at most $\mathrm{dp}(p)$. Show there is a model N and a **S**-decomposition $(\bar{A}, <)$ where equality holds.

The last exercise showed the relation between the depth of a type and the depth of a tree which decomposes a model. Ostensibly, there should be a different notion of depth for each acceptable class K. However, using Lemma XV.1.7 it is possible to prove (cf. [Harrington & Makkai 1985]) that for a countable ω-stable theory computing depth with respect to **S**-decompositions or with respect to **AT**-decompositions gives the same result. Thus, for our purposes the only distinguished case is the eni-depth.

3.7 Exercise. Show that for any shallow theory, T, $\text{eni-dp}(T) \leq \mathrm{dp}(T)$ and the inequality may be strict.

3.8 Examples. Recall from Section XV.3 that there were essentially four types of nonconstant spectra for bounded ω-stable countable theories. The four spectrum functions assigned to $I^*(\aleph_\alpha, AT)$ one of $\alpha+1, \alpha+\omega, (\alpha+1)^\omega$, and $(\alpha+\omega)^\omega$. By choosing $\alpha+1$ to represent the spectra in the finite dimensional case, we are exploiting the group action discussed in Section

XV.3 to simplify the function. We now give four examples of depth two theories with simple spectrum functions. They arise by placing a model of one of the depth one theories in each equivalence class of an equivalence relation with infinitely many classes.

i) Let T_0 be the theory of an equivalence relation with infinitely many infinite classes.
ii) Let T_1 be the theory of an equivalence relation with infinitely many infinite classes, each of which is a model of $\mathrm{Th}(Z, S)$.
iii) Let T_2 be the theory of an equivalence relation with infinitely many infinite classes, each of which contains a model of the theory of infinitely many disjoint unary predicates.
iv) Let T_3 be the theory of an equivalence relation with infinitely many infinite classes, each of which contains a model of the theory of infinitely many disjoint unary predicates where each of the unary predicates contains a model of $\mathrm{Th}(Z, S)$.

These theories are the simplest examples of the following four kinds of theory. T_0 has a finite number of independent isolated depth 0 1-types. T_1 has a finite number of independent depth 0 1-types and at least one is not isolated. T_2 has infinitely many independent depth 0 1-types which are all isolated. T_3 has infinitely many independent nonisolated depth 0 1-types.

The following table indicates the spectrum functions of these four theories. The second column shows the function directly as it is computed from α. The third and fourth columns show that under particular assumptions on α some of the functions coincide.

Theory	$I(\aleph_\alpha, AT)$	*finite* α	*infinite* α
T_0	$\lvert\alpha+\omega\rvert^{\lvert\alpha+1\rvert}$	$\lvert\omega\rvert$	$2^{\lvert\alpha\rvert}$
T_1	$\lvert\alpha+\omega\rvert^{\lvert\alpha+\omega\rvert}$	$2^{\lvert\omega\rvert}$	$2^{\lvert\alpha\rvert}$
T_2	$\lvert\alpha+\omega\rvert^{\lvert\alpha+1\rvert^{\lvert\omega\rvert}}$	$2^{2^{\lvert\omega\rvert}}$	$2^{\lvert\alpha\rvert^{\lvert\omega\rvert}}$
T_3	$\lvert\alpha+\omega\rvert^{\lvert\alpha+\omega\rvert^{\lvert\omega\rvert}}$	$2^{2^{\lvert\omega\rvert}}$	$2^{\lvert\alpha\rvert^{\lvert\omega\rvert}}$

There is one other fundamental example. It is described in detail in Section XVIII.4. The following example shows there are theories without NDOP with each depth up to $|T|^+$.

3.9 Example. Let EER_β denote the theory of β *expanding equivalence relations.* That is, the language contains binary relations E_γ for $\gamma < \beta$ and EER_β asserts that each E_γ is an equivalence relation with infinitely many infinite classes and each $E_{\gamma+1}$ class contains infinitely many E_γ classes. Then EER_β is an ω-stable theory without the dimensional order property. Moreover, $\mathrm{dp}(\mathrm{EER}_\beta) = \beta + 1$ if $\beta < \omega$ and $\mathrm{dp}(\mathrm{EER}_\beta) = \beta + 2$ if $\beta \geq \omega$.

3.10 Exercise. Let T be EER_β and suppose $\beta < \omega$. Show that

$$I^*(\aleph_\alpha, \mathbf{AT}) = \beth_{\beta-1}((\alpha+\omega)^{|\alpha+1|}).$$

3.11 Exercise. Let T be EER$_\beta$ and suppose $\beta \geq \omega$. Show that

$$I^*(\aleph_\alpha, \mathbf{AT}) = \beth_{\beta+1}(|\alpha+\omega|^{|\alpha+1|}).$$

The divergence of results between these two exercises shows the necessity for the peculiar cases in Definition 3.2.

3.12 Exercise. Consider the effect of using the other three cases of Example 3.8 as the 'ground' step of a theory of depth β.

3.13 Exercise. Find an example of a theory with depth $\omega + 1$. (Hint: Consider the disjoint union of theories with finite depth.)

We now show the notion of depth is truly well defined by showing the depth of a strongly regular type p depends only on the parallelism class, indeed, only on the nonorthogonality class of p.

3.14 Theorem. *Let $p \in S(M)$ and $p' \in S(N)$ be regular types where M and N are* **S***-models. If i) p' is a nonforking extension of p or ii) $p' \not\perp p$ then* $\mathrm{dp}(p) = \mathrm{dp}(p')$.

Proof. We prove simultaneously by induction on α that supposing either i) or ii), $\mathrm{dp}(p) \geq \alpha$ if and only if $\mathrm{dp}(p') \geq \alpha$. Suppose we have proved the result for all β less than α. For i), we first show $\mathrm{dp}(p) \geq \alpha$ implies $\mathrm{dp}(p') \geq \alpha$. Choose a realization $\bar{c}$ of p'. Then $M[\bar{c}] \downarrow_M N$. Let $q \in S(M[\bar{c}])$ be orthogonal to M. Then by Theorem VI.2.21, $q \dashv N$. If $\mathrm{dp}(q) < \alpha$, then by induction $\mathrm{dp}(q) = \mathrm{dp}(q^{N[\bar{c}]})$ and $q^{N[\bar{c}]} \dashv N$ so $\mathrm{dp}(p') \geq \mathrm{dp}(q)$. Since we can pick q with each depth $< \alpha$ ($< \beta$ if $\alpha = \beta + 1$ and β is a limit ordinal), $\mathrm{dp}(p') \geq \alpha$. To see $\mathrm{dp}(p') \geq \alpha$ implies $\mathrm{dp}(p) \geq \alpha$, fix $q \in S(N[\bar{c}])$ with $\mathrm{dp}(q) < \alpha$ and $q \dashv N$. Now $N[\bar{c}]$ can be viewed as $N[M[\bar{c}]]$ and $M[\bar{c}] \downarrow_M N$ so by NDOP, $q \not\dashv M[\bar{c}]$ and thus $q \not\perp q_0$ for some regular $q_0 \in S(M[\bar{c}])$. By induction, $\mathrm{dp}(q) = \mathrm{dp}(q_0)$ so $\mathrm{dp}(p) \geq \alpha$.

For part ii), we can invoke i) to assume that $M \prec N$. Then if $\bar{c}$ realizes p', we can find by XII.1.15, using the nonorthogonality and regularity of p and p', a $\bar{d}$ realizing p^N with $N[\bar{c}] \approx N[\bar{d}]$. But then $\mathrm{dp}(p') = \mathrm{dp}(p^N)$ whence by part i) $\mathrm{dp}(p) \geq \alpha$ if and only if $\mathrm{dp}(p^N) \geq \alpha$ if and only if $\mathrm{dp}(p') \geq \alpha$ and we finish.

The following theorem is an immediate corollary of Definition 3.4 and Theorem 2.8.

3.15 Theorem. *If T has the NDOP then every model of T has a normal K-decomposition $(A, <, A_{\langle\,\rangle})$ with* $\mathrm{dp}(A) \leq \mathrm{dp}(T)$.

We would like to obtain an upper bound on the number of K-models of T by saying that Theorem 3.15 defined a function from the set of trees with width at most λ and depth $\mathrm{dp}(T)$ onto the K-models of T with cardinality at most λ. Unfortunately, this isn't quite true. While every K-model of T is K-prime over a normal tree, the model is determined not just by the shape of the tree but by the types that are realized. Thus we have a relation on the class of ordered pairs of trees and models which is not a function. By adding additional structure on the trees we can make this relation into a function.

By the uniqueness of K-prime models, we see that two normal trees $(\bar{A}, <, \bar{A}_{\langle\,\rangle})$ and $(\bar{B}, <', \bar{B}_{\langle\,\rangle})$ represent isomorphic K-models M and N if i) the partial orderings $(\bar{A}, <, \bar{A}_{\langle\,\rangle})$ and $(\bar{B}, <', \bar{B}_{\langle\,\rangle})$ are isomorphic by some isomorphism α and ii) for each $A \in \bar{A}$ and each strongly regular type $p \in S(A)$ such that a successor of A has the form $A[a]$ for some a realizing p, the number of successors of A that are prime over a realization of p is the same as the number of successors of $\alpha(A)$ that are prime over a realization of $\alpha(p)$. We reflect this property in the tree structure by labeling each node $A[a]$ by a unary predicate representing $t(a; A)$.

3.16 Definition. i) A *κ-labeled* tree is a tree $(A, <, A_{\langle\,\rangle})$ such that for each $a \in A$, there is a family $\{U_{a,i} : i < \kappa\}$ of unary predicates and each successor of a satisfies one of these predicates.

ii) A *κ-partially labeled* tree is a tree $(A, <, A_{\langle\,\rangle})$ such that for each $a \in A$ with $\mathrm{dp}_\rho(a) = 1$ there is a family $\{U_{a,i} : i < \kappa\}$ of unary predicates and each successor of a satisfies one of these predicates.

Thus, in a partially labeled tree we label only the 'leaves' or top nodes. Usually the trees can be pruned without loss of generality so that the ρ in this definition can be taken to be 1. Thus, we did not clutter the notation by recording the dependence on ρ.

We will show that it suffices to study partially labeled trees. The following notations make the statement of the result simpler.

iii) Let $L_{\beta,\alpha,\kappa}$ denote the number of κ-labeled trees of depth at most α and power at most $\aleph_\beta$.

iv) Let $P_{\beta,\alpha,\kappa}$ denote the number of κ-partially labeled trees of depth at most α and power at most $\aleph_\beta$.

Note that since all models in, for example, an **S**-representation of a model of a countable theory have cardinality at most $2^{\aleph_0}$, we will always be able to find a κ (depending on K) such that every K-representation corresponds to a κ-labeled tree.

Now an induction like that to prove Lemma 3.3 shows that whether we label all the nodes or just those on the top has no effect on the number of trees. That is,

3.17 Lemma. *For any κ, $L_{\beta,\alpha,\kappa} = P_{\beta,\alpha,\kappa} = T(|\beta + \omega|^\kappa, \alpha)$.*

We tie up the results of this section with the following theorem.

3.18 Theorem. *Let T be a superstable theory without the DOP and with $\mathrm{dp}(T) = \alpha$, then for any β with $\aleph_\beta \geq \lambda_0(\mathbf{I})$*

$$I^*(\aleph_\beta, K) \leq T(|\beta + \omega|^{\lambda_0(I)}, \alpha).$$

Proof. We decompose each model by a depth α tree of models, each with cardinality at most $\lambda_0(I)$. Thus we have a function from the collection of labeled trees with depth α and width at most $\aleph_\beta$ onto the models with cardinality at most $\aleph_\beta$.

3.19 Corollary. *If T is a countable ω-stable theory with infinite depth α*

$$I^*(\aleph_\beta, \mathbf{AT}) \leq T(|\beta + \omega|, \alpha).$$

Proof. Since $\lambda_0(I) = \aleph_0$, the theorem implies: $I^*(\aleph_\beta, \mathbf{AT}) \leq T(|\beta+\omega|^\omega, \alpha)$. But if $\alpha \geq \omega$, $T(|\beta+\omega|^\omega, \alpha) = T(|\beta+\omega|, \alpha)$.

The following exercises refer to the example discussed in the introduction to Chapter XVII. The theory T has a single unary function symbol f; there is a unique point which is mapped to itself by f; all others have infinitely many preimages.

3.20 Exercise. Show that T is ω-stable. (Hint: Show that for any $A \subseteq \mathcal{M}$ and any $m \in \mathcal{M}$ the orbit of m under $\mathrm{Aut}_A(\mathcal{M})$ is determined by a pair $\langle n, a\rangle$ where $\langle n, a\rangle \in \omega \cup \{\infty\} \times A \cup \{*\}$ and $\langle n, a\rangle$ ($\langle \infty, *\rangle$) is assigned to m if $f^n(m) = a$ (there is no such n and a).)

3.21 Exercise. Let $A \subseteq B$ and $a \in M \models T$. If $\mathrm{cl}(a) \cap (\mathrm{cl}(B) - \mathrm{cl}(A)) = \emptyset$ show $a \downarrow_A B$.

3.22 Exercise. Suppose $f(a_0) = f(a_1)$ and $f(a_1) = a_2$. Show $t(a_0; a_1) \dashv a_2$.

3.23 Exercise. Let a be the unique element in $\mathcal{M}$ with $f(a) = a$. Let q_0 denote $t(a; \emptyset)$. For each n find types $p_1 \ldots p_n$ which witness that $\mathrm{dp}(q_0) \geq n$.

3.24 Exercise. Repeat Exercise 3.23 for $q_1 = t(b; a)$ where $f(b) = a$ and $f(a) = a$. Conclude that T is deep.

3.25 Exercise. Show the U-rank of the type q_0 from Exercise 3.23 is ω. Prove no type has higher U-rank.

3.26 Exercise. Show the Morley rank of $x = x$ is ω.

3.27 Exercise. What is the effect on the last four exercises of adding to T the axiom $f^7(x) = f^6(x)$?

3.28 Historical Notes. These results were first proved in [Shelah 1982]. Our presentation was greatly influenced by [Harrington & Makkai 1985] and [Saffe 1981]. The function we have labeled T plays a similar major role in [Baldwin & Shelah 1985]. The proof of Theorem 3.14 by a double induction is due to Saffe [Saffe 1981].

4. Quasi-Isomorphisms and Lower Bounds

In this section we want to compute lower bounds on $I^*(\aleph_\beta, K)$ for a theory T which does not have the DOP and has depth δ. In fact, we will show this function is eventually bounded below by the same function $T(\beta, \alpha)$ which we saw was an upper bound in Section 3. The basic strategy behind such a lower bound argument is simple. We would like to show that if both $(\bar{A}, <, \bar{A}_{\langle\,\rangle})$ and $(\bar{B}, <', \bar{B}_{\langle\,\rangle})$ are K-representations of M then the two trees are isomorphic. We do not need to worry about labeling the trees as in Section 2 if we can achieve the desired lower bound with unlabeled trees. It turns out that for small depths and cardinals less than $\aleph_\omega$ we must take note of the labeling.

Unfortunately, we are unable to guarantee that the map from models to skeletons is actually a function. There may be non-isomorphic normal representations $\bar{A}$ and $\bar{B}$ of a model M. We show, however, that two trees which represent the same model are quasi-isomorphic and that there are sufficiently many non-quasi-isomorphic trees to calculate the spectrum. We can not even attain this result for all models. But we will obtain it for a large enough subset to calculate the number of models when $\aleph_\beta \geq \aleph_\omega$ or the depth of T is infinite. After some preliminary analysis, we defer to the next chapter the exact computation of the lower part of the spectrum for theories with finite depth.

A quasi-isomorphism is simply a correspondence between two trees which is $1-1$ and preserves order but is defined almost everywhere, rather than everywhere. More formally, we say

4.1 Definition. Two trees $A = (A, <, A_{\langle\,\rangle})$ and $B = (B, <, B_{\langle\,\rangle})$ are *μ-quasi-isomorphic*, denoted $A \approx_{q,\mu} B$, if there is a 1-1 relation $h \subseteq A \times B$ such that:

i) If $\langle a, b\rangle$ and $\langle a', b'\rangle$ are in h then $a < a'$ if and only if $b < b'$.
ii) For all $a \in A$, the number of successors of a which are not in $\operatorname{dom} h$ is less than μ.
iii) For all $b \in B$, the number of successors of b which are not in $\operatorname{rng} h$ is less than μ.
iv) If $\operatorname{dp}(a) = 1$ and $\langle a, b\rangle \in h$ then for each successor a' (b') of a (b) there is a successor b' (a') of b (a) with $\langle a', b'\rangle \in h$.

We write $A \approx_q B$ if the μ-ample (see below) trees A and B are μ quasi-isomorphic. In the relevant cases (μ-ample) we will be able to deduce from condition i) that the b referred to in condition iv) also has depth 1.

First, we construct from two K-representations of the same model a pair of $\lambda_0(I)^+$-quasi-isomorphic trees. Then we turn to the question of finding enough trees which are not quasi-isomorphic. We require several lemmas for the first task.

For any tree $\bar{A}$, we denote by $\bar{A}^1$ the subtree consisting of those nodes with nonzero depth. Given representations $\bar{A}$ and $\bar{B}$ of a properly constructed model M, we will first define a bijection between between $\bar{A}^1$ and $\bar{B}^1$ by sending A to $\hat{A}$ if the type of some successor of A over A is nonorthogonal to the type of some successor of $\hat{A}$ over $\hat{A}$ We will show that for an appropriate choice of a model $M_{A\hat{A}}$ containing A and $\hat{A}$, forking determines the same bijection (between 'most' of the successors of A and 'most' of the successors of $\hat{A}$.) To see that this bijection preserves order, we first note that Theorem VI.2.21 shows that order is preserved on the two levels immediately above A and $\hat{A}$. Then we extend this result to the cones above A and $\hat{A}$ by induction on height above A or $\hat{A}$ and the properties of normal representations established in Section 2.

The following notation allows us to define a tree 'lying above' an ideal I. Note that the new tree $\bar{A}_I$ is not a subtree of $\bar{A}$ but is obtained by shrinking an intial segment of $\bar{A}$ to a point. Lemma 4.3 relies heavily on the fact obtained in Theorem 1.10: $M_I \downarrow_I A$.

4.2 Notation. Let I be an ideal in the normal tree $\bar{A}$. We will define a new (normal) tree $\bar{A}_I$. The base node in the new tree is M_I. The other nodes will be B' for $B \in \bar{A} - I$ where B' is defined by the following induction. If $B \in I^+$, $B' = M_I[B]$. If $B \in I^{n+1}$ (The $n+1$st predecessor of B is in I) then $B' = (B^-)'[B]$. $\bar{A}_I = \{B' : B \in \bar{A} - I\}$.

4.3 Lemma. *$\bar{A}_I$ is a normal tree.*

Proof. The first three conditions of Definition 1.3 are clear. It is easy to conclude iv) and v) by induction. We give only the first stage of the induction since it contains all the ideas. For iv), note that if $A \in I^+$, then $J = \{B : B^- = A^- = C\}$ is independent over C and independent from I over C. Thus, $\{M_I[B] : B \in J\}$ is independent over M_I.

For v), let $A^{--} \in I$. Then $A \cup A^- \downarrow_{A^{--}} I$. Thus $A \cup A^- \downarrow_{A^{--}} M_I$ and so $A \downarrow_{A^-} M_I[A^-]$. Now $t(A; A^-) \dashv A^{--}$ implies $t(A; A^-) \dashv M_I$. Thus, since orthogonality is preserved by nonforking extensions, $t(A; M_I[A^-]) \dashv M_I$. Since $A \triangleright_{M_I[A^-]} M_I[A^-][A]$, we conclude $t(M_I[A^-][A]; M_I[A^-]) \dashv M_I$.

In the following theorem we construct a quasi-isomorphism between the nodes of positive depth in two K-repesentations of a K-model M. [Harrington & Makkai 1985] constructs a quasi-isomorphism between the entire representing tree. However, the argument here is much simpler and suffices for much of the spectrum computation. For the remainder we must label the leaves on the trees.

A quasi-isomorphism can fail to be an isomorphism in two ways. First, it is not defined everywhere. Second, it does not preserve height. The first of these problems can be circumvented by dealing with sufficiently bushy trees. The second requires several special tricks. We begin with the first problem. Recall from Definition 3.1 the concept of κ-depth ($\mathrm{dp}_\kappa(a)$) of a node in a tree.

4.4 Definition. A tree $(A, <)$ is called *μ-ample* if for every $a \in A$, $\mathrm{dp}_1(a) = \mathrm{dp}_\mu(a)$.

The following easy result shows that there are the maximal number of non-quasi-isomorphic μ-ample trees. To prove it, we can, for example, specify the set of depths of elements of height 1 in each tree.

4.5 Lemma. *For every $\lambda \geq \mu$, there are 2^λ non-isomorphic, μ-ample trees of width λ.*

4.6 Exercise. Show that if h is a μ-quasi-isomorphism between the μ-ample trees $\bar{A}$ and $\bar{B}$ with $h(A) = B$, the μ depth of A in $\bar{A}$ is the same as the μ depth of B in $\bar{B}$.

We show now that by specifying some easily fulfilled conditions on the representing trees we can guarantee that if two trees represent the same model then the trees obtained by peeling off the leaves are quasi-isomorphic. The key to applying this result is that if $\bar{A}$ has infinite depth, the derived tree has the same depth. Another important idea in this proof is the fact that for each $A \in I^+$, A and a_A are bidominant over M_I. Thus, we can use the triviality of $t(a_A; M_I)$ to deduce properties of A.

4.7 Theorem. *Suppose $\mu \geq \lambda_0(\mathbf{I})^+$ and that $\lambda(\mathbf{I})$ is regular. Let the μ-ample trees $\bar{A}$ and $\bar{B}$ be K-representations of the K-model M. Suppose further that only one nonorthogonality class of K-strongly regular types over A is represented in $\bar{A}$, for each $A \in \bar{A}$ (and similarly for $\bar{B}$). Let $\bar{A}^1$ ($\bar{B}^1$) denote the nodes in $\bar{A}$ ($\bar{B}$) which have depth at least one. Then $\bar{A}^1 \approx_q \bar{B}^1$.*

Proof. (Fig. 3). We first define a map $A \mapsto \hat{A}$ which is a bijection between $\bar{A}^1$ and $\bar{B}^1$ but may not preserve order. Then we show that by throwing away from the domain (range) of the map less than μ successors of any node, we can restrict our map to one which preserves order. Thus, the restricted map is a quasi-isomorphism on the nodes of positive depth. Define $h: A \mapsto \hat{A}$ if for some successor A_1 of $A \in \bar{A}$ and some $B_1 \in \bar{B}$, with $B_1^- = \hat{A}$, $t(A_1; A) \not\perp t(B_1; \hat{A})$. By Lemma 2.8, the transitivity of nonorthogonality on regular types, and the fact that only one nonorthogonality class over A ($\hat{A}$) is realized in M, this map is a bijection.

Fix $A, \hat{A}$. Choose a model $M_{A\hat{A}}$ with $|M_{A\hat{A}}| \leq \mu$ such that $A, \hat{A} \in M_{A\hat{A}}$ and for some ideal $I \subseteq \bar{A}$ and some ideal $J \subseteq \bar{B}$, $M_{A\hat{A}}$ is prime over each of I and J. This model can be chosen by a back and forth. First choose a model, M_0, prime over an ideal of $\bar{A}$ which contains $\hat{A}$ and then an M_1 prime over an ideal of $\bar{B}$ which contains $A \cup M_0$ and so on. Since the property of being an elementary submodel has finite character, this process terminates in ω steps. By Corollary 1.11 the union of the resulting chain satisfies the conditions.

Less than μ successors of either A or $\hat{A}$ are contained in $M_{A\hat{A}} = M_I$. Prune $\bar{A}^1$ and $\bar{B}^1$ by discarding for each A ($\hat{A}$) with depth at least three in $\bar{A}$ ($\bar{B}$) its successors in $M_{A\hat{A}}$. To preserve symmetry discard $h(A)$ if A is discarded and vice versa. In order to show that h preserves order on the pruned trees it suffices to show the following condition. If A_1 was not pruned where $A_1^- = A$ and $A_2 > A_1$ then $h(A_2) > h(A_1)$.

By Theorem 1.10, $\bar{A} \downarrow_I M_I$. Since $A_1 \in \bar{A}$, $A_1 \downarrow_I M_I$. As A_1 does not depend on I over A, we have $A_1 \downarrow_A M_I$. The analogous result holds for the successors of $\hat{A}$. Now by Lemma 2.7, J^+ is a maximal independent subset of realizations of K-strongly regular types over M_J. Since $M_J = M_{A\hat{A}} = M_I$ and J^+ dominates M over M_J, each $A_1 \in I^+$ depends on J^+ over $M_{A\hat{A}}$. By triviality and weight one, there is a unique $\tilde{A}_1 \in J^+$ such that $A_1 \not\downarrow_{M_{A\hat{A}}} \tilde{A}_1$. Now for any other $B \in J^+$ we have $B \downarrow_{B^-} M_{A\hat{A}}$. By Lemma 4.3 we have normal trees $\bar{A}_I$ and $\bar{B}_J$ each with initial node $M_{A\hat{A}}$. Let $(A_1)_{\geq}$ denote

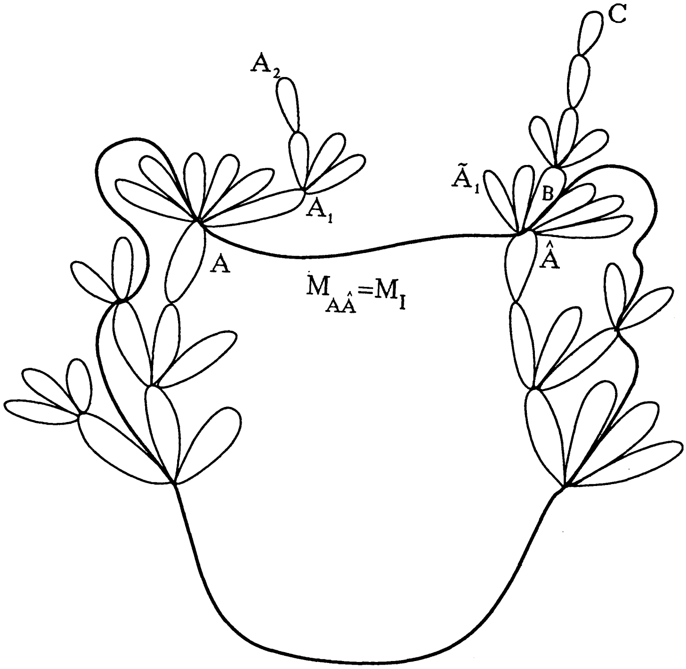

Fig. 3. Theorem XVII.4.7: The construction of quasi-isomorphisms

the cone above A_1 in $\bar{A}_I$ and $(B)_{\geq}$ denote the cone above B in $\bar{A}_J$. Since $A_1 \downarrow_{M_{A\hat{A}}} B$, we have a normal tree $M_{A\hat{A}} \cup (A_1)_{\geq} \cup (B)_{\geq}$. By Lemma 1.7 we see that for any $A_2 > A_1$, and any $C > B$, $t(A_2; A_2^-) \perp t(C; C^-)$. Thus, $h(A_1) = \tilde{A}_1$ and $h(A_2) \geq \tilde{A}_1$. This shows that h preserves order almost everywhere.

We must still verify condition iv) of Definition 4.1 by showing that if A_2 has depth one in $(\bar{A})^1$ and A_2 is in the restricted domain of h, then h is a bijection between the successors of A_2 and the successors of $h(A_2)$. If A_2 is in the restricted domain of h, neither A_2 nor $h(A_2)$ was pruned in the construction. Then, for any successor A_1 of A_2, the argument in the last paragraph shows $h(A_1) > h(A_2)$. Note that for any $A \in \bar{A}$ there is a chain of length n above $A \in \bar{A}$ if and only if there is chain of length n above $h(A)$ in $\bar{B}$. Thus, $h(A_2)$ has depth two in B. Since $h(A_1)$ has depth at least one in B, $h(A_1)$ is an immediate successor of $h(A_2)$.

The fact that a quasi-isomorphism preserves depth is implicit in the argument for condition iv). In general, a quasi-isomorphism need not preserve height.

Now we can obtain the first precise result on the spectrum problem of NDOP theories.

4.8 Theorem. *If T is superstable and has the NDOP but is deep then for every uncountable λ, $I^*(\lambda, K) = 2^\lambda$.*

Proof. Since T has infinite depth, for any $\lambda_0(I)$-ample tree, $\bar{A}$, of depth less than λ^+, we can construct a model M with cardinality λ which is K-prime over a tree isomorphic to $\bar{A}$. If two such trees represent the same model they are quasi-isomorphic by a quasi-isomorphism constructed as in Theorem 4.7. Without loss of generality we can assume that all height 1 types in $\bar{A}$ and $\bar{B}$ are based on a single finite set D. Naming D does not affect the number of models of T in an uncountable cardinal. But since the quasi-isomorphism is given by nonorthogonality and all types in $\bar{A}$ ($\bar{B}$) except those of elements with height 1 are orthogonal to D, we see h must map elements of height 1 to elements of height 1. But since it is easy to construct trees $\bar{A}_X$ for $X \subseteq \lambda$ so that for $A \in \bar{A}_X$ with height 1, $\mathrm{dp}(A)$ in $\bar{A}_X$ is η if and only if $\eta \in X$, we conclude there are 2^λ nonisomorphic models.

The next exercise generalizes one of the major ideas from the previous theorem, the importance of a quasi-isomorphism preserving height. Although this result is crucial for the rest of the argument, its straightforward but tedious proof is omitted.

4.9 Exercise. Show that if there is μ-quasi-isomorphism between two μ-ample trees which preserve height then they are isomorphic.

Thus, we need to find some trick to make quasi-isomorphisms preserve height. We will use two different devices. In, e.g. the ω-stable case, the first works when computing the number of models of power κ for $\kappa \geq \aleph_\omega$. If this trick is not sufficient we will use the second trick and some further analysis.

For the first trick, we restrict the tree $\bar{A}$ by choosing for each $n < \omega$ a set of cardinals, X_n, and requiring that the number of successors of a node of height n and depth 1 be a cardinal in X_n. The next definition describes the kind of trees we will map onto and the function which counts them. We introduce the new parameter γ here to handle the superstable and ω-stable cases simultaneously. We will need to choose γ so that $\aleph_\gamma = \lambda_0(I)$. Thus, for a countable superstable theory, we need $\aleph_\gamma = 2^{\aleph_0}$ and for a countable ω-stable theory, $\gamma = 0$.

4.10 Definition. Let $\bar{X} = \langle X_n : n < \omega \rangle$ be a family of sets of cardinal numbers which lie between $\aleph_\gamma$ and $\aleph_\beta$ such that the X_n are pairwise disjoint $\{\aleph_{\gamma+1}, \aleph_\beta\} \subseteq X_1$, and for each n $|X_n| \geq 2$. The tree $(A, <)$ is *constrained* by $\bar{X}$ if for each $a \in A$ with $\mathrm{ht}(a) = n$ and $\mathrm{dp}(a) = 1$, $|\{b : b^- = a\}| \in X_n$.

The usual kind of induction allows us to count the number of constrained trees of power $\aleph_\beta$. In the ensuing lemma we fix $\aleph_\gamma$ as $\lambda_0(\mathbf{I})$. The constraint on the ordinal β is satisfied by all infinite β when $\lambda_0(\mathbf{I}) = \aleph_0$.

4.11 Lemma. *If $\beta \geq \gamma + \omega$ and α is infinite there are $T(\beta, \alpha)$ trees of power $\aleph_\beta$ which are $\bar{X}$-constrained and $\aleph_{\gamma+1}$-ample but have depth at most α.*

Proof. Although the statement of the theorem is for infinite α, the key to the proof is the analysis for finite α. Since $\beta \geq \gamma + \omega$, we can find at least two cardinals between $\aleph_\gamma$ and $\aleph_\beta$ to put in each X_n. Note, then, that there are at least two of the desired trees with depth one, $|\beta + \omega|$ of depth two, and $T(|\beta + \omega|, n - 1)$ of depth n for each finite $n \geq 2$. Thus, there are $T(\beta, \omega)$ of depth ω. The rest of the induction is routine.

The proof of Lemma 4.11 illustrates an important point in the actual calculation of the number of models. While one may not be able to guarantee that every cardinal less than $\aleph_\beta$ is possible as the number of successors of an arbitrary node (either because of ampleness or some defect in coding), this problem can be surmounted if the tree has infinite depth. The following two exercises emphasize this point.

4.12 Exercise. Complete the proof of Lemma 4.11. (Hint: Consider the depth two case. Suppose $\{\kappa_0, \kappa_1\} \subseteq X_1$. Each function f from 2 to the set of $\kappa < \aleph_\beta$ with $\aleph_\beta \in \operatorname{rng} f$ determines an appropriate tree by assigning for i equal 0 or 1 κ_i successors to $f(i)$ nodes of height one.)

4.13 Exercise. Show that for the case of an ω-stable theory, where $\gamma = 0$, the definition of constrained can be altered to require each $|X_n| = |\beta|$. With this alteration Lemma 4.11 holds for finite α but the change does not suffice to prove Theorem 4.14 for finite α.

Now we can compute the spectrum when both β and α are sufficiently large. Note that if T is a countable ω-stable theory, γ in the following theorem is 0 and so we are just requiring β and α to be infinite. The arithmetic of the following argument depends on two observations. If $\alpha \geq \lambda_0(\mathbf{I})$ then $T(|\beta + \omega|^{\lambda_0(\mathbf{I})}, \alpha) = T(\beta, \alpha)$. If a tree $\bar{A}$ has infinite depth then $\bar{A}^1$ has the same depth.

4.14 Theorem. *Let $\aleph_\gamma = \lambda_0(\mathbf{I})$. Suppose $\beta \geq \gamma + \omega$. If T is a superstable theory, the NDOP holds and* $\operatorname{dp}(T) = \alpha$ *is infinite then $I^*(\aleph_\beta, K) = T(\beta, \alpha)$.*

Proof. From Lemma XVII.3.17, we have the upper bound $T(|\beta + \omega|^{\aleph_\gamma}, \alpha)$. If two $\aleph_{\gamma+1}$-ample trees $\bar{A}$ and $\bar{B}$ represent the same model and we denote by $\bar{A}^1$, $\bar{B}^1$ the nodes with depth at least one then by Theorem 4.7 $\bar{A}^1$ and $\bar{B}^1$ are quasi-isomorphic. If we insist the trees are constrained the quasi-isomorphism preserves height. Applying the second observation before the theorem we see that a lower bound is the number of $\bar{X}$-constrained, $\aleph_{\gamma+1}$-ample trees of power $\aleph_\beta$ and depth α. By Lemma 4.11, this number is $T(\beta, \alpha)$. Since $\alpha \geq \aleph_\gamma$, this equals $T(|\beta + \omega|^{\aleph_\gamma}, \alpha)$ and we finish.

It remains to compute the spectrum for powers $\aleph_\beta$ when $\beta < \gamma + \omega$. For concreteness we restrict the theorems to counting all models of a countable ω-stable theory. We will not obtain the full solution until Section XVIII.5. However, we can solve the problem for theories of infinite depth with the methods we have now. We need the following sharper estimate on the number of ample trees.

4.15 Lemma. *For any $1 \leq m < \omega$ there are $2^{\aleph_m}$ partially 2-labeled trees of power $\aleph_m$ which are $\aleph_1$-ample and have depth less than $m+2$.*

Proof. For any $k > 0$, there are $\aleph_0$ partially 2-labeled trees of cardinality $\aleph_k$ and depth one. Namely, let the nth tree have n nodes of height one labeled by U_0 and $\aleph_k$ labeled by U_1. Since any uncountable tree of depth one is $\aleph_1$-ample, each T_n is. By forming for any subset $X \subseteq \omega$ a tree M_X such that if $a \in M_X$ has height one then $(a_\geq) \approx T_n$ for some $n \in X$ and such that each such subtree occurs $\aleph_1$ times, we create $\beth_1(\aleph_0)$ partially 2-labeled trees which are $\aleph_1$-ample and have depth two. Continuing inductively, there are $\beth_m(\aleph_0)$ partially 2-labeled trees which are $\aleph_1$-ample, have cardinality $\aleph_k$ and depth at most $m+1$. Since $\beth_{m+2}(\aleph_0) \geq 2^{\aleph_m}$ this yields the lemma.

4.16 Theorem. *If T is an ω-stable countable theory without the dimensional order property and* $\mathrm{dp}(T) \geq \omega$ *then for $\beta < \omega$, $I^*(\aleph_\beta, \mathbf{AT}) = 2^{\aleph_\beta}$.*

Proof. Fix $\beta = m$ with $1 \leq m < \omega$. Choose $n = m+2$. For each partially 2-labeled $\aleph_1$-ample tree S of height n we build a normal tree $\bar{A}_S$ of height $n+1$ with $(\bar{A}_S)^1$ isomorphic to S. Each A in $\bar{A}_S$ will have only a single type based on it (to guarantee the uniformity condition in the hypothesis of Theorem 4.7). If the leaf $A \in S$ is labeled by U_0, A will have $\aleph_0$ successors in $\bar{A}_S$; if the label is U_1 the node will have $\aleph_k$ successors in $\bar{A}_S$. We require that if $\mathrm{ht}(A) = k$, then $\mathrm{dp}(t(A; A^-)) = n - k$. Then the quasi-isomorphism between two representations of the same model will preserve height. By Theorem 4.7 if $\bar{A}_{S_0}$ and $\bar{A}_{S_1}$ both represent the same model, $(\bar{A}_{S_0})^1 \approx_q (\bar{A}_{S_1})^1$. Thus, they are isomorphic as trees. To see that the isomorphism preserves labels we need only note that if $h(A_0) = B_0$ then there exist A_0' and B_0' with $t(A_0'; A_0) \not\perp t(B_0'; B_0)$. Since nonorthogonal types in an ω-stable theory have the same dimension modulo $\aleph_0$, A_0 and B_0 have the same label. By Lemma 4.15 we obtain a lower bound of $2^{\aleph_m}$ on the number of models of power $\aleph_m$. Since this is the crudest upper bound, we complete the proof of the theorem.

4.17 Exercise. Let T be a superstable countable theory without the dimensional order property and $\mathrm{dp}(T) \geq \omega$. Show that if $\aleph_\gamma = \lambda_0(\mathbf{S})$ then for $\gamma < \beta < \gamma + \omega$, $I^*(\aleph_\beta, \mathbf{S}) = 2^{\aleph_\beta}$.

There are two difficulties with the method of proof of the last theorem. First, the requirement of considering trees of depth $m+2$ makes it inherently unsuitable if we are trying to find precise estimates for the number of models of a theory with depth m. Secondly, the methods used fail if we try to label finite dimensions. Both of these problems are resolved in the complete solution of the spectrum problem for finite depth theories in Section XVIII.5. The key to the solution is to continue to build $\aleph_1$-ample trees but to preserve labels for finite dimensions. In addition to the methods described here, that proof relies on Bouscaren's analysis in [Bouscaren 1984].

4.18 Historical Notes. These results were originally proved by Shelah in [Shelah 1982]. Our account owes a great deal to the expositions of [Harrington & Makkai 1985] and [Saffe 1983]. The present proof of Theorem 4.7 on the existence of a quasi-isomorphisms is from [Baldwin & Harrington]. The trees discussed in this section play an important role in [Baldwin & Shelah 1985] and in that paper there are other applications of the counting functions discussed here. The case treated in Theorem 4.16 was not discussed explicitly in either [Harrington & Makkai 1985] or [Shelah 1982], but was in [Saffe 1981].