Part B

Dependence and Prime Models

In this part of the book we focus on ways to 'generate' a model of a theory. Of course, the most natural notion of generation is closure under functions. However, we are interested in constructing elementary submodels and it is easy to see that the addition of Skolem functions destroys many model theoretic properties including, in particular, the stability hierarchy. For example, there exist ω-stable theories with no complete ω-stable Skolemization. Thus we turn to more general notions. One such idea is known to model theorists as the 'algebraic closure' of a set A; it consists of those points which lie in finite sets defined by formulas with parameters from A. Except in the extreme case of a strongly minimal set, this notion lacks the exchange (symmetry) property of a good dependence relation. A next attempt is to adjoin those points which realize a principal type over A. Here too, the full symmetry property is lacking but a reasonable substitute (cf. Section IX.3) holds. However, the requirement that every set have a closure with respect to this relation which is a model of T is extremely strong. In fact, the only known nontrivial sufficient condition for every set to have a closure with respect to this relation is the assumption that the theory is countable and ω-stable. It is necessary to consider some further variants in order to find a notion of closure which is both strong enough to be useful and which applies to a wide class of theories. In this part of the book we give a general treatment of several such properties.

Chapter IX
Atomic and Prime Models

The first four chapters of this book discussed generalizations of the notion of independence. We remarked in Chapter II that one feature of this generalization is that we can no longer define 'dependent' as merely 'not independent'. We must instead develop an apparatus to study 'generation'. Since the early 1960's it has been clear that the proper notion of 'submodel generated by A in the category of structures and elementary embeddings' is 'model prime over A'. It turns out, however, that in classifying the models of a first order theory, it does not suffice to investigate arbitrary models of the theory; the detailed study of saturated models of several sorts is essential. For each of these classes of models an appropriate notion of generation (of 'prime over') must be developed. This chapter is devoted to developing such notions.

We will discuss various concepts of prime and atomic models in this book. 'Prime' is a category theoretic concept: M is prime over A in the category K if every K-embedding of A into a member N of K can be extended to a K-embedding of M into N. 'Atomic' is a logical notion. Let $\mathbf{I}$ be a collection of types. Then, M is $\mathbf{I}$-atomic over A if for every $\overline{m} \in M$, there is some $q \in I$ with $q \subseteq t(\overline{m}; A)$ such that $q \vdash t(\overline{m}; A)$. Let K be the category of models of a countable first order theory T with elementary embeddings as morphisms and let $\mathbf{AT}$ be the collection of single formulas. It was realized in the early 1960's that the concepts, 'K-prime over $\emptyset$' and 'countable and $\mathbf{AT}$-atomic over $\emptyset$' are identical. Vaught used this equivalence to prove theorems characterizing prime models and proved that any two prime models of a countable theory are isomorphic. Morley noticed that the atomic component of the equivalence is useful in omitting types and applied this observation in proving his categoricity theorem.

A third aspect of 'primality' turns out to play an important role. Morley's construction of atomic models produces not only a structure which is atomic but one which has been built up in a specific way. This construction of the model corresponds to what we call a 'strictly prime' model. The proof of the uniqueness of strictly prime models is comparatively straightforward. If the countable theory T is stable and admits prime models over all sets we will prove in Chapter X the more difficult assertion that prime and strictly prime coincide and thus deduce the uniqueness of prime models.

In the first section below we sketch the results for the 'classical' case. Then we discuss the situation from a more abstract point of view. In Sections 2 and 3 we develop a set of axioms for primality (or generation) to complement the earlier axioms for independence. In Section 4 we prove the uniqueness of strictly prime models for all our notions simultaneously. In Chapter X we use the properties of independence to show the uniqueness of prime models for these notions in a countable stable theory.

The last two sections of this chapter are devoted to two notions of isolation which do fit precisely into the general scheme of isolation notion we developed in Section 2. In one sense, the failure of these notions to fit the more general rubric is purely technical; they do fit in the scheme of [Shelah 1978]. But there is a thematic reason for the difference. Neither of these notions of 'isolation' admit a natural notion of 'prime'. Section 5 concerns locally atomic models. We show that one can find a locally atomic model over any set if T is a stable theory. We apply this concept in Section 5 to the proof of two cardinal theorems. In Section 6, we describe the notion of **F**-isolation and use it to show that a stable but not superstable theory has 2^λ models of power λ in each uncountable cardinality λ.

1. Elementarily Prime Models

We deal in this section with prime models over sets for countable theories. We first develop Vaught's characterization of prime models over the empty set as atomic models. Then we turn to the existence question for prime models over arbitrary sets. There are two ways to find prime models over the empty set (or indeed over arbitrary countable sets). The first, which appears in [Vaught 1961], uses the omitting types theorem to construct a model which omits the nonprincipal types and thus is atomic. This method depends essentially on the countability hypothesis. The second method, which is used almost exclusively here, is to build up the model guaranteeing that it is atomic during the construction. This method was introduced by Morley [Morley 1965] and this entire chapter is devoted to amplifications of the idea.

1.1 Definition. i) The model M is *prime* over A if every elementary embedding of A into a model N of T can be extended to an elementary embedding of M into N.

ii) The model M is a *prime model* of T if M is prime over the empty set.

iii) The theory T *admits prime models* if there is a prime model over every set A.

For example, the algebraic numbers are the prime model of the theory of algebraically closed fields; the natural numbers are the prime model of 'true arithmetic'. This purely categorical definition is expressed by the following diagram (Fig. 1). However, there are equivalent definitions of a

$$\begin{tikzcd} M \arrow[r, dashed] \arrow[d, phantom, sloped, "\supseteq"] & N \\ A \arrow[ur] & \end{tikzcd}$$

Fig. 1. M is prime over A.

logical character. We need some further notions in order to express them. Recall from Chapter I the notion of an isolated type.

1.2 Definition. Let $p \in S(A)$. Then p is *principal* or *isolated* if there is a single formula $\phi(x;\overline{a}) \in p$ such that $\phi(x;\overline{a}) \vdash p$. That is, for all $\psi \in p$, $\mathrm{Th}(A) \vdash (\forall\overline{x})[\phi(\overline{x};\overline{a}) \rightarrow \psi(\overline{x})]$.

Note that this is exactly the same as saying p is an isolated point in the topology on $S(A)$ induced by Stone duality, that is, the topology where each basic open set is of the form $U_\phi = \{p : \phi \in p\}$.

We denote by **AT** the set of all single first order formulas (with parameters anywhere in $\mathcal{M}$).

1.3 Definition. The model M is *atomic* over A if for each finite sequence $\overline{m} \in M$, $t(\overline{m}; A)$ is an isolated type. M is *atomic* if M is atomic over the empty set.

The immediate significance of these definitions is given by

1.4 Theorem. *M is a prime model of T if and only if M is atomic and countable. Moreover, any two countable atomic models are isomorphic so if T has a prime model it is unique.*

Proof. For one half of the theorem, we rely on the omitting types theorem. If M realizes a nonprincipal type then M is not prime as there is also a model N of T which omits p. For the converse, we are given a countable atomic model M and a model N; we want to elementarily embed M in N. Let $M = \{m_i : i \in \omega\}$. Now construct the embedding f by induction. Say $\phi_0(x)$ generates $t(m_0;\emptyset)$. Since T is complete $T \models (\exists x)\phi_0(x)$. Let $f(m_0)$ be any element of N such that $N \models \phi_0(m_0)$. Suppose we have defined f on $\{m_i : i < n\}$ so that $(M, m_0, \ldots, m_{n-1}) \equiv (M, f(m_0), \ldots, f(m_{n-1}))$. Let $\phi(x_0, \ldots, x_n)$ generate $t(m_0, \ldots, m_n;\emptyset)$. By the induction hypothesis, $N \models (\exists y)\phi(f(m_0), \ldots, f(m_{n-1}), y)$. Choose any witness for this formula as $f(m_n)$. To establish the 'moreover' clause, replace this 'forth' argument by a 'back and forth' argument.

1.5 Exercise. Prove the moreover clause in the last theorem.

1.6 Exercise. Show that if M_1 and M_2 are prime models over the countable set A then $M_1 \approx M_2$ by an isomorphism which fixes A.

We have shown that any theory T has at most one prime model over the empty set. The attempt to describe those theories which have prime models over the empty set leads easily to a sufficient condition (Definition 1.13) for a theory to have a prime model over any set; the uniqueness question is more difficult. We now consider how to construct atomic models over sets. The actual construction plays a vital role. That is, we will show in Section 5 that any two prime constructible models are isomorphic, while leaving open the the possibility that there are prime models which are not constructible. An argument like that for Theorem 1.4 shows that for any set A (even uncountable) two atomic models over A are $L_{\infty,\omega}$-equivalent over A. No such general argument can show the models are isomorphic (even assuming they have the same cardinality). In Chapter X, invoking the notions of freeness as well as isolation, we will show that if T is stable and has prime models over all sets then prime models over sets are unique.

1.7 Definition. A *construction* C over a set A is a sequence $\langle c_j : j \in J\rangle$ such that for each j, $t(c_j; A \cup C_j)$ is isolated. The set B is *constructible* over A if $B = A \cup \langle c_j : j \in J\rangle$ for some construction C.

It is somewhat anomalous that the set B which is constructible over A contains A while the actual range of the constructing sequence may not. The following exercise shows this is only an apparent difficulty.

1.8 Exercise. Show that if B is constructible over A by C, there is a construction C' over A such that $\{c'_j : j \in J\} = B$.

It is essential to remember that a construction is a sequence and not just a set. For example, any embedding of a sequence of order type ω into a model of the theory of dense linear order (without endpoints) is a construction but an embeddding of a sequence of order type $\omega + 1$ is not.

1.9 Lemma. *If B is constructible over A then any elementary monomorphism of A into a model of T can be extended to an elementary embedding of B into that model. Thus, if M is a model which is constructible over A then M is prime over A.*

1.10 Exercise. Prove Lemma 1.9.

Our next goal is to show that if B is constructible over A then B is atomic over A. This relies on the following key property of isolation.

1.11 Lemma (Transitivity Property). *Let $B \subseteq A$. If $t(\overline{d}; A)$ is isolated and for each $\overline{c} \in A - B$, $t(\overline{c}; B)$ is isolated then for each $\overline{c}_1 \in A - B$, $t(\overline{d}^\frown\overline{c}_1; B)$ is isolated. In particular, $t(\overline{d}; B)$ is isolated.*

Proof. For some $\overline{c} \in A - B$, let $\phi(\overline{x}; \overline{c}) \in F(B)$ generate $t(\overline{d}; A)$. Fix $\overline{c}_1 \in A$ and suppose $\psi(\overline{y}_1{}^\frown\overline{y}_2)$ generates $t(\overline{c}_1{}^\frown\overline{c}; B)$. Now, the formula

$$(\exists \overline{y}_2)(\phi(\overline{x}; \overline{y}_2) \wedge \psi(\overline{y}_1; \overline{y}_2))$$

generates $t(\overline{d}^\frown\overline{c}_1; B)$. For, suppose $\chi \in F(B)$ and $\models \chi(\overline{d}^\frown\overline{c}_1)$. Then,

$$\models \psi(\overline{y}_1; \overline{y}_2) \rightarrow (\overline{x})(\phi(\overline{x}; \overline{y}_2) \rightarrow \chi(\overline{x}; \overline{y}_1)).$$

Since it is clear that a union of a chain of structures atomic over A is atomic over A, we can rephrase this as

1.12 Theorem. *If C is constructible over A then C is atomic over A.*

We have seen how to extend a set A to a set B which is atomic over A. Can we find a B which is the universe of a model and with $|B|$ not much greater than $|A|$? The following property of a theory guarantees that such a model B exists with $|B| = |A| + |T|$.

1.13 Definition. The *isolated types are dense* in $S(A)$ if for each formula $\phi(x; \overline{a})$ there is a principal type $p \in S(A)$ with $\phi(x; \overline{a}) \in p$. The theory T is *atomistic* if the isolated types are dense in $S(A)$ for every subset A.

Thus, the theory T is atomistic just if for each $A \subseteq \mathcal{M}$, the Boolean algebra $F(T)$ is atomic.

The following argument is similar to the proof in Lemma III.1.7 and Exercise III.1.8 that a theory is ω-stable if and only if it is stable in all infinite powers.

1.14 Theorem. *If T is a countable ω-stable theory then T is atomistic.*

Proof. If T is not atomistic there is a formula $\phi(\overline{x}; \overline{a})$ such that no atom intersects $\phi(\mathcal{M}; \overline{a})$. Choose by induction $\psi_s(\overline{x}; \overline{a}_s)$ for $s \in 2^{<\omega}$ so that both

$$(\exists \overline{x})[\psi_s(\overline{x}; \overline{a}_s) \wedge \psi_{s\frown 0}(\overline{x}; \overline{a}_{s\frown 0})]$$

and

$$(\exists \overline{x})[\psi_s(\overline{x}; \overline{a}_s) \wedge \neg\psi_{s\frown 0}(\overline{x}; \overline{a}_{s\frown 0})]$$

are true in $\mathcal{M}$. We can find such a $\psi_{s\frown 0}$ because $\psi_s(\overline{a}; \overline{a}_s)$ is not an atom. Let $\psi_{s\frown 1}$ be $\neg\psi_{s\frown 0}$. There are $2^{\aleph_0}$ types over the countable set $\{\overline{a}_s : s \in 2^{<\omega}\}$ so T is not ω-stable.

However, ω-stability is by no means a necessary condition for atomicity. For example, the theories of dense linear order without endpoints and REF_ω are atomistic. On the other hand, CEF_ω is a glaring example of a stable theory which is not atomistic (since it doesn't even have a prime model over any singleton). It is possible to construct examples of $\aleph_0$-categorical theories which are not atomistic.

1.15 Exercise. Show that if T is an $\aleph_0$-categorical theory then T admits prime models over finite sets. That is, for every finite subset, A, of $\mathcal{M}$ there is a prime model of T over A.

1.16 Exercise. Let T be the theory of a dense linear order without endpoints but with a unary predicate P which picks out a dense codense subset. Show that T is $\aleph_0$-categorical but that if $A = P(M)$ is an infinite set there is no prime model over A. (This example is due to Steinhorn.)

1.17 Theorem. *If T is atomistic, then for every A there is a model M of T with $|M| = |A| + |T|$ which is constructible over A.*

Proof. Suppose that C is a substructure of a model of T and enumerate the consistent members of $F_1(C)$ as $\langle \phi_i : i < |C| + |T| \rangle$. Choose a sequence $\langle a_i : i < |C| + |T| \rangle$ such that $t(a_i; C \cup C_i)$ is isolated in $S(C \cup C_i)$ and $\phi_i \in t(a_i; C \cup C_i)$. This is possible because T is atomistic. For any C, let $G(C) = C \cup \{a_i : i < |C| + |T|\}$. Now for the A of the theorem, for each n let $G^{n+1}(A) = G(G^n(A))$. Letting $M = \bigcup\{G^n(A)\}$ completes the construction.

1.18 Exercise. Show that if the countable theory T is atomistic then for every $A \subseteq N \models T$ there is a prime model over A.

1.19 Exercise. Show that if $S(\emptyset)$ is countable then T has a prime model.

1.20 Exercise. Show that if a countable theory T has prime models over every countable set then it has prime models over all sets.

1.21 Exercise. Show that if T is a small countable theory then T admits prime models over all finite sets.

1.22 Historical Notes. Call a model M *algebraically prime* over A if every embedding (not necessarily elementary) of A into a model, N, of T extends to an embedding of M into N. Abraham Robinson introduced the notion of prime models in [Robinson 1956]. However, he worked in the context of model complete theories so the distinction between algebraically and elementarily prime models was obscured. Most of this section is taken from Vaught's beautiful paper [Vaught 1961]. The extension to prime models over sets was made by [Morley 1965]. The contrast between algebraically prime and elementarily prime is explored in [Baldwin & Kueker 1981].

2. The General Notion of Isolation

We give here a semi-abstract treatment of the notion of generation. The key to such a treatment is to describe what it means for a sequence to be in the model 'generated' by a set. The most primitive notion of this sort requires the sequence to be given by a term. Increasingly more general notions require the sequence to be in a finite set definable with parameters from the given set or to realize a principal type generated by a formula with parameters from the given set. Shelah gives in Chapter IV of [Shelah 1978] an axiomatic treatment to include many such notions. Our approach here is to extend directly from the last idea. We will deal with several extensions of the word 'formula' but consider a sequence to be 'generated' by a set if all its relations to that set are determined by a 'formula' over the set.

Our approach to isolation is more concrete than that in [Shelah 1978]. We proceed by discussing three families **I** of 'sets of formulas'. We express the idea that $\overline{a}$ is generated by B by saying $t(\overline{a}; B)$ is implied by some member of **I** whose domain is contained in B. At this level of generality we are able to handle the most important among the notions Shelah discusses. We can generalize somewhat more by replacing 'p is implied by a member of **I**' by some other relation. (Thus, in Section 5 we study the notion given by 'p is locally implied by a member of **I**'.)

As in Chapter II, for each of the exercises in this section all of the preceding axioms may be used unless the exercise expressly says otherwise. We require the following convention.

2.1 Notation. If q has the form $t(\overline{c}; A)$ then $\operatorname{dom}^* q = \operatorname{dom} q = A$; if q has the form $stp(\overline{c}, A)$ then $\operatorname{dom}^* q = A$.

2.2 Definition. A collection of **I**-formulas is one of the following collections:

i) $\mathbf{AT}_\lambda$ is the collection of all types containing fewer than λ first order formulas.
ii) $\mathbf{SET}_\lambda$ is the collection of all types which are over a set of power less than λ.
iii) $\mathbf{S}_\lambda$ is the collection of all types which are almost over a set of power less than λ.

We let **I** vary over $\mathbf{SET}_\lambda$, $\mathbf{AT}_\lambda$, and $\mathbf{S}_\lambda$. Each of these sets of types is given with a cardinal parameter λ which is referred to as $\lambda(\mathbf{I})$. We often abbreviate $\mathbf{AT}_{\aleph_0}$ by **AT** and $\mathbf{S}_{\kappa(T)}$ by **S**. Thus, if **I** is $\mathbf{AT}_\lambda$, an **I**-formula is a type p with $|p| < \lambda$. In particular, if **I** is $\mathbf{AT}_{\aleph_0}$, which we often abbreviate as **AT**, then an **I**-formula is an ordinary first order formula.

These notions correspond to those introduced by Shelah as follows: $\mathbf{AT}_\lambda$ to F^t_λ, $\mathbf{SET}_\lambda$ to F^s_λ, and $\mathbf{S}_\lambda$ to F^a_λ. To obtain some of Shelah's more exotic notions in this framework we must generalize the notion of implication. Thus, the set F^f_λ on page 155 of [Shelah 1978] would come from $\mathbf{SET}_\lambda$ by replacing 'implies' by 'does not fork over'. We have not investigated exactly what properties of the implication notion are needed to make such generalizations work.

In his more abstract treatment Shelah takes the following property as the definition of $\lambda(\mathbf{I})$.

2.3 Exercise. For each collection of formulas **I**, $\lambda(\mathbf{I})$ is the least cardinal such that for each $p \in I$, $|\operatorname{dom}^* p| < \lambda(\mathbf{I})$.

2.4 Exercise. $\mathbf{AT}_{\aleph_0}$ is the notion discussed in Section IX.1.

The following properties are easily seen to hold for each **I**.

2.5 Proposition. *Every set* **I** *of formulas satisfies the following closure conditions.*

i) **I** *is closed under isomorphism.*
ii) *If* $q \in \mathbf{I}$ *then any type (or strong type) obtained by changing the order or names of variables in* q *is also in* **I**.
iii) *Any subtype of a type (strong type) in* **I** *is in* **I** .
iv) *Every type consisting of a single first order formula is in* **I**.
v) *If* $\lambda(\mathbf{I})$ *is regular then* **I** *is closed under increasing unions of length less than* $\lambda(\mathbf{I})$. *More generally, if the consistent type* $p = \bigcup p_i$, *with each* $p_i \in \mathbf{I}$ *for* $i < \operatorname{cf}(\lambda(\mathbf{I}))$, *then* $p \in \mathbf{I}$.

2.6 Exercise. Show every algebraic type is in $\mathbf{I}$.

Note that $\mathbf{AT}_\lambda \subseteq \mathbf{SET}_\lambda$ and if $\lambda > |T|$, $\mathbf{AT}_\lambda = \mathbf{SET}_\lambda$.

The distinctions between the several variations on the term 'isolated' made in the next definition are extremely important. Pay particular attention to the distinction between 'isolated over' and 'isolated by'.

2.7 Definition. i) The type p is $\mathbf{I}$ *-isolated* by $q \in \mathbf{I}$ if $q \vdash p$.

ii) The type p is $\mathbf{I}$*-isolated over* A if for some $q \in \mathbf{I}$, $q \vdash p$ and $\mathrm{dom}^* q$ is contained in A.

iii) The type p is $\mathbf{I}$*-isolated by* A if for some $q \in \mathbf{I}$, $q \vdash p$ and $\mathrm{dom}^* q = A$.

iv) The type p is $\mathbf{I}$ *-isolated* if for some $q \in I$, p is $\mathbf{I}$-isolated by q.

If $p = t(\overline{c}; B)$ is $\mathbf{I}$-isolated over A and $|A| < \lambda(\mathbf{I})$, note that $stp(\overline{c}; A) \vdash p$. (If $\mathbf{I} = \mathbf{SET}_\lambda$ or $\mathbf{AT}_\lambda$ we can replace $stp(\overline{c}; A)$ by $t(\overline{c}; A)$.) Moreover if p is $\mathbf{I}$-isolated by A then $|A| < \lambda(\mathbf{I})$. This last conclusion fails if 'by A' is replaced by 'over A'.

2.8 Exercise. Use Lemma IV.3.12 to show that $t(a; B)$ is $\mathbf{S}$-isolated by $C \subseteq B$ if and only if $stp(a; C) \vdash stp(a; B)$.

There are several monotonicity and continuity properties which are easily seen to hold for any notion of isolation.

2.9 Exercise (Montonicity). If $p \subseteq q \subseteq r$ and r is $\mathbf{I}$-isolated by p then q is $\mathbf{I}$-isolated by p.

2.10 Exercise. (First Continuity Property) If $\langle p_i : i \in \alpha\rangle$ is an increasing sequence of types, each p_i is $\mathbf{I}$-isolated, and $\alpha < \mathrm{cf}(\lambda(\mathbf{I}))$ then $\bigcup p_i$ is $\mathbf{I}$-isolated.

This result easily yields the following more useful formulation.

2.11 Exercise. If $|F| < \mathrm{cf}(\lambda(\mathbf{I}))$ and for each $\overline{f} \in F$, $t(\overline{a}; B \cup \overline{f})$ is $\mathbf{I}$-isolated then $t(\overline{a}; B \cup F)$ is $\mathbf{I}$-isolated.

2.12 Exercise. (Second Continuity Property) If $\langle p_i : i \in \alpha\rangle$ is an increasing sequence of types, p_0 is complete, each p_i is $\mathbf{I}$-isolated over B, and $|B| < \lambda(\mathbf{I})$ then $\bigcup p_i$ is $\mathbf{I}$-isolated over B.

2.13 Exercise. Show the necessity of restricting the cardinality of J or B in the continuity properties. (Hint: consider $t(\sqrt{2}; Q)$ in the theory of dense linear order.)

2.14 Exercise. Suppose $t(\overline{a}; B)$ is $\mathbf{I}$-isolated and $\overline{b} \in B$. Show $t(\overline{a}^\frown \overline{b}; B)$ is $\mathbf{I}$-isolated.

Now we extend the notion of a tuple being isolated over a set to embrace 'the set B is isolated over the set A.' Since each set of $\mathbf{I}$-formulas is closed under permutations of the variables, whether $t(\overline{b}; A)$ is isolated over A depends on the set enumerated by $\overline{b}$, not on the particular enumeration. As usual we extend the notion to infinite sets by imposing a requirement of finite character. Note, however, that just because every finite sequence from B is isolated over A, it does not mean that $t(B; A)$ is actually isolated over A. (Consider the case when $|A|$ and $|B|$ are both much greater than $\lambda(\mathbf{I})$.) Accordingly, we introduce a new term: **I**-atomic.

2.15 Definition. Let $A \subset B$. If each finite sequence $\bar{b}$ from B is **I**-isolated over A then B is **I**-*atomic* over A. If, in addition, B is the universe of a model M, we say M is an **I**-*atomic model* over A.

Although the following notion is introduced as a technical tool, it has a life of its own. A number of notions in stability theory can be profitably investigated by looking at the construction rather than the result. We will see this distinction at several stages in the proof of the uniqueness of prime models. Another example is Shelah's use of F^f_λ as a way of describing independent families of models (cf. Section 6 and [Shelah 1978]).

2.16 Definition. An **I**-*construction* over A is a sequence $E = \{e_j : j \in J\}$ such that for each $j \in J$, $p_j = t(e_j; A \cup E_j)$ is **I**-isolated. When **I** is understood we frequently simply say construction.

We sometimes write an **I**-construction as $\{\langle e_j, B_j\rangle : j \in J\}$, where p_j is **I**-isolated by B_j and $B_j \subset A \cup E_j$, if we need to keep track of the 'witnesses' to the isolation of p_j over $A \cup E_j$.

To avoid double subscripts when quantifying over the elements of an **I**-construction, when $x = e_j$ for some j, we write $Pr(x)$ for the set of e_i with $i < j$ and B_x for the witness to the isolation of $t(x; A \cup Pr(x))$.

The set C is **I**-*constructible* over A if $C = A \cup E$ for some **I**-construction E. Note, however, that a construction is a sequence; thus when we refer to a construction we know both the order of construction and the way in which the p_j were isolated during the construction. We will frequently write C_j for $A \cup E_j$.

2.17 Lemma. *Suppose $\lambda(\mathbf{I})$ is regular. If B is **I**-constructible over $A \subset B$ and $C \subset B$ then there is a C' with $|C'| < |C|^+ + \lambda(\mathbf{I})$ such that $C \subset C' \subseteq B$ and $A \cup C'$ is **I**-constructible over A.*

Proof. For any set $X \subseteq B$ let $X' = \bigcup\{B_x : x \in X\}$; let $X^{n+1} = X \cup (X^n)'$ and let $X^* = \bigcup\{X^n : n < \omega\}$. Note that $|X^*| < \lambda(\mathbf{I}) + |X|^+$. Now C^* is the required subset of B.

As in the previous section, we want the result of any construction to be atomic over the original set. The following axiom specializes to Lemma 1.11 by letting $\mathbf{I} = \mathbf{AT}$ and setting $A = B$.

2.18 Transitivity Axiom. Let $B \subset A$ and let $|B|, |C| < \lambda(\mathbf{I})$. Suppose $t(\bar{d}; A \cup C)$ is **I**-isolated by $B \cup C$ and for each $\bar{c} \in C$, $t(\bar{c}; A)$ is **I**-isolated by B then for each $\bar{c} \in C$, $t(\bar{d}^\frown\bar{c}; A)$ is **I**-isolated by B.

2.19 Exercise. Show that if $\lambda(\mathbf{I})$ is regular then the transitivity axiom is implied by the following simpler formulation. If $t(\bar{d}; A \cup C)$ is **I**-isolated over $A \cup C$ and for each $\bar{c} \in C$, $t(\bar{c}; A)$ is **I**-isolated over B then for each $\bar{c} \in C$, $t(\bar{d}^\frown\bar{c}; A)$ is **I**-isolated over B.

To verify that $\mathbf{SET}_\lambda$ satisfies the transitivity axiom, we need only the following lemma.

2.20 Lemma. *If $B \subseteq A$, $t(\overline{d}; B \cup C) \vdash t(\overline{d}; A \cup C)$ and for each $\overline{c} \in C$, $t(\overline{c}; A)$ is implied by $t(\overline{c}; B)$ then for any $\overline{c}' \in C$, $t(\overline{d}^\frown\overline{c}'; B) \vdash t(\overline{d}^\frown\overline{c}'; A)$.*

Proof. Fix $\overline{c}' \in C$ and a formula $\phi(\overline{x}, \overline{y}; \overline{a})$ such that $\models \phi(\overline{d}, \overline{c}'; \overline{a})$. Since $t(\overline{d}; B \cup C) \vdash t(\overline{d}; A \cup C)$ there is a formula $\psi(\overline{x}, \overline{c}'; \overline{b}, \overline{c}_1)$ and a $\overline{c}_1$ in C with $\models \psi(\overline{d}, \overline{c}'; \overline{b}, \overline{c}_1)$ and $\models \psi(\overline{x}, \overline{c}'; \overline{b}, \overline{c}_1) \rightarrow \phi(\overline{x}; \overline{c}', \overline{a})$. Now, because $t(\overline{c}^\frown\overline{c}_1; B) \vdash t(\overline{c}'^\frown\overline{c}_1; A)$, we can choose a formula $\chi(\overline{y}, \overline{z}; \overline{b}_1)$ such that $\models \chi(\overline{c}', \overline{c}_1; \overline{b}_1)$ and

$$\chi(\overline{y}, \overline{z}; \overline{b}_1) \rightarrow (\forall \overline{x})[\psi(\overline{x}, \overline{y}, \overline{b}, \overline{z}) \rightarrow \phi(\overline{x}, \overline{y}; \overline{a})].$$

Now the formula $(\exists \overline{z})(\chi(\overline{y}, \overline{z}, \overline{b}_1) \wedge \psi(\overline{x}, \overline{y}, \overline{b}, \overline{z}))$ is in $t(\overline{d}^\frown\overline{c}'; B)$ and implies $\phi(\overline{x}, \overline{y}; \overline{a})$. Choosing such a formula for each $\phi(\overline{x}, \overline{y}; \overline{a}) \in t(\overline{d}^\frown\overline{c}'; A)$ shows $t(\overline{d}^\frown\overline{c}'; B) \vdash t(\overline{d}^\frown\overline{c}'; A)$.

2.21 Exercise. Prove that if λ is regular the transitivity axiom holds for $\mathbf{SET}_\lambda$ and for $\mathbf{AT}_\lambda$.

In Section 3 we discuss in more detail the *transposition axiom* (Axiom V.1 of [Shelah 1978], Axiom IX.3.8 here): If $t(\overline{a}^\frown\overline{b}; C)$ is $\mathbf{I}$-isolated by D then $t(\overline{a}; C \cup \overline{b})$ is $\mathbf{I}$-isolated by $D \cup \overline{b}$. The following exercise shows that in the presence of the transposition axiom we obtain another simplification of transitivity.

2.22 Exercise (Weak Transitivity). Show that if $\mathbf{I}$ satisfies the transposition axiom and the weakening of the transitivity axiom obtained by assuming that $t(\overline{d}; A \cup C)$ is isolated by C rather than $B \cup C$ then $\mathbf{I}$ satisfies the transitivity axiom. transitivity of isolation weak transitivity of isolation

In Section 3 we follow the treatment in [Pillay 1983a] and verify the transitivity axiom for $\mathbf{S}_\lambda$ by first verifying the transposition axiom and then the simpler version of transitivity. Illustrating the power of T^{eq}, we give here a direct proof (due to Shelah) of transitivity for $\mathbf{S}_\lambda$.

2.23 Lemma. *For every λ, $\mathbf{S}_\lambda$-isolation satisfies the transitivity axiom.*

Proof. As assumptions we have

i) $stp(\overline{d}; B \cup C) \vdash t(\overline{d}; A \cup C)$ and
ii) $stp(C; B) \vdash t(C; A)$.

We must show that for any $\overline{c} \in C$,

$$stp(\overline{d}^\frown\overline{c}; B) \vdash t(\overline{d}^\frown\overline{c}; A).$$

Translating to T^{eq} by Theorem VIII.1.10 we have

i)′ $t^{\mathrm{eq}}(\overline{d}; \mathrm{cl}(B \cup C)) \vdash t^{\mathrm{eq}}(\overline{d}; \mathrm{cl}(A \cup C))$ and
ii)′ $t^{\mathrm{eq}}(C; \mathrm{cl}(B)) \vdash t^{\mathrm{eq}}(C; \mathrm{cl}(A))$.

Since $\mathrm{cl}(B)$ is algebraically closed, types and strong types over $\mathrm{cl}(B)$ (in L^{eq}) are interchangeable. Applying Lemma IV.3.16 (in T^{eq}) we have

$$t^{\mathrm{eq}}(\mathrm{cl}(B \cup C); B) \vdash t^{\mathrm{eq}}(\mathrm{cl}(B \cup C); A).$$

To use Lemma 2.19, we rewrite and and apply monotonicity to get

i)'' $t^{\mathrm{eq}}(\overline{d}; \mathrm{cl}(B) \cup \mathrm{cl}(B \cup C)) \vdash t^{\mathrm{eq}}(\overline{d}; \mathrm{cl}(A) \cup \mathrm{cl}(B \cup C))$ and
ii)'' $t^{\mathrm{eq}}(\mathrm{cl}(B \cup C); \mathrm{cl}(B)) \vdash t^{\mathrm{eq}}(\mathrm{cl}(B \cup C); \mathrm{cl}(A))$.

By Lemma 2.19 (used in T^{eq}), for any $\overline{e} \in \mathrm{cl}(B \cup C)$,

$$t^{\mathrm{eq}}(\overline{d}^\frown\overline{e}; \mathrm{cl}(B)) \vdash t(\overline{d}^\frown\overline{e}; \mathrm{cl}(A)).$$

Applying Lemma IV.3.16 in the other direction, we have

$$stp(\overline{d}^\frown\overline{e}; B) \vdash t(\overline{d}^\frown\overline{e}; A)$$

and we finish.

We return to the exposition of the general theory of isolation relations.

2.24 Theorem. *Let* $\mathbf{I}$ *satisfy the transitivity axiom and suppose* $\lambda(\mathbf{I}) = \lambda$ *is regular. If* C *is* $\mathbf{I}$*-constructible over* A *then* C *is* $\mathbf{I}$*-atomic over* A.

Proof. We proceed by a series of observations.

i) If $t(\overline{b}; A)$ is $\mathbf{I}$-isolated then $A \cup \overline{b}$ is $\mathbf{I}$-atomic over A. This is immediate from the transitivity axiom. Specifically, a typical element of $A \cup \overline{b}$ has the form $\overline{a}^\frown\overline{b}$ where $\overline{a} \in A$. Certainly, $t(\overline{a}; A)$ is $\mathbf{I}$-isolated over A, say by $\overline{a}$, and by hypothesis, $t(\overline{b}; A)$ is $\mathbf{I}$-isolated over A by some B. If $B' = B \cup \overline{a}$ then by the transitivity axiom, $t(\overline{a}^\frown\overline{b}; A)$ is $\mathbf{I}$-isolated over A by B', as required.

ii) Suppose $A_1 \subseteq A_2 \subseteq A_3$, A_3 is $\mathbf{I}$-atomic over A_2 and A_2 is $\mathbf{I}$-atomic over A_1; then A_3 is $\mathbf{I}$-atomic over A_1. Suppose $\overline{c} \in A_3$ and let $p = t(\overline{c}; A_2)$. Then there is a $B \subseteq A_2$ with $|B| < \lambda$ and p is $\mathbf{I}$-isolated by B. For each $\overline{b} \in B$, let $q_{\overline{b}} = t(\overline{b}; A_1)$ and choose $B_{\overline{b}} \subseteq A_1$ with $|B_{\overline{b}}| < \lambda$ such that $q_{\overline{b}}$ is $\mathbf{I}$-isolated by $B_{\overline{b}}$. Let $B_0 = \cup\{B_{\overline{b}} : \overline{b} \in B\}$. Since λ is regular, $|B_0| < \lambda$. Now p is $\mathbf{I}$-isolated over $B \cup B_0$ by the monotonicity property and for each $\overline{d} \in B$, $t(\overline{d}; A_1)$ is $\mathbf{I}$-isolated over B_0; so by the transitivity axiom, $t(\overline{c}^\frown\overline{d}; A)$ is $\mathbf{I}$-isolated over B_0 for each $\overline{d} \in B$. Thus, by i) A_3 is $\mathbf{I}$-atomic over A_1.

iii) The theorem now follows easily by induction on the length of the construction.

Note that the regularity of $\lambda(\mathbf{I})$ is needed for the second step.

The following notion tells us when in a construction we have realized 'enough' types.

2.25 Definition. The set C is $\mathbf{I}$-*saturated* if for every $B \subseteq C$, whenever $p \in S(B)$ is $\mathbf{I}$-isolated then p is realized in C. A model M is $\mathbf{I}$-*prime* over A if every elementary embedding of A into an $\mathbf{I}$-saturated model, N, can be extended to an elementary embedding of M into N. If M is $\mathbf{I}$-constructible over A and $\mathbf{I}$-saturated we say M is *strictly* $\mathbf{I}$-*prime* over A. If for each $A \subseteq \mathcal{M}$ there is an $\mathbf{I}$-prime model over A, we say T *admits I-prime models.* $\lambda_0(\mathbf{I})$ denotes the cardinality of the $\mathbf{I}$-prime model over $\emptyset$.

Shelah calls the notion we refer to as strictly **I**-prime, **I**-primary. An **I**-saturated model need not be **I**-prime.

Thus, if T is the theory of algebraically closed fields of characteristic zero, the $\mathbf{SET}_{\aleph_0}$-prime model of T is the algebraically closed field with transcendence degree $\aleph_0$ over the rationals.

2.26 Exercise. Show that any model of REF_ω, indeed of any theory, is **AT**-saturated, but REF_ω has no **AT**-prime model over the empty set.

The usual notion of λ-saturation is just **I**-saturation when **I** is $\mathbf{SET}_\lambda$. For $\mathbf{I} = \mathbf{AT}_\lambda$ we get the notion of a λ-compact model. With this definition we can link the notion of **S**-isolation introduced in this section with the concept of strong saturation considered in Section III.2. The next lemma verifies for $\mathbf{S}_\lambda$ the following observation (which obviously holds for $\mathbf{AT}_\lambda$ and $\mathbf{SET}_\lambda$). The model M is **I**-saturated if and only if every consistent **I**-formula over M is realized in M.

2.27 Lemma. *If $\lambda \geq \kappa(T)$, the model M is strongly λ-saturated iff M is $\mathbf{S}_\lambda$-saturated.*

Proof. Suppose first that M is strongly λ-saturated. We have to show that if $A \subseteq M$, $|A| < \lambda$, and q is almost over A then q is realized in M. Let $\overline{c}$ realize q and let $\overline{d}$ realize an extension of q to a type over $A \cup \overline{c}$ which does not fork over A. Then $t(\overline{d}; A \cup \overline{c})$ is finitely satisfiable almost over A. In particular, $t(\overline{d}; A \cup \overline{c})$ is finitely satisfied in M. By the definition of strong λ-saturation, $t(\overline{d}; A \cup \overline{c})$ and a fortiori q is realized in M.

For the converse, suppose $|B| < \lambda$ and $t(\overline{c}; B)$ is finitely satisfied in M. That is, $\overline{c} \downarrow_M B$. Since $\lambda \geq \overline{\kappa}(T)$, we can choose $A \subseteq M$ with $|A| < \lambda$ such that $B \downarrow_A M$ and $\overline{c} \downarrow_A M$. Then by transitivity $\overline{c} \downarrow_A B$. Now let $\overline{d} \in M$ realize $stp(\overline{c}; A)$. Then $\overline{d} \downarrow_A B$ and $\overline{c} \downarrow_A B$ and they realize the same strong type over A, so $\overline{d}$ realizes $t(\overline{c}; B)$ as required.

Recall that the existence of prime models in Section 1 depended on the density of principal types. The next axiom extends this idea to the general case.

2.28 Existence Axiom. If p is an **I**-formula over $A \subseteq B$ then there is a $q \in S(B)$ such that q is **I**-isolated by an **I**-formula q_0 which implies p.

For $\mathbf{SET}_\lambda$ and $\mathbf{AT}_\lambda$ this axiom can be simplified by saying any **I**-formula extends to an **I**-isolated type. The actual formulation is more clumsy because of the difficulty in describing an 'extension' of a type almost over A.

2.29 Exercise. Show the Existence Axiom implies that for every A and every consistent formula $\phi(\overline{x}; \overline{a})$ there is a $p \in S(A)$ which is **I**-isolated over some subset of A and with $\phi(\overline{x}; \overline{a}) \in p$.

2.30 Exercise. Show that if $p \in S(A)$, $B \subseteq A \subseteq C$, and p is **I**-isolated over B then there is a $q \in S(C)$ such that $p \subseteq q$ and q is **I**-isolated.

2.31 Exercise. Show that if T is ω-stable (and countable), T satisfies the Existence Axiom for **AT**.

It is somewhat more difficult to verify the Existence Axiom for $\mathbf{S}_\lambda$. The key observation is that if $A \subseteq B$ and $p \in S(A)$ has no extension to a type in $S(B)$ which forks over A then for any $\overline{c}$ realizing p, $stp(\overline{c}; A) \vdash stp(\overline{c}; B)$.

2.32 Theorem. *For $\lambda \geq \kappa(T)$, $\mathbf{S}_\lambda$ satisfies the Existence Axiom.*

Proof. Let $A \subseteq B$ and let $p = stp(\overline{c}; A)$ for some $\overline{c}$. Let $p_0 = t(\overline{c}; A)$. If suffices to find a type p' with the domain of p' equal to A' with $A \subseteq A' \subseteq B$ and $|A'| < \lambda$ such that $p_0 \subseteq p'$ and no extension of p' to a complete type over B forks over A'. For, then if d realizes a nonforking extension of p' to a type over B, $stp(d; A')$ is the required q_0. To find p', choose inductively a set of pairs (p_i, A_i) with $A = A_0 \subseteq A_i \subseteq B$ and $p_i \subseteq p_{i+1}$ such that p_{i+1} forks over $A_i = \operatorname{dom} p_i$ and for each i, $|A_i| < \kappa(T) + \lambda$. Using the local character of forking it is easy to construct such a sequence. However, by the definition of $\kappa(T)$ it must cease with some (p_i, A_i) for $i < \kappa(T)$ and p_i is an appropriate p'.

2.33 Exercise. If T is superstable, prove the $\mathbf{S}_{\kappa(T)}$-isolated types are dense in $S(A)$ for each A. (Hint: For each formula $\psi(\overline{x}; \overline{a})$ such that $\phi \rightarrow \psi$, minimize $R_C(\phi(\overline{x}; \overline{b}))$.)

2.34 Exercise. If $\mathbf{I}$ satisfies the Existence Axiom, C is $\mathbf{I}$-saturated if every $p \in S(C)$ which is $\mathbf{I}$-isolated is realized in C.

2.35 Exercise. Show that if C is $\mathbf{AT}$-saturated and T is atomistic then C is a model of T.

We want to construct for any A, an $\mathbf{I}$-saturated B containing A and, if possible, of the same cardinality as A. To describe such a construction we require one further invariant. This invariant allows us to discuss the construction of S-saturated models in terms of the realization of ordinary types.

2.36 Definition. $\mu(\mathbf{I})$ is the least cardinal ρ such that any set D which satisfies the following condition on D and ρ is $\mathbf{I}$-saturated. If $B \subseteq D$, $|B| < \rho$, and $q \in S(B)$ is $\mathbf{I}$-isolated then q is realized in D. If no such ρ exists, $\mu(\mathbf{I}) = \infty$.

$\mu(\mathbf{I})$ is an entirely different notion from the $\mu(T)$ discussed in Section III.4.

2.37 Exercise. Show $\mu(\mathbf{AT}_\kappa) \leq \kappa$ and $\mu(\mathbf{SET}_\kappa) \leq \kappa$.

Thus $\mu(\mathbf{I}) \leq \lambda(\mathbf{I})$ in two of the common examples. However, for $\mathbf{S}_\lambda$, this inequality is no longer always true. The obstruction arises from the fact that a complete type p over a set A may be implied by many different strong types over finite subsets of A.

2.38 Exercise. Show that if T is REF_ω, then $\mu(\mathbf{S}) = \aleph_1$.

2.39 Lemma. $\mu(\mathbf{S}) = \kappa(T) + \aleph_1$.

Proof. Let $\lambda = \kappa(T) + \aleph_1$. Fix a D such that if $B \subseteq D$, $q \in S(B)$, and $|B| < \lambda$ and q is S-isolated then q is realized in D. Let $C \subseteq D$ and suppose $p \in S(C)$ is S-isolated by A. We must show p is realized in D. By Lemma 2.27 choose an extension $\hat{p}$ of p to $S(D)$ which is also S-isolated. Without loss of generality, we may assume $\hat{p}$ is S-isolated over A. Let $\overline{a} \in \mathcal{M}$ realize $\hat{p}$. Now choose $\langle e_i : i \leq \omega \rangle \subseteq D$ such that $t(e_i; A \cup E_i) = t(\overline{a}; A \cup E_i)$. Since for each $i < \omega$, $|A \cup E_i| < \lambda$, we may choose the $\overline{e}_i \in D$. Now, as in Theorem V.2.2, $\overline{e}_\omega$ realizes p.

Now we show that if the Existence Axiom holds, T admits **I**-prime models.

2.40 Theorem. i) *If* **I** *satisfies the Existence Axiom and* $\mu(\mathbf{I}) < \infty$ *then every set* A *is contained in an* **I**-*saturated* B *which is* **I**-*constructible over* A *and with* $|B| \leq |A + 2|^{\mu(\mathbf{I})+|T|}$.
ii) *If* $\mu(\mathbf{I}) = \aleph_0$ *then* $|B| \leq |A| + \aleph_0$.

Proof. i) For any set C define C' as follows. Enumerate all types p such that for some D with $D \subseteq C$, and $|D| < \mu(\mathbf{I})$, $p \in S(D)$. Note that the number of such types is less than $\max(|C|^{<\mu(\mathbf{I})}, \mu(\mathbf{I})^{|T|})$. Build a sequence $E = \langle e_i : i < \alpha \rangle$ by letting each e_i realize an extension of the ith such type to a complete type over $E_i \cup C$ which is **I**-isolated. Such an extension is guaranteed by the Existence Axiom. Let $C' = C \cup E$. Now define by induction $\langle C_\gamma : \gamma < \mu(\mathbf{I})^+ \rangle$ with $C_0 = A$ and $C_{\gamma+1} = C'_\gamma$ and taking unions at limits. By the definition of $\mu(\mathbf{I})$, $C_{\mu(\mathbf{I})^+}$ is **I**-saturated. Note that by induction $|C_\gamma| \leq |\gamma| \times |A|^{\mu(\mathbf{I})+|T|}$ so $|C_{\mu(\mathbf{I})^+}| \leq |A + 2|^{\mu(\mathbf{I})+|T|}$.

ii) Under the additional hypothesis, $|C'| \leq \max(|C|, \mu(\mathbf{I})) = |C| + \aleph_0$. Thus, $|B| \leq |A| + \aleph_0$.

Of course, for **AT** if we can construct **I**-saturated models containing A at all, we can require their cardinality to be the same as $\max\{|A|, |T|\}$. But such a strong cardinality restriction is easily seen to fail if we extend even to $\mathbf{AT}_{\aleph_1}$.

2.41 Exercise. Show that if T is countable, $\lambda_0(\mathbf{S}) \leq 2^{\aleph_0}$.

2.42 Corollary. *If* **I** *satisfies the Existence Axiom then for every set* A *there is a strictly* **I**-*prime model* M *over* A *with* $|M| \leq (|A| + 2)^{|T|+\mu(\mathbf{I})}$.

Note that if **I** satisfies the Existence Axiom, every **I**-saturated structure is a model. In particular, if B is **I**-prime over A, then B is a model.

2.43 Exercise. The $\mathbf{AT}_{\aleph_1}$-prime model over the empty set of the theory of infinitely many independent unary predicates has power of the continuum.

2.44 Exercise. The $\mathbf{S}_{\aleph_0}$-prime model of REF_ω has power $2^{\aleph_0}$.

2.45 Exercise. Show if T is an uncountable theory such that for every $A \subseteq \mathcal{M}$, $F(A)$ is atomistic then T admits **AT**-prime models over all sets.

It is easy to prove the following result by induction.

2.46 Theorem. i) *If B is* **I**-*constructible over A and $C \supseteq A$ is* **I**-*saturated then there is an elementary embedding of B into C.*

ii) *If B is* **I**-*constructible over A, and B is* **I**-*saturated then B is* **I**-*prime over A.*

We can also show that every **I**-prime stucture is **I**-atomic.

2.47 Theorem. *If B is* **I**-*prime over A, then B is* **I**-*atomic over A.*

Proof. Choose an **I**-saturated set C containing A which is **I**-constructible as in Theorem 2.40. Then B can be embedded into C and C is **I**-atomic over A by Theorem 2.24 so B is **I**-atomic over A.

It is not nearly so easy to show that if B is **I**-prime over A then B is **I**-constructible over A. This has been shown only for countable stable theories which admit prime models over all sets. Indeed, the following question remains open. Let T be a countable stable theory and let M be prime over A (in the usual, **AT**, sense). Must M be **AT**-constructible over A?

2.48 Exercise (Sacks). The countable first order theory T is *quasi-totally transcendental* (q.t.t.) if for each A the subset of $S(A)$ consisting of points whose Morley rank is less than ∞ is dense in $S(A)$. Show that if T is q.t.t. **AT** still satisfies all the properties it was shown to satisfy for an ω-stable theory T in this section. (Only the Existence Axiom is nontrivial.) Conclude that T admits prime models over sets.

2.49 Exercise. Suppose $p \in S(A)$ is **AT**-isolated and the multiplicity of p is finite. Show that for any $\overline{c}$ realizing p, $stp(\overline{c}; A)$ is realized in every model containing A.

2.50 Exercise. Show that an increasing union of sets which are **I**-atomic over A is **I**-atomic over A. Conclude that if $A \subseteq N$ there is a maximal subset of N which is atomic over A. Show that the analogous result, replacing atomic by constructible fails.

2.51 Historical Notes. The importance of discussing several notions of isolation was discovered by Shelah. He presents two such families is [Shelah 1970] and [Shelah 1978]. This section is primarily a translation of Sections IV.1 and IV.2 of [Shelah 1978]. Shelah's Chapter IV contains not only a larger collection of isolation relations but a valuable chart summing up the properties of the various notions.

3. Bookkeeping Axioms for Isolation Relations

This brief section introduces some 'bookkeeping' axioms for isolation notions. We separate them from the axioms in Section 2 because they are somewhat more technical and are required only for the uniqueness as opposed to the existence of prime models. In the main these axioms deal with the continuing problem of how our various properties behave with respect to the concatenation of two sequences. That is they attempt to capture the relation between the isolation of the types $t(\overline{c}; A)$ and $t(\overline{d}; A)$ separately and the isolation of $t(\overline{c}\frown\overline{d}; A)$. We began this discussion for isolation with the transitivity axiom in Section 2. The necessity to justify these axioms could be regarded as a defect in our notation.

We show that under certain conditions if N is prime over A and $\overline{b} \in N$ then N is prime over $A \cup \overline{b}$. Note that the obvious statement of a symmetry principle for isolation, 'If $t(\overline{a}; A \cup \overline{b})$ is **I**-isolated then $t(\overline{b}; A \cup \overline{a})$ is **I**-isolated' is false. This principal obviously fails if $\overline{a} \in A$ and $\overline{b}$ realizes a nonisolated type over $A \cup \overline{a}$. The following example is a little more complex.

3.1 Exercise. Consider Ehrenfeucht's example [Vaught 1961] of a theory with 3 countable models: $\mathrm{Th}(Q, <, 0, 1, 2, \ldots)$. Let M be the model of T with a the least upper bound of $\{0, 1, 2, \ldots\}$ and $b > a$. Show $t(b; a)$ is **AT**-isolated but $t(a; b)$ is not.

3.2 Exercise. Find an example of two sequences $\overline{a}$ and $\overline{b}$ and a set A such that both $t(\overline{a}; A)$ and $t(\overline{b}; A)$ are **AT**-isolated but $t(\overline{a}\frown\overline{b}; A)$ is not.

We can, however, weaken this version of the symmetry property and obtain a principle both useful and true.

3.3 Symmetry Axiom for Isolation. If $t(\overline{a}; B)$ is **I**-isolated by $C \subseteq B$ and $t(\overline{b}; B)$ is **I**-isolated by $D \subseteq B$ then $t(\overline{a}; B \cup \overline{b})$ is **I**-isolated by C iff $t(\overline{b}; B \cup \overline{a})$ is **I**-isolated by D.

Remember that we formulated **I**-isolation in terms of '$\vdash$' and a collection of **I**-formulas. The difficulty with symmetry comes not from '$\vdash$' but from the difficulty in making the set of formulas in the hypothesis an **I**-formula. The key to seeing that the symmetry principle holds is to realize the notation $t(A; C) \vdash t(A; C \cup D)$ disguises the role of D. Remember that $t(A; C \cup D)$ may be thought of as a collection of sentences with names for the elements of A, C, and D. The side of the semicolon on which a particular name appears changes our view of the name but not its role in the set of sentences. Thus, $\mathrm{Diag}(A \cup B)$ and $t(A; B)$ contain the same formulas. In more detail, consider the meaning of the statements:

$$t(A; B) \vdash t(A; B \cup C) \text{ and } t(C; B) \vdash t(C; B \cup A).$$

The first means:

$$T \cup \mathrm{Diag}(B \cup C) \cup t(A; B) \vdash t(A; B \cup C).$$

The second means:

$$T \cup \mathrm{Diag}(B \cup A) \cup t(C; B) \vdash t(C; B \cup A).$$

Consideration of the latter two statements should easily show you that the first two are equivalent. In our earlier notation all of these statements mean $t(A; B) \perp^w t(C; B)$.

3.4 Exercise. Verify that **AT** satisfies this axiom.

3.5 Lemma. $\mathbf{AT}_\lambda$ *satisfies the symmetry axiom.*

Proof. From the hypotheses of the symmetry axiom, we have $p \vdash t(\overline{a}; B)$ and $q \vdash t(\overline{b}; B \cup \overline{a})$ where $p \subseteq t(\overline{a}; C)$, $q \subseteq t(\overline{b}; D)$ for some C, D contained in B and $|p|, |q| < \lambda$. That is,

$$p \cup \mathrm{Diag}(B) \vdash t(\overline{a}; B)$$

$$q \cup \mathrm{Diag}(B \cup \overline{a}) \vdash t(\overline{b}; B \cup \overline{a}).$$

But then,

$$p \cup \mathrm{Diag}(B \cup \overline{b}) \vdash t(\overline{a}; B \cup \overline{b})$$

since $\mathrm{Diag}(B \cup \overline{b}) \vdash q$ and $p \cup \mathrm{Diag}(B) \vdash \mathrm{Diag}(B \cup \overline{a})$. Thus, $t(\overline{a}; B \cup \overline{b})$ is $\mathbf{AT}_\lambda$-isolated by C.

Note that in the proof we did not use the hypothesis that $t(\overline{b}; B)$ is $\mathbf{I}$-isolated; it is required for the converse.

Invoking T^{eq} we proved the strong form of the transitivity axiom for $\mathbf{S}_\lambda$ in Lemma 2.23. Using a rudimentary form of symmetry for strong types, we will verify the weak version of transitivity for $\mathbf{S}_\lambda$. Recall from Exercise 2.22 that in the presence of transposition the two forms of transitivity are equivalent. Now we formally introduce the axiom of transposition and verify that it holds for $\mathbf{AT}_\lambda$, $\mathbf{SET}_\lambda$, and $\mathbf{S}_\lambda$. With the aid of transposition we will show that symmetry holds for $\mathbf{S}_\lambda$. We require two preliminary lemmas.

3.6 Lemma. $stp(\overline{a}; B) \vdash stp(\overline{a}; B \cup \overline{c})$ *implies* $stp(\overline{c}; B) \vdash stp(\overline{c}; B \cup \overline{a})$.

Proof. The hypothesis yields that for any $\overline{c}'$ realizing $stp(\overline{c}; B)$, $\overline{c}' \downarrow_B \overline{a}$. The lemma follows since strong types are stationary.

3.7 Lemma. *If* $t(\overline{d}; A \cup C)$ *is* $\mathbf{S}_\lambda$*-isolated by* C *and for each* $\overline{c} \in C$, $t(\overline{c}; A)$ *is* $\mathbf{S}_\lambda$*-isolated by* B *then for each* $\overline{c} \in C$, $t(\overline{d}^\frown\overline{c}; A)$ *is* $\mathbf{S}_\lambda$*-isolated by* B.

Proof. We want to show that $stp(\overline{d}^\frown\overline{c}; B) \vdash t(\overline{d}^\frown\overline{c}; A)$ for each $\overline{c} \in C$. By Lemma 3.6, it suffices to show that for each $\overline{a} \in A$,

$$stp(\overline{a}; B) \vdash stp(\overline{a}; B \cup C \cup \overline{d}).$$

To see this, fix $\overline{a} \in A$. By hypothesis, $stp(\overline{c}; B) \vdash stp(\overline{c}; A)$ so by Lemma 3.6, $stp(\overline{a}; B) \vdash stp(\overline{a}; B \cup \overline{c})$. But, $stp(\overline{d}; C) \vdash stp(\overline{d}; A \cup C)$ so clearly $stp(\overline{d}; B \cup C) \vdash stp(\overline{d}; A \cup C)$. For any $\overline{a} \in A$, by Lemma 3.6 again, $stp(\overline{a}; B \cup C) \vdash stp(\overline{a}; B \cup C \cup \overline{d})$. But then $stp(\overline{a}; B) \vdash stp(\overline{a}; B \cup C \cup \overline{d})$.

We often make use of the observation that $t(\overline{a}^\frown\overline{b})$ and $t(\overline{a}; \overline{b})$ carry the same information. When dealing with isolation we must be a little fussy. We encode this fussiness in the transposition axiom.

3.8 Transposition Axiom. If $t(\overline{a}^\frown\overline{b}; C)$ is $\mathbf{I}$-isolated by D then $t(\overline{a}; C \cup \overline{b})$ is $\mathbf{I}$-isolated by $D \cup \overline{b}$.

3.9 Lemma. *The transposition axiom holds for* $\mathbf{SET}_\lambda$.

Proof. Let $p = t(\overline{a}^\frown\overline{b}; C)$ and fix $q \in S(D)$ such that $q \vdash p$. Let q' be $\{\phi(\overline{x}; \overline{b}) : \phi(\overline{x}; \overline{y}) \in q\}$. Clearly $q' \vdash t(\overline{a}; C \cup \overline{b})$.

3.10 Exercise. Verify that $\mathbf{AT}_\kappa$ satisfies the transposition axiom.

The verification of this axiom for $\mathbf{S}_\lambda$ is only slightly more difficult.

3.11 Lemma. $\mathbf{S}_\lambda$ *obeys the transposition axiom.*

Proof. Note that $stp(\overline{a}; D \cup \overline{b}) = stp(\overline{a}'; D \cup \overline{b})$ implies (Lemma IV.3.19) $stp(\overline{a}^\frown\overline{b}; D) = stp(\overline{a}'^\frown\overline{b}; D)$. This, in turn, implies $t(\overline{a}^\frown\overline{b}; C) = t(\overline{a}'^\frown\overline{b}; C)$ which implies $t(\overline{a}; C \cup \overline{b}) = t(\overline{a}'; D \cup \overline{b})$ and we finish.

3.12 Lemma. *The symmetry axiom holds for* $\mathbf{S}_\lambda$.

Proof. We have $stp(\overline{a}; C) \vdash stp(\overline{a}; B \cup \overline{b})$ and $stp(\overline{b}; D) \vdash stp(\overline{b}; B)$. We want to show $stp(\overline{b}; D) \vdash stp(\overline{b}; B \cup \overline{a})$. Clearly, $stp(\overline{a}; C) \vdash stp(\overline{a}; B \cup \overline{b})$ implies $stp(\overline{a}; B) \vdash stp(\overline{a}; B \cup \overline{b})$. But then, Lemma 3.6 yields $stp(\overline{b}; B) \vdash stp(\overline{b}; B \cup \overline{a})$. As, $stp(\overline{b}; D) \vdash stp(\overline{b}; B)$ we have the result.

If $\lambda(\mathbf{I})$ is regular, we can extend the transposition axiom.

3.13 Lemma (Generalized Transposition). *Suppose* $\mathbf{I}$ *satisfies the transposition axiom,* $|C| < \lambda(I)$, $\lambda(\mathbf{I})$ *is regular, and* $\overline{a} \cup C$ *is* $\mathbf{I}$*-atomic over* B *then* $t(\overline{a}; B \cup C)$ *is* $\mathbf{I}$*-isolated.*

Proof. By the transposition axiom, for each $\overline{c} \in C$ $t(\overline{a}; B \cup \overline{c})$ is $\mathbf{I}$-isolated. By the first continuity property (Exercise 2.10) we have the result.

With the aid of the transitivity axiom, we can prove a converse to the transposition axiom which allows us to decompose sequences for isolation just as we did for independence in Chapter II.

3.14 Lemma. *Suppose* $\mathbf{I}$ *satisfies the transposition and transitivity axioms. Then the following are equivalent.*

i) $t(\overline{a}^\frown\overline{b}; A)$ *is* $\mathbf{I}$*-isolated over* B.
ii) $t(\overline{a}; A \cup \overline{b})$ *is* $\mathbf{I}$*-isolated over* $B \cup \overline{b}$ *and* $t(\overline{b}; A)$ *is* $\mathbf{I}$*-isolated over* B.

Proof. The result is immediate from the axioms cited.

3.15 Historical Notes. Most of this material is from Chapter IV of [Shelah 1978]. The proof of transitivity for $\mathbf{S}$ without recourse to T^{eq} is taken from Chapter 8 of [Pillay 1983a]. The proof here of symmetry for $\mathbf{AT}_\lambda$ seems to be the natural attack. The more complicated argument used here for $\mathbf{S}_\lambda$ was used by Shelah in both cases.

4. Uniqueness Of Strictly Prime Models

In this section we prove, assuming that $\lambda(\mathbf{I})$ is regular, that if M and N are both $\mathbf{I}$-constructible over A and $\mathbf{I}$-saturated, that is, if M and N are strictly $\mathbf{I}$-prime over A, then M and N are isomorphic over A. In Chapter X, we will invoke properties of the freeness relation to show that this result extends to prime models over A if T is a countable stable theory which admits prime models. In particular, these arguments yield that prime models for two of the categories we have discussed in detail, **AT**-prime and **S**-prime, are unique, at least when the theory is countable and stable.

Since strictly prime models are prime it is immediate that two strictly prime models over a set A can each be embedded in the other. Thus, by the Schröder-Bernstein theorem they have the same cardinality. We want to extend this Schröder-Bernstein phenomena from the category of set mappings to the category of elementary embeddings. We will construct a back-and-forth to create the isomorphism.

There are three steps to the argument showing that there is a unique (up to isomorphism over A) **I**-saturated, **I**-constructible model over A. We first note that this result holds outright if the construction is not too long. Then we show how to break an arbitrary construction into pieces to which the first argument applies. Finally, we stitch the pieces together by the general methods of back-and-forths. The next two lemmas combine to prove, in Theorem 4.3, the 'short' case of the theorem. They follow easily from the generalized transposition property (Lemma 3.13) and we leave them as an exercise.

4.1 Lemma. *If $E = \langle e_i : i < \alpha\rangle$ is an* **I***-construction over A, $C \subseteq E$, $|C| < \lambda(\mathbf{I})$, and $\lambda(\mathbf{I})$ is regular then $\langle e_i : i < \alpha\rangle$ is an* **I***-construction over $A \cup C$.*

The following somewhat weaker conclusion also holds: $D = E - C = \langle d_j : j < \delta\rangle$ is constructible over $A \cup C$.

4.2 Lemma. *If $E = \langle e_i : i < \alpha\rangle$ is an* **I***-construction over A and $B \subseteq E$ with $|B| < \lambda(\mathbf{I}))$ and $\lambda(\mathbf{I})$ is regular then B is* **I***-constructible over A.*

4.3 Theorem. *If M and N are strictly* **I***-prime over A and $|M| \leq \lambda(\mathbf{I})$ then $M \approx N$.*

4.4 Exercise. Prove Lemma 4.1, Lemma 4.2, and Theorem 4.3.

4.5 Exercise. If $|E| < \lambda(\mathbf{I})$, $\lambda(\mathbf{I})$ is regular, and E is constructible over A then E is constructible over A under any wellordering of E with ordinal less than $\lambda(\mathbf{I})$.

4.6 Exercise. Show that although the rationals may be constructed as an **AT**-prime model of the theory of dense linear order, there are wellorderings of type greater than ω under which they are not constructible.

4.7 Exercise. Suppose $|A|$, $|B| < \lambda(\mathbf{I})$ and $\lambda(\mathbf{I})$ is regular. Show that if $B \subseteq N$ for some N which is **I**-prime over $M \cup A$, then if N' is a strictly **I**-prime model over $M \cup B$ and $A \subseteq N'$, N' is strictly **I**-prime over $M \cup A$.

We are able to show the isomorphism exists in Theorem 4.3 because each initial segment of a short construction is sufficiently nice. For the general case, it suffices by [Kueker 1970] to find a set, J, of partial isomorphisms between M and N each with cardinality less than M such that J is closed under increasing unions with cardinality less than $\mathrm{cf}(|M|)$ and J satisfies the following back and forth conditions.

$$(\forall g \in J)(\forall m \notin \operatorname{dom} g)(\exists g_1 \in J)(g_1 \supseteq g \wedge m \in \operatorname{dom} g_1)$$
$$(\forall g \in J)(\forall n \notin \operatorname{rng} g)(\exists g_1 \in J)(g_1 \supseteq g \wedge n \in \operatorname{rng} g_1).$$

The actual construction of J will require further restrictions on the domains of the partial isomorphisms. That is, we must find some nice subsets of a construction such that if the domain and range of a partial isomorphism are nice then it can be extended. The crucial step in establishing uniqueness is to see that any subset of a construction is contained in a nice set which is not much larger than itself. The criteria for B being a nice subset of the construction of E over A is that both B should be constructible over A and $E - B$ should be constructible over $A \cup B$. The following definition provides a more concrete way to identify such subsets.

4.8 Definition. If $\{\langle e_i, B_i\rangle : i < \alpha\}$ is a construction over A, then a subset $C \subseteq E$ is *closed* (relative to the construction E) if for each $e_i \in C$, $B_i \subseteq C \cup A$.

If C is a closed subset of the construction E then we can reorder the construction and put C first. More formally:

4.9 Notation. i) Let $E = \langle e_i : i < \alpha\rangle$ and suppose $C \subseteq E$. Then $<_C$ is the following linear order on α.

a) If $e_i \in C$ and $e_j \in E - C$ then $i <_C j$.
b) Restrict the ordering $<$ to C and to $E - C$.

ii) Let γ be the ordinal of $<_C$ restricted to C and write $C = \{c_i : i < \gamma\}$. Let D denote $E - C = \{d_i : i < \delta\}$. Define maps $f : \delta \mapsto \alpha$ and $g : \gamma \mapsto \alpha$ by $f(i) = j$ if $e_j = d_i$ and $g(i) = j$ if $e_j = c_i$.

It is easy to see that if C is a closed subset of the construction E over A then $C \cup A$ is constructible over A by $\{\langle c_i, B_{g(i)}\rangle : i <_C \gamma\}$. We proceed to the somewhat harder fact that in this situation $E \cup A$ is constructible over $A \cup C$.

4.10 Lemma. *If C is a closed subset of the construction E over A then $E - C = D$ is constructible over $A \cup C$ by $\{\langle d_j, B_{f(j)}\rangle : j < \delta\}$.*

Proof. We have to show that for each $j < \delta$, $p_j = t(d_j; A \cup C \cup D_j)$ is $\mathbf{I}$-isolated by $B_{f(j)}$. For this, we fix j and show by induction on $k < \gamma$ that $t(d_j; D_j \cup A \cup C_k \cup B_{f(j)})$ is $\mathbf{I}$-isolated by $B_{f(j)}$. Since $C_0 = \emptyset$, the ground case of the induction is trivial. The second continuity property described in Exercise 2.12 guarantees that the condition is preserved at limit ordinals. So suppose $k = l + 1$. If c_k preceded d_j in the original ordering the result is clear by monotonicity. So suppose $f(j) < g(k)$ and thus that $B_{f(j)} \subseteq C_k \cup D_j \cup A$ with $t(c_k; A \cup C_l \cup D_{j+1})$ $\mathbf{I}$-isolated. By induction, $t(d_j; D_j \cup C_l \cup A)$ is $\mathbf{I}$-isolated by $B_{f(j)}$. Applying 3.3 (the symmetry axiom for isolation) $t(d_j; D_j \cup C_k \cup A)$ is $\mathbf{I}$-isolated by $B_{f(j)}$ as required.

4.11 Corollary. *If E is constructible over A and $C \subseteq C' \subseteq E$ with C and C' closed, then C' is closed in the construction $\langle d_j : j < \delta \rangle$ of $A \cup D$ over $A \cup C$. Thus $A \cup C'$ is constructible over $A \cup C$.*

Proof. By Lemma 4.10, $t(d_j; A \cup C \cup D_j)$ is **I**-isolated by $B_{f(j)}$. Since $<_C$ agrees with $<$ on D, $B_{f(j)} \subseteq D_j \cup C \cup A$.

It is even easier to see that if E is constructible over A, C is closed in E, and $D \subseteq E - C$ is closed in the $<_C$-ordered construction of $E - C$ over $A \cup C$ then $C \cup D$ is closed in E.

We now state the main theorem of this section. We then sketch the proof, indicating two lemmas which we still must prove. Then we will prove those two lemmas.

4.12 Theorem. *Let E and F be **I**-prime and **I**-constructible over A. If $\lambda(\mathbf{I})$ is regular then E is isomorphic to F over A.*

Proof. We noticed in Theorem 4.3 that it suffices to consider the case $|E| > \lambda(\mathbf{I})$. Let J be the collection of partial A-isomorphisms between E and F such that each $g \in J$ satisfies the following conditions.

i) $\operatorname{dom} g$ is a closed subset of E.
ii) $\operatorname{rng} g$ is a closed subset of F.
iii) $|\operatorname{dom} g| < |E|$

We must show that J is closed under the union of increasing chains with cardinality less than $\operatorname{cf}(|E|)$ and that J satisfies the following back and forth conditions.

$$(\forall g \in J)(\forall e \notin \operatorname{dom} g)(\exists g_1 \in J)(g_1 \supseteq g \wedge e \in \operatorname{dom} g_1)$$

$$(\forall g \in J)(\forall f \notin \operatorname{rng} g)(\exists g_1 \in J)(g_1 \supseteq g \wedge f \in \operatorname{rng} g_1).$$

The closure under increasing chains is immediate. In order to treat the successor stages we have to show each point can be embedded in a small closed set (Lemma 4.12) and then use the results on short sequences to extend the map to the closed sets. The following lemma is a slightly refined version of Lemma 2.17.

4.13 Lemma. *Suppose $\lambda(\mathbf{I})$ is regular. Let $E = \{\langle e_i, B_i \rangle : i < \alpha\}$ be a construction over A and $D \subseteq E$ with $|D| < \lambda(\mathbf{I})$. There exists a closed subset C of E with $D \subseteq C$ and $|C| < \lambda(\mathbf{I})$.*

Proof. Define $\langle C_i : i < \omega \rangle$ by induction: $C_0 = D$, $C_{i+1} = \cup\{B_j : e_j \in C_i\}$ and let $C = \bigcup_{i<\omega} C_i$. Clearly, if $\lambda(\mathbf{I}) > \aleph_0$ then $|C| < \lambda(\mathbf{I})$. If $\lambda(\mathbf{I}) = \aleph_0$, define a partial order $<$ on C by: e_i is immediately below e_j if and only if $e_i \in B_j$. It is easy to see that $(C, <)$ is a finite branching tree with no infinite branch, so, by König's lemma, the tree has only finitely many nodes and so $|C| < \aleph_0$.

There is one more crucial lemma (Lemma 4.14) which enables us to conclude Theorem 4.12. To apply Lemma 4.14 to the proof of Theorem

4.12, let B from Lemma 4.14 be $A \cup \operatorname{rng} g = A \cup \operatorname{dom} g$, where we identify the domain and range of g via g.

4.14 Lemma. *Suppose $\lambda(\mathbf{I})$ is regular. Let E and F be $\mathbf{I}$-constructible and $\mathbf{I}$-saturated over B and $e \in E$. There is a partial B-isomorphism g between E and F with $e \in \operatorname{dom} g$ and such that both $\operatorname{dom} g$ and $\operatorname{rng} g$ are closed, and $|\operatorname{dom} g| < \lambda(\mathbf{I})$.*

Proof. We will define an increasing sequence of maps $\langle g_i : i < \omega \rangle$ such that, denoting $\operatorname{dom} g_i$ by C_i and $\operatorname{rng} g_i$ by D_i, we have $e \in C_0$, $|C_i|$, $|D_i| < \lambda(\mathbf{I})$, and C_{2i} (D_{2i+1}) is closed in E (F). Choose by Lemma 4.13 a closed subset C_0 containing e with $|C_0| < \lambda(\mathbf{I})$. Let g_0 map C_0 into F and let D_{-1} be empty. Suppose g_k, C_k and D_l for $k \leq 2i$, $l < 2i$ have been chosen as required. We will construct D_{2i}, D_{2i+1}, C_{2i+1}, and C_{2i+2}. Let D_{2i} be the range of g_{2i}. Choose by Lemma 4.13 a set D_{2i+1} containing D_{2i} which is closed and with $|D_{2i+1}| < \lambda(\mathbf{I})$. Since F is constructible over B and $|D_{2i}| < \lambda(\mathbf{I})$, by Lemma 4.1 F is constructible over $B \cup D_{2i}$. Since $|D_{2i+1} - D_{2i}| < \lambda(\mathbf{I})$ by Lemma 4.2, $D_{2i+1} - D_{2i}$ is constructible over $B \cup D_{2i}$. So by Theorem 2.46 there is an $(B \cup D_{2i})$-embedding of $B \cup D_{2i+1}$ into E. Call this map, which extends g_{2i}, g_{2i+1}^{-1}. Let C_{2i+1} be the image of this map and construct C_{2i+2} by Lemma 4.13. Now, $g = \bigcup g_i$ is the required map.

4.15 Historical Notes. A full discussion of the history of the uniqueness theorem for prime models is given in Section X.3 We only mention here that the straightforward proof for strictly prime models is due to Ressayre (unpublished).

5. *Locally Atomic Models*

One of the principal difficulties in the study of general stable theories is the lack of prime models over arbitrary sets. This section is devoted to a substitute notion, locally atomic models. We develop a notion of $\mathbf{L}$-atomic such that every subset of $\mathcal{M}$ for a countable stable theory can be embedded in an $\mathbf{L}$-saturated model. Unfortunately, there is no natural algebraic version of this notion of atomicity. That is, there does not seem to be a reasonable meaning for the phrase locally prime. However, as we will see in this section, locally atomic models are very useful for proving two cardinal theorems. They work well for this purpose because locally atomic models allow one to omit nonprincipal Δ-types for finite sets of formulas Δ and this is exactly what is needed for two cardinal theorems. Moreover, we will be able to use the existence theorem proved in this section to partially extend results about ω-stable theories to small superstable theories. Throughout this section T is stable.

The development here can not be given as a straightforward example of the type of isolation relation considered in Section IX.2. Shelah [Shelah

1978] developed a rubric which included both examples. However, to simplify the earlier exposition we used a scheme into which this notion does not fit very well. In this context we must modify our scheme somewhat. We define the class of **L**-*formulas* to be exactly the set **AT** of first order formulas. We vary the notion of isolation by substituting a weakened form of implication.

5.1 Definition. Let T be a countable stable theory. The type $p \in S(A)$ is **L**-*implied* (written $\vdash_L$) by $p|B$ for some countable $B \subseteq A$ if for every formula $\phi(\overline{x};\overline{y})$ there is a formula $\psi_\phi(\overline{x};\overline{b}) \in p|B$ such that $\psi_\phi(\overline{x};\overline{b}) \vdash p_\phi$. We also say $p|B$ *locally isolates* (**L**-isolates) p. Let $\lambda(\mathbf{L}) = \aleph_1$. As in Section IX.2, we have a corresponding notion of **L**-*atomic.* The structure A is **L**-saturated if every consistent **L**-formula over A is realized in A.

We noticed in Section IX.2 the equivalence (for the **I** discussed there) of two notions of **I**-saturation: i) Each type implied by a consistent **I**-formula is realized; ii) Each **I**-formula is realized. Since an **L**-isolated type need not be implied by a single **L**-formula this equivalence does not hold here. This divergence accounts for the nonuniformity of the proofs of Theorem X.1.18, Corollary X.1.19, etc. in the next chapter. Under the notion of **L**-saturation which we have chosen every model of a stable theory is **L**-saturated.

The choice of $\lambda(\mathbf{L})$ as $\aleph_1$ makes $\lambda(\mathbf{L})$ the bound on the size of an isolating set so it shares the crucial property of the $\lambda(\mathbf{I})$ for the other **I**. This definition of local isolation differs slightly from the definition of $F^l_{\aleph_1}$ in [Shelah 1978], IV.2. Pillay pointed out that the two definitions are equivalent. The following exercise verifies this assertion.

5.2 Exercise. Show that the type $p \in S(A)$ is **L**-implied by $p|B$ for some countable $B \subseteq A$ if for every $\phi(\overline{x};\overline{y})$ there is a formula $\psi_\phi(\overline{x};\overline{b}) \in p|B$ such that $\psi_\phi(\overline{x};\overline{b}) \vdash p^+_\phi$ where $p^+_\phi = \{\phi(\overline{x};\overline{a}) : \phi(\overline{x};\overline{a}) \in p\}$. (Hint: Show that if $\psi_\phi \vdash p^+_\phi$ and $\psi'_\phi \vdash p^+_{\neg\phi}$ then $\psi_\phi \wedge \psi'_\phi \vdash p_\phi$.)

Note that p^+_ϕ differs from p_ϕ since it does not contain instances of the negation of ϕ.

Although the isolating formulas for **L**-isolation involve only finitely many parameters we can in general isolate the type of an element only by using countably many elements.

5.3 Exercise. Show that if p is **L**-isolated over B, then p is $\mathbf{AT}_{\aleph_1}$ isolated over B.

Just as in Definition 2.7 we can define the notions of p being locally isolated by B or over B.

5.4 Exercise. Verify that **L**-isolation satisfies the continuity and monotonicity properties elaborated in Exercises 2.10 through 2.14.

To show that every subset of a model of a countable stable theory can be embedded in an **L**-saturated model we show that **L** satisfies the existence and transitivity axioms. For this task we use the local rank defined in Definition III.1.10.

5.5 Lemma. *For any set A and any formula $\chi(\overline{x};\overline{a})$ over A, there is a type $p \in S(A)$ with $\chi(\overline{x};\overline{a}) \in p$ and with p locally isolated by a countable subset of A.*

Proof. Fix $\chi(\overline{x};\overline{a}) \in F(A)$ and an enumeration $\langle \phi_i(\overline{x};\overline{y}) : i < \omega \rangle$ of $F(T)$. We define by induction a sequence of types p_n such that $p = \bigcup_{n<\omega} p_n$ locally implies a complete type over A and $\chi(\overline{x};\overline{a}) \in p$. First, let $p_0 = \{\chi(\overline{x};\overline{a})\}$. If p_n has been defined, let $p_{n+1} = p_n \cup \{\psi_n(\overline{x};\overline{a}_n)\}$ where ψ_n is chosen so that $R(p_{n+1}, \phi_n)$ is minimal among all consistent extensions of p_n over A. It suffices to show that for any $\psi(\overline{x};\overline{c}) \in F(A)$, if $p \cup \psi(\overline{x};\overline{c})$ is consistent then for some $n < \omega$, $p_n \vdash \psi(\overline{x};\overline{c})$. For, then p has a unique consistent extension $q \in S(A)$ and $p \vdash_L q$. To see this, fix such a $\psi(\overline{x};\overline{c})$ and let $\psi(\overline{x};\overline{y})$ be ϕ_m. If $p_{m+1} \cup \neg\psi(\overline{x};\overline{c})$ is also consistent then both that type and $p_{m+1} \cup \psi(\overline{x};\overline{c})$ have ranks at least as great as $R(p_{m+1}, \phi_m)$ by the choice of p_{m+1}. But this contradicts the definition of $R(p_{m+1}, \phi_m)$.

5.6 Exercise. Show that for uncountable T, we can define **L**-isolation as we did for countable languages but that we must take $|T|^+$ for $\lambda(\mathbf{L})$.

5.7 Lemma. **L**-*isolation satisfies the transitivity axiom.*

Proof. Assume we have $B \subseteq A$ such that $p = t(\overline{d}; A \cup C)$ is locally isolated over $B \cup C$ and for each $\overline{c} \in C$, $t(\overline{c}; A)$ is locally isolated over B. We must show that for any $\overline{c}' \in C$, $t(\overline{d}^\frown\overline{c}'; A)$ is locally isolated over B. That is, for any formula $\phi(\overline{x},\overline{y};\overline{z})$ we must find a formula $\psi(\overline{x},\overline{y};\overline{b})$ over B such that if $\models \phi(\overline{d},\overline{c}';\overline{a})$ with $\overline{a} \in A$ then $\models \psi(\overline{x},\overline{y};\overline{b}) \rightarrow \phi(\overline{x},\overline{y};\overline{a})$. We have a formula $\chi(\overline{x};\overline{b}_1,\overline{c}_1)$ such that $\chi(\overline{x};\overline{b}_1,\overline{c}_1) \rightarrow \phi(\overline{x};\overline{c},\overline{a})$ where $\overline{b}_1 \in B$ and $\overline{c}_1 \in C$ since $t(\overline{d}; A \cup C)$ is locally isolated over $B \cup C$. Moreover, there is a $\phi_1(\overline{y},\overline{z};\overline{b}_2)$ such that $\models \phi_1(\overline{c},\overline{c}_1;\overline{b}_2)$ and

$$\models [\phi_1(\overline{y},\overline{z};\overline{b}_2) \rightarrow ((\forall\overline{x})\chi(\overline{x},\overline{b}_1,\overline{z}) \rightarrow \phi(\overline{x},\overline{y},\overline{a}_1))]$$

since $t(\overline{c}^\frown\overline{c}_1; A)$ is locally isolated over B. Now, $(\exists\overline{z})\phi_1(\overline{y},\overline{z},\overline{b}_2) \wedge \chi(\overline{x},\overline{b}_1,\overline{z})$ is the required formula $\psi(\overline{x},\overline{y};\overline{b})$ where $\overline{b} = \overline{b}_1{}^\frown\overline{b}_2$.

Recall Definition 2.36 of $\mu(\mathbf{I})$. The following exercise depends crucially on our definition (Definition 5.1) of **L**-saturation.

5.8 Exercise. Show that $\mu(\mathbf{L}) = \aleph_0$.

Now by a proof like that of Theorem 2.40 we can deduce

5.9 Theorem. *If T is a countable stable theory then every set A is contained in a model M with $|M| \leq |A| + \aleph_0$ which is* **L**-*saturated and* **L**-*atomic over A.*

L-isolation does not satisfy the transposition property (Axiom 3.8). Thus, the proof of the uniqueness of strictly prime models given in Section 4 does not work for this notion.

5.10 Exercise. Verify the failure of the transposition property by considering the theory, T, of an infinite set. If $M \models T$ and $m \in M$, then $t(M;\emptyset)$ is **L**-atomic but $t(m; M - \{m\})$ is not **L**-isolated.

5.11 Exercise. Show that if T is the theory of infinitely many independent unary predicates, T has $2^{\aleph_0}$ non-isomorphic models which are all **L**-constructible over the empty set.

Now we use this construction to prove the two cardinal theorem for stable theories. This will complete the proof of Morley's categoricity theorem from Section I.3.

5.12 Definition. The theory T *admits the pair of cardinals* (κ, λ) if for some formula $\phi(\overline{x}; \overline{m})$ and some model M of T, $|M| = \kappa$ and $|\phi(M, \overline{m})| = \lambda$. We also say, T has a (κ, λ) model.

5.13 Theorem. *If a countable stable theory admits some pair* (κ, λ) *with* $\kappa > \lambda$ *then it admits any pair* (κ', λ') *with* $\kappa' \geq \lambda'$.

Proof. Assuming the hypothesis, it is easy to show by the Löwenheim-Skolem theorem that there is a pair of models M, N of T with $|M| = |N| = \lambda'$ such that N is a proper submodel of M but $\phi(N, \overline{n}) = \phi(M, \overline{n})$ for some $\overline{n} \in N$. We will show how to construct from any such pair of models a third model M' which is a proper elementary extension of M but with $\phi(M', \overline{n}) = \phi(N, \overline{n})$. With this in hand it is easy to construct by induction on $\alpha < \kappa'$ a chain of models M_α such that for all α, $\phi(M_\alpha, \overline{n}) = \phi(N, \overline{n})$. The union of this chain demonstrates that T admits (κ', λ').

To construct M', let a be an arbitrary element of $M - N$ and let c realize the nonforking extension over M of $t(a; N)$. We claim that for any formula $\psi(x, \overline{m}, y)$ over M, if $\models (\exists y)(\psi(c, \overline{m}, y) \wedge \phi(y, \overline{n}))$ then for some c' in $\phi(N, \overline{n})$, $\models \psi(c, \overline{m}, c')$. To see this, let d define $t(a; N)$. For every $\overline{n}' \in N$, if $\models (\exists y)(\psi(a, \overline{n}', y) \wedge \phi(y, \overline{n}))$ then for some $c_0 \in N$, $\models \psi(a, \overline{n}', c_0) \wedge \phi(c_0, \overline{n})$. Thus

$$(\forall \overline{z})\,[d\,((\exists y)\psi(x, \overline{z}, y) \wedge \phi(y, \overline{n}))\,[\overline{z}, y] \rightarrow (\exists y)(\phi(y, \overline{n}) \wedge d(\psi(x, \overline{z}, y))[\overline{z}, y])]$$

is true. Since this formula also holds of c we establish the claim.

Now let M' be locally atomic over $M \cup c$. If some element d of $M' - N$ satisfies $\phi(d, \overline{n})$, then by local atomicity there is a formula $\psi(c, \overline{m}, x)$ such that $\models \psi(c, \overline{m}, x) \rightarrow \phi(x, \overline{m})$ and $\models \psi(c, \overline{m}, x) \rightarrow x \neq n$ for each n in $\phi(N, \overline{n})$. But this contradicts the choice of c and we finish the theorem.

5.15 Exercise. Verify the first sentence of the proof of Theorem 5.14. (Hint: Let M witness that T admits (κ, λ). Choose an elementary submodel of M which contains $\phi(M, \overline{m})$ but has cardinality λ. Add to the language a unary predicate to pick out this submodel and then apply the Löwenheim-Skolem theorem.)

5.16 Exercise. Prove that any countable stable theory which admits a two cardinal model admits (ω_1, ω). Apply the technique from Theorem 5.14 to build a chain of models but replace the use of locally atomic models by an application of the omitting types theorem.

We can extend the two cardinal theorem to uncountable languages as follows. Recall that Vaught's two cardinal theorem for cardinals far apart, [Vaught 1965], asserts that if a theory T has a (κ, λ) model with $\kappa \geq \beth_\omega(\lambda)$ then T has a (κ', λ') model for any pair $\kappa' \geq \lambda' \geq |T|$. The key step in Vaught's proof is to show that in the Skolem theory of T the set of sentences which express the following properties is consistent. First, $\{\langle c_n : n < \omega \rangle\}$ is a set of order indiscernibles in $L(T)$. Second, if U denotes the distinguished predicate and $\overline{c}, \overline{c}'$ are increasing sequences of the indiscernibles which satisfy U, then for any Skolem function τ, $\tau(\overline{c}) = \tau(\overline{c}')$. Since every reduct of a stable theory T to a countable language is stable, we can easily establish the consistency of this set of sentences by appealing to the two cardinal theorem for countable stable theories. We have outlined the proof of the following theorem.

5.17 Theorem. *If a stable theory admits some pair (κ, λ) with $\kappa > \lambda$ then it admits any pair (κ', λ') with $\kappa' \geq \lambda' \geq |T|$.*

We pointed out in Exercise 5.6 that the notion of locally atomic models extends to uncountable languages. Using this observation there is another approach to the two cardinal theorem for uncountable theories.

5.18 Exercise. Derive Theorem 5.17 by a proof in the style of the proof given for Theorem 5.14.

5.19 Historical Notes. The study of two cardinal theorems began with Vaught's proof [Vaught 1961] that an $\aleph_1$-categorical theory could not have a two cardinal model. He deduced this result from the transfer principle $(\kappa, \lambda) \mapsto (\aleph_1, \aleph_0)$ which he showed holds for an arbitrary countable first order theory in that same paper. Vaught [Vaught 1965] and Morley [Morley 1965a] then independently showed that for a countable theory T, $(\beth_\omega, \omega) \mapsto (\kappa, \omega)$ for any κ. Attempts to extend these results to arbitrary first order theories led via the Chang two cardinal theorem [Chang & Keisler 1973] to problems which are more set theoretic than model theoretic in character. Shelah [Shelah 1969] and Lachlan [Lachlan 1972] restored the subject to the realm of model theory by discovering the relevance of assuming that the theory is stable. The general outline of the proof of Theorem 5.14 comes from [Lachlan 1972]. The exact construction here is from [Lascar 1973]; a similar version of the proof using conservative extensions and compactness rather than definability occurs in [Baldwin 1975]. The concept of a locally atomic model, though not the name, occurs in [Lachlan 1972] and [Shelah 1971]. Lachlan dealt only with countable languages while Shelah included the uncountable case. There are various extensions of this notion in [Shelah 1978]. The two cardinal theorem for uncountable stable theories was originally proved by Shelah in his thesis invoking an absoluteness argument. Harnik outlines five different proofs of the theorem in [Harnik 1975]. Alan Mekler brought this fairly direct proof to my attention when he discovered it; Harnik had earlier noted its existence in the cited paper.

6. *The Number of Models of Strictly Stable Theories*

This section is devoted to a proof that a strictly stable (i.e. a stable but not superstable theory) has 2^λ models of power λ if $\lambda > |T|$ and λ is regular. In Chapter VIII of [Shelah 1978], the result is proved by much more complicated extensions of the methods here for arbitrary $\lambda > |T|$. The idea of Shelah's proof was discussed in Section I.5.

Our proof will require three ingredients. The first is the existence of a certain indiscernible tree if T is stable but not superstable. This was proved in Theorem III.4.25. The second is the development of the properties of a further notion of isolation. The third is certain combinatorial facts about stationary sets.

We begin by describing a notion of isolation which does not fit particularly well with the scheme we have described in Section 2; however, we are already very familiar its properties. Throughout this section we assume that T is stable.

6.1 Definition. The type $p \in S(A)$ is $\mathbf{F}_\lambda$ isolated if there is a subset B contained in A with $|A| < \lambda$ such that p does not fork over B. We will write $\mathbf{F}$ for $\mathbf{F}_{\aleph_0}$.

Shelah calls this notion $\mathbf{F}_\lambda$-isolation.

Note that we can squeeze this notion into our general scheme by taking as the set of isolating formulas the collection $\mathbf{AT}_\lambda$ but replacing the relation of provability by the notion q 'pseudoproves' p if p is a nonforking extension of q. With this in mind we can continue to use the terminology defined in Sections 2 and 3.

6.2 Exercise. Show that any p is $\mathbf{F}_{\kappa(T)}$-isolated.

Thus, when discussing countable models the only interesting case is that of $\mathbf{F}_{\aleph_0}$ and then only when T is not superstable. It is easy to see that the basic properties of an isolation relation discussed in section 2 hold for $\mathbf{F}_\lambda$-isolation.

6.3 Exercise. Show that for any λ, $\mathbf{F}_\lambda$-isolation satisfies the monotonicity and continuity properties discussed in Exercises 2.9 through 2.14.

6.4 Exercise. Deduce the symmetry, transitivity, and transposition properties of $\mathbf{F}_\lambda$-isolation from the analogous statements in Chapter III.

The anomaly of suddenly treating as a dependence notion the concept we have regarded as a notion of independence for much of the book has no clear explanation. There is one property of the other isolation relations which is not shared by $\mathbf{F}_\lambda$ and this provides some insight. For any of the other isolation relations, $\mathbf{I}$, there is a cardinal $\mu(\mathbf{I}) < \infty$ which enables us (Theorem 2.40), in the presence of the Existence Axiom, to embed each set in an **I**-saturated model. In this case no such cardinal exists and, in fact, there are no $\mathbf{F}_\lambda$-saturated models.

However, the Existence Axiom, 2.28, is easily seen to hold for $\mathbf{F}_\lambda$. Thus, a variant of the proof of Theorem 2.40 shows the existence of $\mathbf{F}_\lambda$-constructible models over arbitrary sets.

6.5 Lemma. *If T is stable then over A there is an $\mathbf{F}_\lambda$-constructible model.*

Proof. Enumerate the types over A and realize them in order by an element which does not fork over its predecessors in the sequence. Iterating this procedure ω times produces a model of T.

Note that this model is by no means prime in the normal sense.

6.6 Exercise. Show that there are two models of the theory of infinitely many independent unary predicates which are each **F**-constructible over the empty set such that neither can be embedded in the other.

Before proving the main theorem, we require the following combinatorial notions.

6.7 Definition. A subset C of the cardinal λ is *closed and unbounded (cub)* if

i) for every $\alpha < \lambda$ there is a $\beta \in C$ with $\alpha < \beta$, and
ii) if $X \subseteq C$ and $|X| < \lambda$ then $\sup(X) \in C$.

A subset S of λ is *stationary* if S intersects every *cub*.

The key facts needed about these sets are contained in the following lemma. The proof of this lemma can be found in a number of works on set theory, e.g. [Kunen 1980].

6.8 Lemma. *Let $\lambda > \aleph_0$ be regular.*

i) *The set, Λ_0, of $\alpha < \lambda$ with cofinality ω is stationary.*
ii) *For any strictly increasing continuous function (in the order topology on λ) g from λ to itself, the set*
$$C = \{i < \lambda : \textit{for all } j < i,\ g(j + \omega) < i\}$$
is a cub.
iii) *There exist a family of λ pairwise disjoint stationary subsets of λ.*
iv) *There exist a family $\langle W_i : i < 2^\lambda \rangle$ of stationary subsets of λ such that if $i \neq j$ then $W_i - W_j$ is stationary. The same result holds replacing λ by Λ_0.*

Statements i) and ii) of the lemma are fairly straightforward from the definitions. Statement iii) is a well known theorem of Ulam (See Theorem 1.3.2 in the Appendix of [Shelah 1978] or Corollary 6.12 of [Kunen 1980].) The following exercises indicate how iv) can be fairly easily deduced from iii).

6.9 Exercise. Show there exist a family of 2^λ pairwise incomparable subsets of λ. (That is, subsets satisfying iv) but with $W_i - W_j \neq \emptyset$ replacing the requirement that $W_i - W_j$ is stationary.) (Hint: Almost disjoint sets suffice. Two sets are almost disjoint if their intersection is finite.)

6.10 Exercise. Combine the previous exercise with iii) to obtain iv).

To aid the comprehension of the following complicated argument we fix a number of notational conventions for this section.

6.11 Notation. Elements of $\lambda^{<\omega}$ (finite sequences of ordinals $< \lambda$) are denoted by s or t; elements of λ^ω (countable sequences of ordinals $< \lambda$) are denoted by σ or τ; elements of $\lambda^{\leq\omega} = \lambda^{<\omega} \cup \lambda^\omega$ are denoted by ν or η.

We denote by I the structure $\langle \lambda^{\leq\omega}, \subseteq, < \rangle$ with the indicated universe, with $\subseteq$ interpreted as initial segment relation between functions, and with $<$ interpreted as lexicographic order.

The following assertion can be easily derived from Theorem III.4.25.

6.12 Lemma. *If T is stable but not superstable then for every infinite cardinal λ there is a model M, which contains a set $A = \{\overline{a}_\eta : \eta \in \lambda^{\leq\omega}\}$ with the following property. For each $B \subseteq A$ and each $\eta \in \lambda^\omega$, if B contains $\{\overline{a}_{\eta|n} : n < \omega\}$ then $t(\overline{a}_\eta; B)$ is not* **F***-isolated. Moreover, A is an independent tree in the sense of Definition II.2.25.*

We fix the tree A from Lemma 6.12 for the remainder of this section.

6.13 Theorem. *If T is stable but not superstable then for every regular $\lambda > |T|$, $I(\lambda, T) = 2^\lambda$.*

Proof. Recall that Λ_0 denotes the set of ordinals less than λ which have cofinality ω. Let W be the set of 2^λ pairwise incomparable subsets of Λ_0 guaranteed by Lemma 6.8. For each set of ordinals $w \in W$ we define a model M^w of T as follows. For each $\delta \in \Lambda_0$ fix one strictly increasing function $\sigma_\delta \in \lambda^\omega$ which witnesses that $\mathrm{cf}(\delta) = \omega$. For $w \in W$, let

$$I^w = \lambda^{<\omega} \cup \{\sigma_\delta : \delta \in w\}.$$

Let M^w be **F**-constructible over $A^w = \{\overline{a}_\nu : \nu \in I^w\}$. Now if w, u are distinct elements of W, we will deduce a contradiction from the assumption that there is an isomorphism between M^w and M^u.

In order to effect this contradiction we will approximate the tree I^w by the set of functions with bounded range. For any $i < \lambda$, $I_i^w = \bigcup_{\beta < i} \beta^{\leq\omega} \cap I^w$. Similarly, $A_i^w = \{\overline{a}_\eta : \eta \in I_i^w\}$.

Now the crucial fact is contained in the following assertion.

6.14 Claim. *For any $w \in W$, A^w is* **F***-atomic over A_δ^w if and only if $\delta \notin w$.*

Proof. This follows routinely from the fact that A is an independent tree. For, if $B \subseteq A^w$, $t(B; A_\delta^w)$ is **F**-isolated if and only if the downward closure of B, i.e. $\{\overline{a}_\sigma : \sigma \subseteq \tau \wedge \overline{a}_\tau \in B\}$, intersects A_δ^w in a finite set. But, if $\delta \notin w$, it easy to check that this condition holds. Moreover, the condition clearly fails if $\delta \in w$.

Let M^w be constructed by the sequence $\{c_\alpha^w : \alpha < \lambda\}$. Suppose that $t(c_\alpha^w; A^w \cup C_\alpha^w)$ is $\mathbf{F}$-isolated by $\overline{b}_\alpha^w$. (Remember C_α^w is $\{c_\beta^w : \beta < \alpha\}$.) Now we construct the closed unbounded subsets of Λ_0 whose intersection will lead to the contradiction. Let G^w denote the set of α such that for each $\beta < \alpha$, $\overline{b}_\beta^w \subseteq A_\alpha^w \cup C_\alpha^w$.

6.15 Exercise. Show G^w is the set of α such that $A_\alpha^w \cup C_\alpha^w$ is the universe of an $\mathbf{F}$-construction.

Now, it is easy to see that G^w, G^u and the set S of those α such that g maps $A_\alpha^w \cup C_\alpha^w$ onto $A_\alpha^u \cup C_\alpha^u$ are closed unbounded sets. Thus there is an ordinal δ in $G^w \cap G^u \cap S \cap w$ and so not in u. This δ will yield the contradiction.

We first argue that M^u is $\mathbf{F}$-atomic over $A_\delta^u \cup C_\delta^u$. Claim 6.14 implies, that A^u is $\mathbf{F}$-atomic over A_δ^u. Since $\delta \in G^u$, C_δ^u is $\mathbf{F}$-atomic over A_δ^u. We can prove from the definition of G^u by induction on the length of C_δ^u that $C_\delta^u \downarrow_{A_\delta^u} A^u$. Now by transitivity of the nonforking relation, A^u is $\mathbf{F}$-atomic over $A_\delta^u \cup C_\delta^u$. By transitivity of isolation we conclude M^u is $\mathbf{F}$-atomic over $A_\delta^u \cup C_\delta^u$. In particular, $t(g(\overline{a}_{\sigma_\delta}^w); A_\delta^u \cup C_\delta^u)$ is $\mathbf{F}$-isolated. But this contradicts the assumption that g is an isomorphism. For, since $\delta \in S$, we can assume that g maps $A_\delta^u \cup C_\delta^u$ to $A_\delta^w \cup C_\delta^w$ and $t(\overline{a}_{\sigma_\delta}; A_\delta^w \cup C_\delta^w)$ is not $\mathbf{F}$-isolated. As, if it were, we could apply the fact that $\delta \in G^w$ to see that C_δ^w is $\mathbf{F}$-atomic over A_δ^w which implies by Theorem II.2.5 that $C_\delta^w \cup a_\delta^w$ is $\mathbf{F}$-atomic over A_δ^w. But this contradicts the fundamental fact about the construction, Claim 6.12, that $t(\overline{a}_\delta; A_\delta^w)$ is not $\mathbf{F}$-isolated since $\delta \in w$.

6.16 Corollary. *If T is a stable but not superstable theory the function $I(\kappa, AT)$ which gives the number of models of T of power κ is increasing.*

Proof. Observe that the previous proof actually shows that if λ is a regular cardinal less than or equal to κ then $I(\kappa, AT) \geq 2^\lambda$. Thus, for regular cardinals we have the maximum number of models and the function does not decrease at a singular cardinal.

6.17 Historical Notes. This proof given here is the proof of Theorem VIII.2.7 of [Shelah 1978]. The contrast with proofs using Ehrenfeucht-Mostowski models which dominate Chapter VIII of [Shelah 1978] is discussed in Section I.5. Pillay [Pillay 1981] provides a very nice exposition of the Ehrenfeucht-Mostowski version of the proof when λ is regular. Hodges [Hodges 198?] is writing an exposition of Shelah's approach to the nonstructure results. Conversations with Jürgen Saffe greatly aided my writing of this section.