ON THE REGULARITY AND RIGIDITY THEOREMS AND PROBLEMS FOR THE SOLUTIONS OF SOME CLASS OF THE DEGENERATE ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This is a short survey paper based on the topics of the author's research along with his collaborators. Main topics presented in the paper are on the regularity and rigidity theorems for the solutions of the elliptic differential equations. In particular, the author poses several open problems in these topics for further study. The paper contains six sections. Regularity theorems for the elliptic equations of the non-divergence form with rough coefficients are presented in Section 1. In Section 2, we introduce and summarize some recent developments on the rigidity problems and theorems for the solution of some linear degenerate elliptic equations. A typical example is the rigidity problem for the solution of the Dirichlet problem of the Laplace-Beltrami operator on the unit ball in $\mathbb{C}^n$ with Bergman metric. In Section 3, the rigidity theorems and problems for harmonic maps between two complete non-compact Kähler manifolds are discussed. In Section 4, we summarize the approximation formula for the potential function of the Kähler-Einstein metric. In Section 5, we summarize some rigidity theorems for the degenerate Monge-Ampère equations as well as some characterization theorems for some strictly pseudoconvex pseudo-Hermitian manifolds. Finally, in Section 6, we summarize some recent results on the bottom of the spectrum of the Laplace-Beltrami operators on Kähler manifolds.

1. BOUNDARY VALUE PROBLEM FOR UNIFORM ELLIPTIC PDES

Let Ω be a bounded domain in $\mathbb{R}^n$, and let $A(x) = [a^{ij}(x)]$ be an $n \times n$ symmetric matrix-valued function on Ω. Let

$$\mathcal{L}_A = \sum_{i,j=1}^{n} a^{ij} \frac{\partial^2}{\partial x_i \partial x_j}. \tag{1.1}$$

Then we say that $\mathcal{L}$ is elliptic if there are non-negative functions $\lambda(x)$ and $\Lambda(x)$ such that

$$\lambda(x) I_n \le A(x) \le \Lambda(x) I_n, \quad x \in \Omega. \tag{1.2}$$

We say that $\mathcal{L}_A$ is uniformly elliptic if there are two positive constants λ_0 and Λ_0 such that

$$\lambda_0 \le \lambda(x) \le \Lambda(x) \le \Lambda_0, \quad x \in \Omega. \tag{1.3}$$

We say that $\mathcal{L}_A$ is degenerate elliptic if $\mathcal{L}$ is not uniformly elliptic on Ω. For

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any $0 < \alpha \le 1$, we let $C^{0,\alpha}(\Omega) = C^{0,\alpha}(\overline{\Omega})$ denote the set of all continuous functions u on Ω such that

$$\|u\|_{C^{0,\alpha}(\Omega)} = \sup\Big\{|u(x)| + \frac{|u(x)-u(y)|}{\|x-y\|^\alpha} : x, y \in \Omega,\ x \ne y\Big\} < \infty. \tag{1.4}$$

For any positive integer k and $0 < \alpha \le 1$, we let $C^{k,\alpha}(\Omega) = C^{k,\alpha}(\overline{\Omega})$ denote the set of all functions u having all partial derivatives up to order k and satisfying

$$\begin{aligned} &\|u\|_{C^{k,\alpha}(\Omega)} \\ = \;& \|u\|_{C^k(\overline{\Omega})} + \sup\Big\{\frac{\sum_{|\beta|=k}\Big|D^\beta u(x) - D^\beta u(y)\Big|}{\|x-y\|^\alpha} : x, y \in \Omega, x \ne y\Big\} < \infty, \end{aligned} \tag{1.5}$$

where

$$D^\beta u(x) = \frac{\partial^{|\beta|} u}{\partial x^\beta}(x), \quad \beta = (\beta_1, \cdots, \beta_n) \quad \text{and } \ |\beta| = \sum_{j=1}^n \beta_j. \tag{1.6}$$

With the above notations, one may prove the following proposition.

Proposition 1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{k,\alpha}$ boundary. Then $(C^{k,\alpha}(\Omega); \|\cdot\|_{C^{k,\alpha}})$ forms a Banach space for any non-negative integer k and $0 < \alpha < 1$.*

1.1. **Existence, uniqueness and regularity.** The following theorem is a well-known result in the elliptic theory of linear PDEs, which can be found in the book of Gilbarg and Trudinger [23] and in the book of Evans [20].

Theorem 2. *Let $k \ge 2$ be an integer. Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{k,\alpha}$ boundary $\partial\Omega$. Let $\mathcal{L}$ be a uniformly elliptic operator with $a^{ij} \in C^{k-2,\alpha}(\Omega)$ and if $\phi \in C^{k,\alpha}(\partial\Omega)$ and $f \in C^{k-2,\alpha}(\Omega)$. Then the Dirichlet boundary value problem:*

$$\begin{cases} \mathcal{L}_A u = f, & \text{in } \Omega \\ u = \phi, & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

has a unique solution $u \in C^{k,\alpha}(\Omega)$ for any $0 < \alpha < 1$.

• If there are no any smooth assumptions on a^{ij}, then the boundary value problem for the non-divergence form elliptic equation (1.7) may not have a weak solution because integration by parts may not make sense.
• If $a^{ij} \in C(\overline{\Omega})$, then (1.7) has a unique weak solution (see [23]). Even if a weaker condition, $a^{ij} \in VMO(\Omega)$, (1.7) still has a unique solution (see [12, 13]). S. Byun and L. Wang wrote a series papers on how to weaken the condition on a^{ij} so that (1.7) has a unique weak solution and provide regularities (see [5, 6, 7]). Here, we only state a theorem by Chiarenza, Frasca and Longo in [12, 13].

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary and $1 < p < \infty$. Let $\mathcal{L}_A$ be the uniform elliptic operator satisfying (1.2) and (1.3) with $a^{ij} \in VMO(\Omega)$. Then the boundary value problem (1.7) with $\phi = 0$ on $\partial\Omega$ and $f \in L^p(\Omega)$ has a unique solution $u \in W^{2,p}(\Omega)$ and*

$$\|u\|_{W^{2,p}(\Omega)} \le C_{n,p,\Omega}\|f\|_{L^p(\Omega)} \tag{1.8}$$

where $C_{n,p,\Omega}$ is constant depending only on p, n, Ω, λ_0 and Λ_0.

It is well known that u may not belong to $W^{2,p}(\Omega)$ when $f \in L^p(\Omega)$ when $p = 1$ and ∞ even if $a^{ij} = \delta_{ij}$. Instead, one may replace L^∞ by BMO and L^1 by H^1. The following theorem was proved by Chang and Li in [9].

• A function a on Ω is said to be in Dini(Ω) if

$$\|a\|_{Dini(\Omega)} = \int_0^1 \frac{\omega(a,t)}{t} dt, \tag{1.9}$$

where

$$\omega(a,t) = \sup\{|a(x) - a(y)| : |x - y| \le t, \ x, y \in \Omega\}. \tag{1.10}$$

Theorem 4. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $\mathcal{L}_A$ be the uniform elliptic operator satisfying (1.2) and (1.3). Then*

(i) If $\partial\Omega$ is $C^{1,1}$ and $a^{ij} \in Dini(\Omega)$, then

$$\Big\|\frac{\partial^2}{\partial x_i \partial x_j}\mathcal{L}_A^{-1}(f)\Big\|_{BMO_r(\Omega)} \le C_{A,\Omega}\|f\|_{BMO_z(\Omega)} \tag{1.11}$$

(ii) If $\partial\Omega$ is Lipschitz ($C^{0,1}$) and $a^{ij} \in VMO(\Omega)$, then

$$\Big\|\frac{\partial^2}{\partial x_i \partial x_j}\mathcal{L}_A^{-1}(f)\Big\|_{H^1_r(\Omega)} \le C_{A,\Omega}\|f\|_{H^1_z(\Omega)} \tag{1.12}$$

where $BMO_r(\Omega)$ is the restriction of $BMO(\mathbb{R}^n)$ on Ω and $BMO_z(\Omega)$ is a subset of $BMO(\mathbb{R}^n)$ with zero value on $\mathbb{R}^n \setminus \Omega$. One can define $H^1_r(\Omega)$ and $H^1_z(\Omega)$ respectively in a similar manner.

• **Uniqueness:** Uniqueness can be obtained by the Maximum Principle.

• **Existence:** Existence can be obtained by the method of continuity plus an *a priori* estimate. We will explain the idea here; details can be found in the book of Gilbarg and Trudinger [23] and references therein.

• **Method of Continuity:**

Theorem 5. *Let T_0 and T_1 be two densely defined, bounded linear operators from Banach space X_1 to X_2. Assume that there is a constant C such that*

$$\|u\|_{X_1} \le C\|((1-t)T_0 + tT_1)u\|_{X_2}, \quad \text{for all } u \in Dom(T_0) \cap Dom(T_1) \tag{1.13}$$

and all $t \in [0,1]$. If $T_0 : X_2 \to X_2$ is onto, then $T_1 : X_1 \to X_2$ is onto.

Proof. Write

$$T_t = (1-t)T_0 + tT_1. \tag{1.14}$$

Then

$$\|T_t u\|_{X_2} = \|(1-t)T_0 u + tT_1 u\|_{X_2}.$$

Let $I = \{t \in [0,1] : T_t(X_1) = X_2\}$. Then $0 \in I$. It is sufficient to $1 \in I$. Assume that $t_1 \in I$. Then $T_{t_1}^{-1} : X_2 \to X_1$ is bounded, and $\|T_{t_1}^{-1}\| \le C$. Notice that

$$T_t = T_{t_1} + (t - t_1)(T_1 - T_0) = T_{t_1}[I + (t - t_1)T_{t_1}^{-1}(T_1 - T_0)]. \tag{1.15}$$

One can easily see that if

$$|t - t_1|\|T_{t_1}^{-1}\|(\|T_0\| + \|T_1\|) < 1, \tag{1.16}$$

then $t \in I$. By (1.11), if T_t is invertible, then $\|T_t^{-1}\| \le C$. After repeating the above process $N =: [C(\|T_0\| + \|T_1\|] + 1$ times, one has that $1 \in I$. The proof is complete. □

• *A priori* estimate (1.13) for T_t with $T_0 = \Delta$ and $T_1 = \mathcal{L}_A$.

Let

$$\tilde{\lambda}_0 = \min\{\lambda_0, 1\} \text{ and } \tilde{\Lambda}_0 = \min\{\Lambda_0, 1\}. \tag{1.17}$$

Then

$$\tilde{\lambda}_0 I_n \le t I_n + (1-t)A \le (t + \Lambda_0(1-t))I_n = \tilde{\Lambda}_0 I_n. \tag{1.18}$$

Notice the fact that

$$t\Delta + (1-t)\mathcal{L}_A = \mathcal{L}_{tI_n + (1-t)A}. \tag{1.19}$$

To prove $t\Delta + (1-t)\mathcal{L}_A$ satisfies *a priori* estimate (1.13), by (1.18) and (1.19), it suffices to prove *a priori* estimate for $\mathcal{L}_A$ with A satisfying (1.2) and (1.3).

We will sketch the main idea of the proof of (1.8) as originally proved in [12] and [13] and the idea of the proofs of (1.11) and (1.12) in [9] using integral operators. Without loss of generality, we may only consider the case $n \ge 3$.

1.2. **Potential functions.**

• It is well known that the the Newton potential for Δ on $\mathbb{R}^n$ is

$$N(x,y) = -\frac{1}{(n-2)\omega_n}\frac{1}{|x-y|^{n-2}}, \quad x, y \in \mathbb{R}^n \tag{1.20}$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$, which means that

$$\Delta_x N(x,y) = \delta_y \tag{1.21}$$

where δ_y is the Dirac mass concentrated at y. Let

$$\|x-y\|^2_{A(w)} =: \sum_{i,j=1}^n a^{ij}(w)(x_i - y_i)(x_j - y_j), \quad x, y \in \mathbb{R}^n \tag{1.22}$$

with $[a^{ij}(w)]$ being the inverse matrix of $A(w)$.

• When $A = [a^{ij}(x)]$ is a constant positive definite matrix, then the potential function (Green function) for $\mathcal{L}_A$ on $\mathbb{R}^n$ is

$$N_A(x,y) = -\frac{1}{(n-2)\omega_n\sqrt{\det A}}\|x-y\|_A^{-(n-2)}. \tag{1.23}$$

This means that

$$(\mathcal{L}_A)_x N_A(x,y) = \delta_y, \quad y \in \mathbb{R}^n. \tag{1.24}$$

• When $A = [a^{ij}(x)]$ is not a constant matrix, then the Green function for $\mathcal{L}_A$ on Ω is much more complicated. However, one may divide the domain into many small pieces and use constant matrices to approximate $A(x)$ in each small piece. In order to do this, we will use the following notation:

$$\Gamma(w;\xi) = -\frac{1}{(n-2)\omega_n\sqrt{\det A(w)}}\|\xi\|_{A(w)}^{-n+2}, \quad w\in\Omega,\ \xi\in\mathbb{R}^n \tag{1.25}$$

and

$$\Gamma_i(w;\xi) = \frac{\partial\Gamma(w;\xi)}{\partial\xi_i} \quad \Gamma_{ij}(w;\xi) = \frac{\partial^2\Gamma(w;\xi)}{\partial\xi_i\partial\xi_j}, \quad 1\le i,j\le n. \tag{1.26}$$

For $w\in\Omega$ and $x,y\in\mathbb{R}^n$, we will use the following notations:

$$\Gamma(w;x,y) = \Gamma(w;x-y) \tag{1.27}$$

and similarly, $\Gamma_i(w;x,y)=\Gamma_i(w;x-y)$, $\Gamma_{ij}(w;x,y)=\Gamma_{ij}(w;x-y)$.

• If we write

$$\mathcal{L}_{A(x)}u(y) = \sum_{i,j=1}^n a^{ij}(x)\frac{\partial^2 u(y)}{\partial y_i\partial y_j}, \quad x,y\in\Omega, \tag{1.28}$$

then for $u\in C_0^2(\Omega)$, for any fixed $w\in\Omega$, one has that

$$u(x) = \int_\Omega \Gamma(w;x-y)\mathcal{L}_{A(w)}u(y)dy, \quad x\in\Omega. \tag{1.29}$$

By applying the Divergence theorem, if one applies $\frac{\partial^2}{\partial x_i\partial x_j}$ to (1.29), one has that

$$\begin{aligned}(1.30)\quad & \frac{\partial^2 u(x)}{\partial x_i\partial x_j} \\ = &\ \text{p.v.}\int_\Omega \Gamma_{ij}(w;x-y)\mathcal{L}_{A(w)}u(y) + \mathcal{L}_{A(w)}u(x)\int_{|t|=1} t_j\Gamma_i(w,t)d\sigma(t).\end{aligned}$$

Notice that

$$\begin{aligned}\mathcal{L}_{A(x)}u(y) &= (\mathcal{L}_{A(x)}-\mathcal{L}_{A(y)})u(y)+\mathcal{L}_{A(y)}u(y) \\ &= \sum_{i,j=1}^n (a^{ij}(x)-a^{ij}(y))\frac{\partial^2 u(y)}{\partial y_i\partial y_j}+\mathcal{L}_{A(y)}u(y).\end{aligned} \tag{1.31}$$

Replacing w by x in (1.30), and combining it with (1.31), one has

$$\begin{aligned}\frac{\partial^2 u(x)}{\partial x_i\partial x_j} &= \text{p.v.}\int_\Omega \Gamma_{ij}(x;x-y)\mathcal{L}_{A(x)}u(y) \\ &\quad +\mathcal{L}_{A(x)}u(x)\int_{|t|=1} t_j\Gamma_i(x,t)d\sigma(t) \\ &= \text{p.v.}\int_\Omega \Gamma_{ij}(x;x-y)\sum_{k,\ell}(a^{k\ell}(x)-a^{k\ell}(y))\frac{\partial^2 u(y)}{\partial y_i\partial y_j}dy \\ &\quad +\text{p.v.}\int_\Omega \Gamma_{ij}(x;x-y)\mathcal{L}_{A(y)}u(y)dy \\ &\quad +\mathcal{L}_{A(x)}u(x)\int_{|t|=1} t_j\Gamma_i(x,t)d\sigma(t).\end{aligned} \tag{1.32}$$

If we let

$$T_{ij}(f)(x) = \text{p.v.} \int_{\Omega} \Gamma_{ij}(x; x-y) f(y), \tag{1.33}$$

then it was verified that T_{ij} is a standard Calderòn-Zygmund operator (see [12] and [9] for details). Thus,

$$\begin{aligned} & \text{p.v.} \int_{\Omega} \Gamma_{ij}(x; x-y) \sum_{k,\ell} (a^{k\ell}(x) - a^{k\ell}(y)) \frac{\partial^2 u(y)}{\partial y_i \partial y_j} dy \\ &= \sum_{k,\ell=1}^{n} [M_{a^{k\ell}}, T_{ij}] \frac{\partial^2 u}{\partial y_k \partial y_\ell}. \end{aligned} \tag{1.34}$$

It has been proved in [17], [43], [2], [16] and references therein that

$$\|[M_{a^{k\ell}}, T_{ij}]\|_{L^p \to L^p} \le C_{p,n} \|a_{ij}\|_{BMO(\Omega)} \tag{1.35}$$

for all $1 < p < \infty$. Moreover, it was proved in [9] that

$$\|[M_{a^{k\ell}}, T_{ij}]\|_{BMO_z \to BMO_r} \le C_\Omega \|a_{ij}\|_{LMO(\Omega)} \le C_\Omega \|a_{ij}\|_{Dini(\Omega)} \tag{1.36}$$

and

$$\|[M_{a^{k\ell}}, T_{ij}]\|_{H^1_z \to H^1_r} \le C_\Omega \|a_{ij}\|_{LMO(\Omega)} \le C_\Omega \|a_{ij}\|_{Dini(\Omega)}, \tag{1.37}$$

where $B_r(x) =: B(x,r)$ and

$$\|a\|_{LMO(\Omega)} =: \sup_{B_r(x) \subset \Omega} \Big\{ \frac{|\log |B_r(x)||}{|B_r(x)|} \int_{B_r(x)} |f(y) - f_{B_r(x)}| dy \Big\}. \tag{1.38}$$

We use a partition of unity: Choose $\psi_k \in C^\infty(\mathbb{R}^n)$ with $\text{supp}(\psi_k) \subset B(x_k, 2\epsilon)$ and $\psi_k = 1$ on $B(x_k, \epsilon)$ and $0 \le \psi_k \le 1$ with $x_k \in \Omega$ so that

$$\sum_{k=1}^{N} \psi_k = 1, \quad \text{on } \overline{\Omega}. \tag{1.39}$$

Let u be the unique solution of (1.7). Then

$$u = \sum_{k=1}^{N} u \psi_k, \quad z \in \overline{\Omega}. \tag{1.40}$$

We separate our argument into the following two cases:

(i) Interior estimate when $\text{supp}(\psi_k) \cap \partial\Omega = \emptyset$,

(ii) Boundary estimate when $\text{supp}(\psi_k) \cap \partial\Omega \neq \emptyset$.

• For the interior estimate: One can apply the formula (1.32) to the function $u\psi_k$, then use (1.34)–(1.37) to get an interior estimate.

• For the boundary estimate: One can not apply the formula (1.32) directly to the function $u\psi_k$. Instead, one may use the standard technique of flattening the boundary through changes of variables. The operator $\mathcal{L}_A$ will be changed to $\mathcal{L}_B + \sum_{j=1}^n h_j(x) \frac{\partial}{\partial x_j}$, where $\mathcal{L}_B$ remains uniform elliptic. Without loss of generality, one may assume that $B = A$, $\Omega_k = \{x \in \mathbb{R}^n : \|x\| < 2\epsilon, \quad x_n > 0\}$ and $u^k =: u\psi_k = 0$ on $\partial\Omega_k$ with $x_n \neq 0$. Let

$$T(x,y) = x - \frac{2x_n}{a_{nn}(x)} (a_{n1}(y), \cdots, a_{nn}(y)) = (T_1, \cdots T_n) \tag{1.41}$$

and

$$T_{nj}(x) = \delta_{nj} - \frac{2}{a_{nn}(x)} a_{nj}(x). \tag{1.42}$$

Without loss of generality, one may also assume that $u(x', 0) = 0$ for $\|x'\| < 2\epsilon$. It was proved in [13] that one has the following formula for u^k holds:

$$\begin{aligned} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} &= \text{P.V.} \int_{\Omega_k} \Gamma_{ij}(x; x-y) \sum_{k,\ell} (a^{k\ell}(x) - a^{k\ell}(y)) \frac{\partial^2 u(y)}{\partial y_i \partial y_j} dy \\ &\quad + \text{P.V.} \int_{\Omega_k} \Gamma_{ij}(x; x-y) \mathcal{L}_{A(y)} u(y) dy \\ &\quad + \mathcal{L}_{A(x)} u(x) \int_{|\xi|=1} t_j \Gamma_i(x, \xi) d\sigma(\xi) + I_{ij}(u)(x), \end{aligned} \tag{1.43}$$

where

$$\begin{aligned} I_{ij} &= \int_{\Omega_k} \Gamma_{ij}(x; T(x) - y) \sum_{k,\ell} (a^{k\ell}(x) - a^{k\ell}(y)) \frac{\partial^2 u(y)}{\partial y_i \partial y_j} dy \\ &\quad + \int_{\Omega_k} \Gamma_{ij}(x; T(x) - y) \mathcal{L}_{A(y)} u(y) dy), \end{aligned} \tag{1.44}$$

$$\begin{aligned} I_{in} = I_{ni} &= \int_{\Omega_k} \sum_{j=1}^{n} \Gamma_{ij}(x; T(x) - y) T_{nj}(x) \\ &\qquad \times \sum_{k,\ell} (a^{k\ell}(x) - a^{k\ell}(y)) \frac{\partial^2 u(y)}{\partial y_i \partial y_j} dy \\ &\quad + \int_{\Omega_k} \Gamma_{ij}(x; T(x) - y) T_{nj}(x) \mathcal{L}_{A(y)} u(y) dy, \end{aligned} \tag{1.45}$$

for $1 \le i, j \le n-1$, and

$$\begin{aligned} I_{nn} &= \int_{\Omega_k} \sum_{i,j=1}^{n} \Gamma_{ij}(x; T(x) - y) T_{ni}(x) T_{nj}(x) \\ &\qquad \times \sum_{k,\ell} (a^{k\ell}(x) - a^{k\ell}(y)) \frac{\partial^2 u(y)}{\partial y_i \partial y_j} dy \\ &\quad + \int_{\Omega_k} \Gamma_{ij}(x; T(x) - y) T_{ni}(x) T_{nj}(x) \mathcal{L}_{A(y)} u(y) dy. \end{aligned} \tag{1.46}$$

Using the above formulae (1.35)–(1.37), (1.43)–(1.46), the embedding $W^{2,p} \subset W^{1,p}$ being compact and interior estimate, one can obtain *a priori* estimates stated in (1.11) and (1.12).

2. Degenerate linear PDEs

There are many kinds of degenerate linear elliptic PDEs, here, we will introduce one which is from complex analysis and complex geometry.

2.1. **Basic knowledge on domains in $\mathbb{C}^n$.** First, we introduce some notations and definitions:

• A bounded domain $\Omega \subset \mathbb{C}^n$ with C^2 boundary is pseudoconvex (strictly pseudoconvex) if there is a function, called defining function, $\rho \in C^2(\mathbb{C}^n)$ so that

(i) $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$,

(ii) $\partial\Omega = \{z \in \mathbb{C}^n : \rho(z) = 0\}$,

(iii) $\nabla\rho(z) \neq 0$ on $\partial\Omega$

(vi) $L_\rho(z, w) := \sum_{i,j=1}^n \frac{\partial^2\rho(z)}{\partial z_i \partial \overline{z}_j} w_i \overline{w}_j \geq 0$ (> 0) for all $w \in H_z(\partial\Omega)$ and $w \neq 0$, where

$$H_z(\partial\Omega) = \{w \in \mathbb{C}^n : \sum_{j=1}^n \frac{\partial\rho(z)}{\partial z_j} w_j = 0\}. \tag{2.1}$$

• A real-valued function $u \in C^2(\Omega)$ is said to be strictly plurisubharmonic (plurisubharmonic) in Ω if the complex Hessian matrix $H(u)(z)$ is positive definite (positive semi-definite) for all $z \in \Omega$, where

$$H(u)(z) = \left[\frac{\partial^2 u(z)}{\partial z_i \partial \overline{z}_j}\right] \tag{2.2}$$

is $n \times n$ self-adjoint matrix.

The following is a summary for some properties of pseudoconvex domains in $\mathbb{C}^n$, one may find them in the books of Chen and Shaw [14] and Krantz [40] and (iv) below can be found in [8] and [61].

Proposition 6. *The following statements hold:*

(i) Every bounded domain in complex plane is strictly pseudoconvex;

(ii) Every convex (strictly convex) domains are pseudoconvex (strictly pseudoconvex);

(iii) Every smoothly bounded pseudoconvex domain D, there is a defining function ρ so that $u = -\log(-\rho)$ is strictly plurisubharmonic ($H(u)$ is positive definite) on D.

(iv) If D is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$, then there is a defining function $\rho \in C^\infty(\overline{D})$ so that $H(\rho)(z)$ is positive definite on $\overline{D}$.

(v) For any weakly smoothly bounded pseudoconvex domain D in $\mathbb{C}^n$ there is $\epsilon_D \in (0, 1]$ and function $\rho \in C^{\epsilon_D}(\overline{D})$ with $\rho < 0$ on D, $\rho(z) = 0$ on ∂D and $H(\rho)(z)$ is positive definite. In fact, we can make $H(\rho) \geq I_n$.

(vi) $A(0, 1, 2) = \{z \in C^n : 1 < |z| < 2\}$ (with $n > 1$) is not a pseudoconvex domain.

• The following question is a very important in several complex variables.

Question: For a given smoothly bounded pseudoconvex domain D in $\mathbb{C}^n$, does there exist a smooth plurisubharmonic defining function for D ?

The above question has been studied by several authors (see the book of Krantz [40] and the book of Chen and Shaw [14]). Well-known Worm domain in $\mathbb{C}^2$ constructed by Diedrich and Fornaess [19] is a counterexample to the above question. The Diederich-Fornaess Worm domain is defined as follows:

$$D = \left\{ (z_1, z_2) \in \mathbb{C}^n : \left| z_1 + e^{i \log |z_2|^2} \right| < 1 - \Phi(\log |z_2|^2) \right\},$$

where $\Phi \in C^\infty(-\infty, \infty)$ vanishes identically on some interval $[-r, r]$ of positive length. Moreover, Φ can be chosen so that D is pseudoconvex, but D does not admit a smooth plurisubharmonic defining function.

• A real-valued function $u \in C^2(\Omega)$ is said to be a strictly plurisubharmonic exhaustion function for Ω if

1) $\Omega = \{z \in \mathbb{C}^n : u(z) < \infty\}$
2) $\partial\Omega = \{z \in \mathbb{C}^n : u(z) = \infty\}$
3) Complex hessian $H(u)(z) = \left[\frac{\partial^2 u(z)}{\partial z_i \partial \overline{z}_j}\right]$ is positive definite for $z \in \Omega$.

Then metric $g = \sum_{i,j=1}^n u_{i\overline{j}} dz_i \otimes d\overline{z}_j$ is a Kähler metric, in many cases, it is also a complete metric on Ω.

• Laplace-Beltrami operator associated to the metric g above is given as:

$$\Delta_g = \Delta_u := -4 \sum_{i,j=1}^n u^{i\overline{j}} \frac{\partial^2}{\partial z_i \partial \overline{z}_j}, \quad [u^{i\overline{j}}]^t = H(u)^{-1}. \tag{2.3}$$

• We know that $-\Delta_u$ is elliptic, but, in general, it is not uniformly elliptic on Ω.

EXAMPLE 1. *Let $\Omega = B_n$ be the unit ball in $\mathbb{C}^n$ and $\rho(z) = |z|^2 - 1$ and*

$$u(z) = -\log(-\rho(z)) = -\log(1 - |z|^2), \quad z \in B_n. \tag{2.4}$$

Then

$$H(u) = \frac{1}{1 - |z|^2}\left[\delta_{ij} + \frac{\overline{z}_i z_j}{1 - |z|^2}\right], \quad u^{i\overline{j}} = (1 - |z|^2)(\delta_{ij} - z_i \overline{z}_j). \tag{2.5}$$

Then u is strictly plurisubharmonic exhausted function for B_n and the metric induced by u is the well-known Bergman metric as well as Kähler-Einstein metric. Moreover,

$$\Delta_u = -4(1 - |z|^2) \sum_{i,j=1}^n (\delta_{ij} - z_i \overline{z}_j) \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \tag{2.6}$$

is a degenerate elliptic operator, the degeneracy for Δ_u is on $\partial\Omega$.

2.2. **Dirichlet boundary value problem/homogeneous case.**

$$\begin{cases} \Delta_u h(z) = 0, & z \in \Omega \\ h(z) = \phi(z), & z \in \partial\Omega \end{cases} \tag{2.7}$$

• Question: For $\phi \in C^\infty(\partial\Omega)$, what regularity does the solution u of (2.7) satisfy?

• When $\Omega = B_n$, the unit ball in $\mathbb{C}^n$, the Dirichlet boundary value problem (2.7) becomes:

$$\begin{cases} \sum_{i,j=1}^n (\delta_{ij} - z_i \overline{z}_j) \frac{\partial^2 h(z)}{\partial z_i \partial \overline{z}_j} = 0, & z \in \Omega \\ h(z) = \phi(z), & z \in \partial\Omega \end{cases} \tag{2.8}$$

has the unique solution (see Hua [35], Rudin [77]):

$$h(z) = P[\phi](z) = \int_{\partial B_n} \frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}} \phi(w) d\sigma(w), \quad z \in B_n. \tag{2.9}$$

• When $n = 1$, Δ_g is not degenerate since

$$\sum_{i,j=1}^{n} (\delta_{ij} - z_i \bar{z}_j) \frac{\partial^2 h}{\partial z_i \partial \bar{z}_j} = 0 \iff \frac{\partial^2 h(z)}{\partial z \partial \bar{z}} = 0. \tag{2.10}$$

Therefore, the uniform elliptic theory implies that

Proposition 7. *If $n = 1$ and if $\phi \in C^\infty(\partial B_n)$, then $h = P[\phi] \in C^\infty(\overline{B}_n)$.*

• When $n > 1$, $\sum_{i,j=1}^{n} (\delta_{ij} - z_i \bar{z}_j) \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$ is not uniformly elliptic in B_n. By the results in [24] and [70], one has that

Proposition 8. *If $\phi \in C^\infty(\partial B_n)$ with $n > 1$, there are two functions $f, g \in C^\infty(\overline{B}_n)$ so that the unique solution h of (2.8) or (2.9) satisfies:*

$$h(z) = f(z) + g(z)(1-|z|^2)^n \log(1-|z|^2), \quad z \in B_n. \tag{2.11}$$

2.3. **Rigidity of smooth solutions.** By (2.11) in Proposition 2.3, one has that

- $P[\phi] \in C^{n-\epsilon}(\overline{B}_n)$ if $\phi \in C^n(\partial B_n)$.

An example given by Garnett and Krantz [41] and by Graham [24] shows that

- $P[\phi] \notin C^2(\overline{B}_2)$ if even $\phi(z) = |z_1|^2$ is smooth on ∂B_2.
- **Question:** Assume $\phi \in C^\infty(\partial B_n)$ so that $P[\phi] \in C^\infty(\overline{B}_n)$. What can one says about ϕ?

• R. C. Graham [24] gave a perfect answer of the above question. He proved the following celebrated theorem.

Theorem 9. *If $P[\phi](z) \in C^\infty(\overline{B}_n)$, then $P[\phi](z)$ must be pluriharmonic, real part of holomorphic function in B_n.*

• When D is polydisc in $\mathbb{C}^n$, the following theorem was proved by Li and Simon in [67].

Theorem 10. *Let $D = D(0,1)^n$ be the unit polydisc and the subharmonic defining function for $D(0,1)$ be $\rho_j(z_j) \in C^\infty(\overline{D(0,1)})$. Let*

$$u(z) = -\sum_{j=1}^{n} \log(-\rho_j(z_j))$$

introduce the Kähler metric u. If $h \in C(\overline{D(0,1)}^n)$ is u-harmonic, then $h(z)$ must be harmonic in each variable z_j for $1 \le j \le n$.

Furthermore, one may ask the following questions:

• **Question** *1) Does the above theorem of Graham hold for Δ_u in B_n with more general potential function u?*

2) Does it hold for more general strictly pseudoconvex domain in $\mathbb{C}^n$?

• The questions were studied by C. R. Graham and J. Lee in [32]. They studied the problem on a smoothly bounded strictly pseudoconvex domain D in $\mathbb{C}^2$ with some symmetry (invariant under one-parameter group) as well as the potential satisfies similar symmetry. They proved the above theorem of Graham holds on those special domains. In particular, they consider the rotation symmetric defining function $\rho(z) = \psi(|z|^2)$ for the unit ball B_n. Then $u(z) = -\log(-\psi(|z|^2))$ is strictly plurisubharmonic on B_n if and only if

(GL) $\psi \in C^\infty[0,1], \quad \psi(1) = 0, \quad \psi'(t) > 0 \quad (-\psi)\psi'' + (\psi')^2 > 0$ on $[0,1]$.

Let u be the Kähler metric induced by $u = -\log(-\psi(|z|^2))$ satisfying (GL). Then the following theorem was proved by Graham and Lee in [32]

Theorem 11. *If $u = -\log(-\psi(|z|^2))$ and ψ satisfying (GL), then*
(i) For $n = 2$, if $h \in C^\infty(\overline{B}_2)$ and $\Delta_u h = 0$ in B_2, then $h(z)$ must be pluriharmonic.
(ii) For any $n > 2$, there is a ψ satisfying (GL) and $u(z) = -\log(-\psi(|z|^2))$, and there is a $h \in C^\infty(\overline{B}_n)$ with $\Delta_u h = 0$ in B_n, but $h(z)$ is not pluriharmonic in B_n.

• In order to understand the above theorem of Graham and Lee better, a joint work by the author and D. Wei in [70], we obtained the following results:

1) We give a formula to construct $\rho = \psi(|z|^2)$ satisfying (GL).

Proposition 12. *If $\psi : [0,1] \to (-\infty, 0]$ satisfies (GL), then there is negative function $B_0 \in C^\infty([0,1])$ so that*

$$\psi(t) = \psi(0) \exp\Big[\frac{\psi'(0)}{\psi(0)} \int_0^t \exp\Big(\int_0^{t_1} \frac{B_0(s)}{s-1} ds\Big) dt_1\Big]. \tag{LW}$$

Remark: If $-\psi(0) = \psi'(0) = -B_0(s) = 1$, (LW) gives $\psi(t) = t - 1$.

2) We give a characterization on $\rho(z) = \psi(|z|^2)$ when the theorem of Graham type holds; when it fails. Let u be the metric induced by the plurisubharmonic function $u = -\log(-\psi(|z|^2))$, where ψ is given by (LW). We also assume that $\psi(t)$ is real analytic at $t = 0$ and $t = 1$. The following theorem was proved in [70].

Theorem 13. *For $n > 1$, if $h \in C^n(\overline{B}_n) \cap C^\infty(\partial B_n)$ is $u-$harmonic in B_n, then $h \in C^\infty(\overline{B}_n)$.*

• By the example of Graham and Lee, in general h is not pluriharmonic for all $u = -\log(-\rho)$ with ρ satisfying (LW).

• We will formulate a necessary and sufficient condition to guarantee the theorem of Graham type holds.

For any negative function $B_0 \in C^\infty([0,1])$. which is real analytic at $t = 0$ and $t = 1$, after a normalization, we may write

$$B_0(t) = -1 + \sum_{j=1}^\infty b_j (t-1)^j. \tag{LW1}$$

• In [70], the author and Wei were able to formulate a condition Q_n such that coefficients $b_1, \cdots, b_{n-2} \in Q_n$ if and only if that $h(z) \in C^n(\overline{B}_n)$ and $\Delta_u h(z) = 0$ on B_n imply $h(z)$ is pluriharmonic in B_n.

• The statement of our theorem for general n is a little bit complicated. It is very clean when $n = 3$.

Theorem 14. *Let $\mathbb{Q}_3 = \{-\frac{1}{2}(1+p)(1+q) : p, q \in \mathbb{N}\}$. Then $b_1 \notin \mathbb{Q}_3$ if and only that for any $h \in C^3(\overline{B}_3)$ is u-harmonic in B_3 implies h is plurisubharmonic in B_3.*

• Let $b_0 = -1$. For $n \geq 3$ and $k \geq 1$, one has the following recursively relations [70].

$$\text{(RF)} \qquad \begin{aligned} &(n-k-2)g_{k+2} \\ = \quad &\Big[p+q+k+(n-1)b_1(k+1)+pq\Big]\frac{g_{k+1}}{k+2} \\ &+\sum_{j=2}^{k}\Big[(n-1)j(b_{k+2-j}+b_{k+1-j})-jpqb_{k+1-j}\Big]\frac{g_j}{k+2} \\ &+b_k\frac{(n-1-pq)}{(k+2)(n-1)}pq \end{aligned}$$

and

$$g_2 = \frac{(1+p+q+pq)}{2(n-2)(n-1)}pq, \quad n \geq 3. \tag{2.12}$$

Let $\mathbb{Q}_n$ be the set of $(b_1, \cdots, b_{n-2})$ so that the right hand side of (RF) is zero where $n = k+2$, p and q are any positive integers. This set is uniquely determined by (RF). The following theorem was proved by the author and Wei in [70].

Theorem 15. *Let $B_0 \in C^n([0,1])$ be negative with $b_0 = -1$. Then the following two statements are equivalent:*

(i) $(b_1, \cdots, b_{n-2}) \notin \mathbb{Q}_n$;

(ii) If $h \in C^n(\overline{B}_n)$ is u-harmonic in B_n then h is pluriharmonic in B_n.

Remark: Question about whether one has Graham type's theorem for domain D without symmetry. A natural example for such domain D is the following convex domain

$$D(A) = \{z \in \mathbb{C}^n : \rho(z) = |z|^2 + \mathrm{Re}\sum_{j=1}^{n} a_j z_j^2 - 1 < 0\} \quad A_j \in (-1,1). \tag{2.13}$$

The boundary of $D(A)$ is real ellipsoid in $\mathbb{C}^n$. Kähler metric induced by $u = -\log(-\rho)$ is complete without any symmetry. It is natural to ask: Does the Graham type theorem hold for $(D(A), \Delta_u)$? This problem is still open. In [85], Wei proved some partial result on this problem.

3. Rigidity theorem and problems for harmonic maps

Let M^m and N^n be two Kähler manifolds with Kähler metrics $h = h_{i\bar{j}} dz_i d\bar{z}_j$ and $g = g_{\alpha\bar{\beta}} dw^\alpha d\overline{w}^\beta$, respectively. Let $u : M \to N$ be a map from M to N. When M and N are compact, we say that u is harmonic if u minimizes the the energy functional

$$E[u] =: \int_M e[u] dv_h, \tag{3.1}$$

where

$$e[u](z) = \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} h^{i\bar{j}} g_{\alpha\bar{\beta}} (\partial_i u^\alpha \overline{\partial_j u^\beta} + \partial_{\bar{j}} u^\alpha \overline{\partial_{\bar{i}} u^\beta}). \tag{3.2}$$

Let $\Gamma^s_{t\gamma}$ be the Christoffel symbols of the Hermitian metric g on N, and let $u = (u^1, u^2, \cdots, u^n) : M \to N \subset \mathbb{C}^n$ be a map. We can define harmonic map in local coordinates as follows which works if M is not compact:

(a) We say that u is *harmonic* if the tension field

$$\tau^s[u] = \triangle_M u^s + \sum_{t,\gamma=1}^{n} \sum_{i,j=1}^{m} \Gamma^s_{t\gamma} h^{i\bar{j}} \partial_i u^t \overline{\partial}_j u^\gamma = 0, \quad \text{for } 1 \le s \le n \tag{3.3}$$

where $\triangle_M = h^{i\bar{j}} \partial^2_{i\bar{j}}$ and $(h^{i\bar{j}})$ is the inverse matrix of the matrix $(h_{i\bar{j}})$.

(b) We say that u is *pluriharmonic* if

$$\partial\overline{\partial} u^s + \sum_{t,\gamma} \Gamma^s_{t\gamma} \partial u^t \overline{\partial} u^\gamma = 0, \quad \text{for } 1 \le s \le n. \tag{3.4}$$

(c) We say that u is *holomorphic.* if $\overline{\partial} u^s = 0$ for $1 \le s \le n$.

• **Implication relations.** Since N is Kähler, it is well-known that any pluriharmonic maps are harmonic, and any holomorphic or anti-holomorphic maps are pluriharmonic.

(d) We say that Kähler manifold (N, g) has strongly negative (semi-negative) curvature in the sense of Siu if curvature tension $R_{i\bar{j}k\bar{\ell}}$ satisfy the following condition:

$$\sum_{i,j,k,\ell=1}^{n} R_{i\bar{j}k\bar{\ell}} \xi_{i\bar{j}} \overline{\xi_{k\bar{\ell}}} > 0 \quad (\text{respectively} \ge 0) \tag{3.5}$$

for any nonzero matrix $(\xi_{i\bar{j}})$ of the form:

$$\xi_{i\bar{j}} = A_i \overline{B_j} - C_i \overline{D_j}. \tag{3.6}$$

• **Strong rigidity theorem of Siu:** When both M and N are compact, Siu [79] gave the following celebrated strong rigidity theorem on harmonic map.

Theorem 16. *Let (M^m, h) and (N^n, g) be two compact Kähler manifolds with (N, g) having strongly negative curvature in the sense of Siu. Then any harmonic map $u : M \to N$ with the rank of du at one point being greater than or equal to four must be holomorphic or antiholomorphic.*

• Note that the last condition excludes the case of complex dimension one when the theorem is obviously false.

• The key of the proof of Siu's super rigidity theorem is his $\partial\overline{\partial}$-Bochner formula:

$$\partial\overline{\partial}(g_{\alpha\overline{\beta}}u^{\alpha}_{\overline{i}}\overline{u}^{\beta}_{\overline{j}}d\overline{z}_i\wedge dz_j)=R_{\alpha\overline{\beta}\gamma\overline{\delta}}u^{\alpha}_{\overline{i}}\overline{u}^{\beta}_{j}u^{\gamma}_{k}\overline{u}^{\delta}_{\overline{\ell}}d\overline{z}_i\wedge dz_j\wedge dz_k\wedge d\overline{z}_\ell - g_{\alpha\overline{\beta}}D\overline{\partial}u^{\alpha}\wedge\overline{D}\partial\overline{u}^{\beta}. \tag{3.7}$$

• **Idea of Siu's proof:** When M is a compact manifold, the integration of the left hand side, after wedging a (m-2) power of the Kähler form, is zero from integration by parts. It was shown in [79], that both terms of the right hand side are non-negative when u is a harmonic map and the curvature of N is strongly negative, and therefore they are pointwise zero. This fact coupling with the the rank assumption on du shows that u must be holomorphic or antiholomorphic (cf. [79]).

• **General question:** A general question one may ask is when a harmonic map u is holomorphic or antiholomorphic if N is Kähler with strongly negative curvature.

When M is a complete (noncompact) manifold, N is strongly negative is not enough guarantee the harmonic map becomes holomorphic or antiholomorphic. Let us see a simple example. Let $N=B^n$ the unit ball in $\mathbb{C}^n$, g is Bergman or Kähler-Einstein metric

(3.8)

$$g=-\sum_{i,j=1}^{n}\frac{\partial^2\log(1-|z|^2)}{\partial z_i\partial\overline{z}_j}dz_i\otimes d\overline{z}_j=\sum_{i,j=1}^{n}\Big[\frac{\delta_{ij}}{1-|z|^2}+\frac{\overline{z}_iz_j}{(1-|z|^2)^2}\Big]dz_i\otimes d\overline{z}_j$$

Then curvature tensor for (N,g) is:

$$R_{i\overline{j}k\overline{\ell}}=(g_{i\overline{j}}g_{k\overline{\ell}}+g_{k\overline{j}}g_{i\overline{\ell}}). \tag{3.9}$$

Therefore,

$$R_{i\overline{j}k\overline{\ell}}\xi_{i\overline{j}}\overline{\xi_{k\overline{\ell}}}=|\sum_{i\overline{j}}\xi_{i\overline{j}}|^2+g_{i\overline{\ell}}g_{k\overline{j}}\xi_{i\overline{j}}\overline{\xi_{k\overline{\ell}}}>0 \text{ if } [\xi_{i\overline{j}}]\neq 0. \tag{3.10}$$

and the Christofle's symbols for (N,g) is:

$$\Gamma^s_{t\gamma}[w]=(1-|w|^2)^{-1}(\overline{w}^{\gamma}\delta_{ts}+\overline{w}^t\delta_{\gamma s}). \tag{3.11}$$

Therefore, the boundary value problem:

(3.12)

$$\begin{cases}\tau^{\alpha}[u]=\Delta_M u^{\alpha}+\sum_{i,j=1}^{m}h^{i\overline{j}}\sum_{t,\gamma=1}^{n}\frac{(\overline{u}^{\gamma}\delta_{ts}+\overline{u}^t\delta_{\gamma s})\partial_i u^t\overline{\partial}_j u^{\gamma}}{(1-|u|^2)}=0, \text{ if } z\in D\\ \qquad u=\phi(z), \text{ if } z\in\partial D\end{cases}$$

with $\phi:\partial D\to\mathbb{C}^n$ and $|\phi(z)|\le c_0<1$ on ∂D always has a solution if Δ_M is uniform elliptic. Such u can not be either holomorphic or anti-holomorphic if ϕ is not a boundary value of a holomorphic map or anti-holomorphic map.

With some extra condition, the following natural question was studied by the author and Ni [64].

Problem 1. *Let h, g denote the Bergman metrics on B_m and B_n, respectively; and let $u:(B_m,h)\to(B_n,g)$ be a proper harmonic map so that u can* *be extended to C^1 map up to the boundary ∂B_m. Is u either holomorphic or anti-holomorphic?*

A closely related problem of Problem 1 is the existence and regularity of proper harmonic maps, namely

Problem 2. *Let $\phi : \partial B_m \to \partial B_n$ be a smooth map. Does there exist a proper harmonic map u so that $u = \phi$ on ∂B_m? If such harmonic map u exists what can one say about the regularity of u ?*

For the real hyperbolic space, Peter Li and L. F. Tam initiated the systematic study of the existence, uniqueness and regularity of proper harmonic maps from the unit ball D^m in $\mathbb{R}^m$ to D^n in $\mathbb{R}^n$ with respect to the hyperbolic metrics (cf. [48, 49, 50]). In [48, 49], among other things, they proved that if $\phi : S^{m-1} \to S^{n-1}$ is a C^1 map with energy density $e(\phi)(x) \neq 0$ for all $x \in S^{m-1}$ (here $e(\phi)$ is defined with respect to the standard metrics on S^{2m-1} and S^{2n-1}) then there is a unique proper harmonic map extension $u : D^m \to D^n$ with boundary value ϕ. Moreover, if $\phi \in C^m(S^{m-1}, S^{n-1})$ then $u \in C^{m-1,\alpha}(\overline{D}^m, \overline{D}^n)$ for any $\alpha < 1$. They also proved that if $e(\phi) \neq 0$ on S^{m-1} then the energy density $e[u]$ of the harmonic map u with respect to the hyperbolic metric satisfying

$$\lim_{x\to S^{m-1}} e[u](x) = \lim_{x\to S^{m-1}} h^{ij} g_{k\ell} \frac{\partial u^k}{\partial x_i} \frac{\partial u^\ell}{\partial x_j}(x) = m, \quad x \in S^{m-1}. \tag{3.13}$$

where $h = h_{ij}dx_idx_j$ is the hyperbolic metric for D^m and $g = g_{ij}dy_idy_j$ is the hyperbolic metric for D^n, and (h^{ij}) is the inverse matrix of (h_{ij}).

For the complex case when the domain and target manifolds are rank one symmetric space of noncompact type, the problem was first studied by H. Donnelly [18]. He generalized the above existence and uniqueness results of Li-Tam to the setting with some necessary contact conditions on the boundary map ϕ.

When $e(\phi)$ vanishes on S^{m-1}, the existence of a proper harmonic extension becomes less tractable, partial progress was made by J. Wang [84], where he proved the existence under the assumption that $e(\phi)$ has finitely many zeros on S^{m-1} and ϕ is locally rotationally symmetric around those points.

The answer with lower regularity solution to Problem 1 is negative. The example was constructed by the author and Ni in [64].

Theorem 17. *For any $0 < \epsilon < 1$, there is a proper harmonic map $u \in C^{2-\epsilon}(\overline{B}_2)$ from B_2 to B_3 with respect to the Bergman metric, which is neither holomorphic nor anti-holomorphic.*

This theorem tells us, in general, a proper harmonic map is not necessarily holomorphic or anti-holomorphic. It is natural to find out what are the necessary and sufficient conditions under which the proper harmonic map is holomorphic or anti-holomorphic. The following definition was given in [64]:

•(e) We say that a map $u : B_m \to B_n$ is k-harmonic with respect to the origin 0 if the restriction of u is harmonic on the intersection of B_m and to any k-dimensional complex linear subspace through the origin.

• **Remark:** It is clear that a map is harmonic if and only if it is m-harmonic with respect to the origin, and pluriharmonic map are k-harmonic with respect to the origin for all $1 \le k \le m$.

The following theorem was proved in [64].

Theorem 18. *Let $u \in C^2(\overline{B}_m, \overline{B}_n)$ $(m > 1)$ be a proper map from B_m to B_n with respect to the Bergman metrics. Then the following statements are equivalent.*

(i) u is either holomorphic or anti-holomorphic;

(ii) $(m-1)$-harmonic with respect to the origin;

(iii) u is harmonic; and $\mathcal{L}u$ is orthogonal to u on ∂B_m where $\mathcal{L} = (\delta_{ij} - z_i\overline{z}_j)\frac{\partial^2}{\partial z_i \partial \overline{z}_j}$;

(iv) u is harmonic and

$$\lim_{r\to 1^-} e[u](rz) = m \quad \text{on} \quad \{z \in \partial B_m : E_b[u] = |\overline{\partial}_b \overline{u}(z)|^2 + |\overline{\partial}_b u(z)|^2 \neq 0\},$$

where the energy density $e[u]$ is given by

$$e[u](z) = \frac{(1-|z|^2)}{(1-|u(z)|^2)^2}(\delta_{ij} - z_i\overline{z}_j)(\delta_{\alpha\beta}(1-|u|^2) + \overline{u}^\alpha u^\beta) \times (\partial_i u^\alpha \overline{\partial_j u^\beta} + \partial_{\overline{j}} u^\alpha \overline{\partial_{\overline{i}} u^\beta}).$$

Here we sum i, j from 1 to m, and sum α, β from 1 to n.

Combining Theorem 3.3 and a recent work of X. Huang [31] on the rigidity of proper holomorphic maps we have the following corollary.

Corollary 19. *Let $u \in C^2(\overline{B}_m, \overline{B}_n)$ $(m > 1)$ be a proper $(m-1)$-harmonic map with respect to the origin from $B_m \to B_n$ with respect to the Bergman metric with $n \le 2m-2$. Then there are $\phi \in Aut(B_m)$ and $\psi \in Aut(B_n)$ such that either $\psi \circ u \circ \phi$ or $\psi \circ \overline{u} \circ \phi$ is a holomorphic linear map.*

Let Ω_1 and Ω_2 be two smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. Let ρ is smooth strictly plurisubharmonic defining function for Ω_1 and r for Ω_2. Let

$$h = -\frac{\partial^2 \log(-\rho)}{\partial z_i \partial \overline{z}_j} dz_i \otimes d\overline{z}_j, \quad g = -\frac{\partial^2 \log(-r)}{\partial w^i \partial \overline{w}^j} dw^i \otimes d\overline{w}^j$$

• The rigidity problem for proper harmonic map from $u : (\Omega_1, h) \to (\Omega_2, g)$ was studied by the author and Simon [68]. They generalize Theorem 3.3 in this setting.

With arguments of the proof of Theorem 3.3 and Fefferman's asymptotic expansion of the Bergman kernel function of a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ (cf. [21]) and the result in [68], one has the following corollary in [68].

Corollary 20. *Let Ω_1 and Ω_2 be smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^m$ $(m > 1)$ and C^n, respectively. Let $u \in C^2(\overline{\Omega}_1, \overline{\Omega}_2)$ be a proper $(m-1)$-harmonic map with respect to the origin from Ω_1 to Ω_2 with respect to the Bergman metrics on Ω_1 and on Ω_2. Then u is either holomorphic or antiholomorphic.*

• Open Questions

One can see that if $u : (B_m, h) \to (B_n, g)$ is a harmonic map with h is Bergman or Kähler-Einstein metric, then u satisfies a system of n semi-linear degenerate partial differential equations and we know solution of (3.12) can be up to $C^{m-\epsilon}(\overline{B}_m)$ with some necessary boundary condition on ϕ [64]. The natural question arose as follows:

QUESTION. *Let h, g denote the Bergman metrics on B_m and B_n, respectively; and let $u : (B_m, h) \to (B_n, g)$ be a harmonic map so that $u \in C^n(\overline{B}_m)$ with $m > 1$. Is u either holomorphic or anti-holomorphic?*

4. Kähler-Einstein metric/Monge-Ampère equations

Let D be a domain in $\mathbb{C}^n$. Let $u \in C^2(D)$ be a strictly plurisubharmonic function on D. Let

$$u_{i\bar{j}}(z) = \frac{\partial^2 u(z)}{\partial z_i \partial \overline{z}_j}, \quad 1 \le i, j \le n. \tag{4.1}$$

Then $\sum_{i,j=1}^n u_{i\bar{j}} dz_i \otimes d\overline{z}_j$ defines a Kähler metric on D since

$$d\omega = d \sum_{i,j=1}^{n} u_{i\bar{j}} dz_i \wedge d\overline{z}_j = 0. \tag{4.2}$$

Fact 1: If $g = g_{i\bar{j}} dz_i \otimes d\overline{z}_j$ is a Kähler metric, then the Ricci curvature for g is

$$R_{k\bar{\ell}} = -\frac{\partial^2}{\partial z_k \overline{z}_\ell} \log \det[g_{i\bar{j}}]. \tag{4.3}$$

After a normalization, we have the following definition:

Definition 21. *We say that Kähler metric $g = g_{i\bar{j}} dz_i \otimes d\overline{z}_j$ is Einstein metric if $R_{k\bar{\ell}} = -(n+1) g_{k\bar{\ell}}$.*

Fact 2: If u is a strictly plurisubharmonic solution of the Monge-Ampère equation:

$$\det H(u)(z) = e^{(n+1)u}, \quad z \in D \tag{4.4}$$

then the metric $u_{i\bar{j}} dz_i \otimes d\overline{z}_j$ defines a Kähler-Einstein metric.

The following theorem was proved by Cheng and Yau (1980) in [15].

Theorem 22. *Let D be a smoothly bounded domain in $\mathbb{C}^n$. Then*

(i) If D is pseudoconvex, the Einstein equation (4.4) with boundary condition:

$$u = +\infty \quad \text{on } \partial D \tag{4.5}$$

has a unique plurisubharmonic solution on D, and metric $u_{i\bar{j}} dz_i \otimes d\overline{z}_j$ defines a complete metric on D

(ii) If D is strictly pseudoconvex, then $\rho(z) = -e^{-u(z)} \in C^{n+3/2}(\overline{D})$

Remark: The smoothness assumption in Part (i) of the theorem of Cheng and Yau has been replaced by a very weak condition by Mok and Yau [74]. Part (ii) of the theorem of Cheng and Yau has been sharpen by J. Lee and Melrose in [46]. They gave the following asymptotic expansion theorem on ρ.

Theorem 23. *Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let u be the plurisubharmonic solution of (4.4) and (4.5), and let $\rho(z) = -e^{-u}$. Then for given a defining function $\rho_0(z) \in C^\infty(\overline{D})$ of D, there are functions $a_j \in C^\infty(\overline{D})$ such that*

$$\rho(z) = \rho_0(z)[a_0(z) + \sum_{j=1}^{\infty} a_j(z)[\rho_0^{(n+1)} \log(-\rho_0(z)))^j], \tag{4.6}$$

where $a_0(z) > 0$ on ∂D. In particular, one has $\rho \in C^{n+2-\epsilon}(\overline{D})$.

• **Approximating** ρ. Question about how approximate the solution $\rho(z) = -e^{-u}$ was first studied by C. Fefferman in [22] who studied the following fully non-linear operator:

$$J[\rho](z) = -\det \begin{bmatrix} \rho & (\overline{\partial}\rho) \\ (\overline{\partial}\rho)^* & H(\rho) \end{bmatrix}, \quad \overline{\partial}\rho(z) = [\frac{\partial \rho}{\partial \overline{z}_1}, \cdots, \frac{\partial \rho}{\partial \overline{z}_n}]. \tag{4.7}$$

The relation between $J[\rho]$ and $\det H(u)$ was given by Cheng and Yau [15] when $J[\rho] = 1$, the general case was given in [58], which can be stated as the following theorem.

Proposition 24. *If $\rho(z) = -\exp(u(z))$, then*

$$\det H(u) = J[\rho]e^{(n+1)u} \tag{4.8}$$

Therefore, one has that

$$\begin{cases} \det H(u) = e^{(n+1)u} \text{ in } D \\ \qquad u = +\infty \text{ on } \partial D. \end{cases} \iff \begin{cases} \det J[\rho] = 1 \text{ in } D \\ \qquad \rho = 0 \text{ on } \partial D. \end{cases} \tag{4.9}$$

In (4.6), one can write

$$a_0(z) = \sum_{j=0}^{\infty} a_{0j}(z)\rho_0(z)^j \tag{4.10}$$

Question about how to compute $a_{0,j}$ in (4.10) in terms of ρ_0 explicitly has been studied by C. Fefferman [22] and R. Graham [25] and others. Graham [25] provided an iteration formula to evaluate $a_{0,j}$. An alternative formula for $a_{0,j}$ or approximation for ρ in terms of ρ_0 was given by the author [63] as follows.

Theorem 25. *Let $r(z)$ be a smooth negative defining function for D so that $\ell(\rho) := -\log(-r(z))$ is strictly plurisubharmonic in D. Let*

$$\rho_0(z) = r(z), \quad \rho_{j+1}(z) = \rho_j(z)J(\rho_j)^{-1/(n+1)}e^{-B_j} \tag{4.11}$$

with

$$B_j(z) = \frac{tr(H(\ell(\rho_j))^{-1}H(\log J(\rho_j))}{(j+2)(n-j)(n+1)}. \tag{4.12}$$

Then

$$J(\rho_{j+1})(z) = 1 + O(\delta(z)^{j+2}), \quad j = 0, 1, \cdots, n-1 \tag{4.13}$$

and

$$\delta(z) = dist(z, \partial D), \quad a_0(z) = \frac{\rho_n(z)}{\rho_0(z)} \tag{4.14}$$

Moreover, if

$$B_n = \frac{tr(H(\ell(\rho_n))^{-1} H(\log J(\rho_n)))}{(n+2)(n+1)} \ell(\rho_n(z)) \tag{4.15}$$

then

$$J(\rho_{n+1}) = 1 + O(\delta(z)^{n+2} \log \delta(z)). \tag{4.16}$$

• **Condition on Ricci lower bound:** By (4.3), (4.4) and Proposition 4.4, one has the following corollary.

Corollary 26. *Let u be strictly plurisubharmonic in D and $\rho(z) = -e^{-u}$. Let $g = \sum_{i,j=1}^n u_{i\bar{j}} dz_i \otimes d\bar{z}_j$ be the Kähler metric induced by u. Then there is a relation between the Ricci curvature $R_{i\bar{j}}$ and plurisubharmonicity of $-\log J[\rho]$ as follows:*

$$R_{i\bar{j}} \geq -(n+1) g_{i\bar{j}} \text{ in } D \iff -\log J[\rho] \text{ is plurisubharmonic in } D. \tag{4.17}$$

• **Plurisubharmonicity for $\rho(z)$.**

Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. Let u be the plurisubharmonic potential function for Kähler-Einstein metric (the solution of (4.9)). Let $\rho(z) = \rho_D(z) =: -e^{-u(z)}$.

Question. What pseudoconvex domain D has plurisubharmonic $\rho_D(z)$?

• A simple example is $D = B_n$, the unit ball in $\mathbb{C}^n$, where $u(z) = -\log(1 - |z|^2)$ and $\rho(z) = |z|^2 - 1$ is strictly plurisubharmonic in B_n.

• The result was proved by the author in [63].

Theorem 27. *If $D(A)$ is domain defined by (2.13) whose boundary is real ellipsoid in $\mathbb{C}^n$. Let $\rho(z) = -e^{-u(z)}$ with u is the potential function for Kähler-Einstein metric on $D(A)$. Then $\rho(z)$ is strictly plurisubharmonic on $D(A)$.*

Since u is the plurisubharmonic solution of (4.9), one can easily see that $\rho(z)$ is strictly plurisubharmonic in D if and only if $\det H(\rho)(z) > 0$ on D. The following theorem was proved in [63] and in [70] which is helpful to verify if $\det H(\rho) > 0$ on D.

Theorem 28. *Let $\rho(z) = -e^{-u(z)}$ with u is the potential function for Kähler-Einstein metric on D. Then $\det H(\rho)(z)$ attains its minimum over $\overline{D}$ at some point in ∂D.*

• **Ricci-flat Kähler metric.** By (4.3), one can easily see that if u is strictly plurisubharmonic solution of

$$\det H(u) = e^{f(z)}, \quad z \in \mathbb{C} \tag{4.18}$$

with f is plurisubharmonic, then the Ricci curvature for the Kähler metric

$$g = -\sum_{i,j=1}^{n} \frac{\partial^2 \log(-\rho)}{\partial z_i \partial \overline{z}_j} dz_i \otimes d\overline{z}_j \tag{4.19}$$

is flat. In particular, when $f = 0$, one has

$$\det H[u] \equiv 1, \quad z \in \mathbb{C}^n. \tag{4.20}$$

It is easy to verify that if

$$u(z) = \sum_{i,j=1}^{n} c_{i\bar{j}} z_i \overline{z}_j + \sum_{j=1}^{n} (b_j z_j + \bar{b}_j \overline{z}_j) + c \tag{4.21}$$

with $\left[c_{i\bar{j}}\right]$ is positive definite matrix and $\det[c_{i\bar{j}}] = 1$, then u is a solution of (2.20). Thus, a natural open question arises:

Problem Under what geometric condition, any plurisubharmonic solution $u \in C^2(\mathbb{C}^n)$ of $\det H(u) \equiv 1$ on $\mathbb{C}^n$ is a quadratic form (4.21).

• Without a strong geometric condition on $u_{i\bar{j}} dz_i \otimes d\overline{z}_j$, the answer of the above problem is negative even if $u_{i\bar{j}} dz_i \otimes d\overline{z}_j$ is complete.

• This problem related to the Jacobian conjecture:

Jacobian Conjecture: Let $\psi : \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic polynomial map so that $\det \psi'(z) \equiv 1$ on $\mathbb{C}^2$. Then ψ is one-to-one and onto.

• We know there are a counter example if we replace polynomial by entire holomorphic map (see the book of Rudin [77]).

EXAMPLE 2. *There is a holomorphic map $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ such that*
(i) $\phi(\mathbb{C}^2) \neq \mathbb{C}^2$;
(ii) $\det \phi'(z) \equiv 1$ on $\mathbb{C}^2$.

Let $u(z) = |\phi(z)|^2$. Then

$$\det H(u)(z) = |\det \phi'(z)|^2 = 1$$

But u is not a quadratic polynomial. This means that one must assume g is complete metric or much stronger condition on g so that the solution of (4.20) can be quadratic. In general, this problem is widely open. Moreover, Jacobian conjecture is also widely open.

5. Degenerate Monge-Ampère equation/rigidity

Let M be a complex manifold of dimension n. If M is a pseudoconvex domain given by a C^∞ positive defining function τ defined on M, i.e. $\tau : M \to [0, 1)$ is onto and $\tau \in C^\infty(M)$ is strictly plurisubharmonic in M. The following surprising result was proved by W. Stoll in [80] and D. Burns in [3] and P.-M. Wong [87].

Theorem 29. *Let M be a complex manifold of dimension n. Let $\tau : M \to [0,1)$ be a smooth strictly plurisubharmonic onto map. If*

$$\det H(\log\tau) = 0, \quad \text{for all } z \in M \text{ with } \tau(z) \neq 0. \tag{5.1}$$

Then M is biholomorphic to B_n.

There is another way to write Theorem 5.1 by using the Fefferman operator and Monge-Ampère operator which was given by the author in [59].

Corollary 30. *Let $\rho \in C^\infty(M)$ be a negative finite strictly plurisubharmonic defining function for M. If*

$$\frac{\det H(\rho)}{J(\rho)} = \text{constant on } M. \tag{5.2}$$

Then M is biholomorphic to the ball $B(0,m)$ in $\mathbb{C}^n$, where

$$m = \max\{-\rho(z) : z \in M\}. \tag{5.3}$$

Proof. Since ρ is strictly plurisubharmonic in M, we have $J(\rho) > 0$ on M. Let $z_0 \in M$ be such that $m = -\rho(z_0)$. Then $\partial\rho(z_0) = 0$ and

$$J(\rho)(z_0) = m \det H(\rho)(z_0). \tag{5.4}$$

Let

$$\tau(z) = \rho(z) + m. \tag{5.5}$$

Then $\tau : M \to [0,m)$ is smooth, onto and strictly plurisubharmonic. Since

$$|\partial\tau|^2_\tau = \sum_{i,j=1}^n \tau^{i\bar{j}}\tau_i\tau_{\bar{j}} = |\partial\rho|^2_\rho \quad \text{and} \quad \det H(\rho) = \det H(\tau), \tag{5.6}$$

we have

$$J[\tau] = -\det H(\tau)[\tau - |\partial\tau|^2_\tau] = -m\det H(\rho) + J(\rho). \tag{5.7}$$

Notice that

$$J(\tau) = -\tau^{n+1}\det H(\log\tau), \tag{5.8}$$

one has

$$J(\tau) = 0 \iff \det H(\log\tau) = 0 \quad M \setminus \tau^{-1}(0). \tag{5.9}$$

Therefore,

$$\frac{\det H(\rho)}{J(\rho)} \equiv \text{constant on } M \iff J(\tau) \equiv 0 \quad \text{on } M. \tag{5.10}$$

This completes the proof of the corollary by Theorem 5.1. □

• Based on the related problems in Pseudo-Hermitian CR geometry, one may ask the following question: Does the condition (5.2) in Corollary 5.2 can be replaced by a weaker condition $\det H(\rho)/J(\rho)$ =constant on ∂M ? The problem has been studied by the author in [59]. The following theorem was proved there.

Theorem 31. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function $\rho \in C^3(\overline{D}) \cap C^\infty(D)$ such that $u(z) = -\log(-\rho(z))$ is strictly plurisubharmonic in D. Let $g = u_{i\bar{j}} dz_i \otimes d\overline{z}_j$ be the Kähler metric induced by u. If the Ricci curvature has the lower bound:*

$$R_{i\bar{j}} \geq -(n+1) g_{i\bar{j}} \tag{5.11}$$

and if

$$\frac{\det H(\rho)}{J(\rho)} \equiv \text{constant on } \partial D, \tag{5.12}$$

then

$$\frac{\det H(\rho)}{J(\rho)} \equiv \text{constant on } D. \tag{5.13}$$

In particular, combining the above and a theorem in [58], we have the following corollary.

Corollary 32. *Let D be a smoothly bounded strictly pseudoconvex domain $\mathbb{C}^n$. Let u be the potential function for Kähler-Einstein metric for D and let $\rho(z) = -e^{-u}$. If (5.12) holds, then there is a biholomorphic map $\phi : D \to B_n$ so that $\det \phi'(z)$ is constant.*

• **Remark.** The boundary condition in Theorem 5.3 can be connected to the pseudo scalar curvature when one views $(\partial D, \theta)$, $\theta = (\partial\rho - \overline{\partial}\rho)/(2i)$, as a pseudo-Hermitian CR manifold (see Theorem 6.11 in the last section of this article).

Based on the existence and uniqueness of the Kähler-Einstein metric of Cheng and Yau [15], Theorems 5.1 and 5.2 of Stoll, Burns and Wong and other motivations in [58], one may naturally ask the following question.

Question: Let u be the potential function for Kahler-Einstein metric for a smoothly bounded pseudoconvex domain D. What is nice extra condition on u so that D is biholomorphic to B_n?

In order to study the above question, let us study what can be a necessary condition first, which may help us to search for a sufficient condition.

Let $\phi : D \to B_n$ be a biholomorphic map. Let $v(z) = -\log(1 - |\phi|^2)$. Then

$$\det H(v) = |\det \phi'(z)|^2 e^{(n+1)v} \tag{5.14}$$

and $\log |\det \phi'(z)|^2$ is pluriharmonic in D. Moreover, we let

$$\tau(z) = 1 - e^{-v(z)} = |\phi(z)|^2 \tag{5.15}$$

Then $\tau : D \to [0, 1)$ is strictly plurisubharmonic and onto. Moreover, $\log \tau$ is plurisubharmonic in D and

$$\det H(\log \tau)(z) = 0, \quad \text{if} \quad \tau(z) \neq 0. \tag{5.16}$$

Conversely, we have the following theorem was proved by the author in [58].

Theorem 33. *Let v be strictly plurisubharmonic in D so that*

$$\det H(v) = g(z)e^{(n+1)v} \text{ in } D; \quad v = +\infty \text{ on } \partial D \tag{5.17}$$

with $m = \min\{v(z) : z \in D\} = 0$ and $\log g(z)$ is pluriharmonic. If $\log \tau$ is plurisubharmonic near ∂D with $\tau(z) = 1 - e^{-v(z)}$ then D is biholomorphic to B_n.

• **Remark:** In fact, in the above theorem, the condition $\log g$ is pluriharmonic can be reduced to $-\log g$ is plurisubharmonic or the Ricci curvature, associate metric induced by u satisfying: $R_{i\bar{j}} \geq -(n+1)g_{i\bar{j}}$.

• **Remark.** Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let $f_1(z), f_2(z) \in C^\infty(\overline{D})$ be positive functions on D. If u_j is the plurisubharmonic solution of

$$\det H(u) = f_j(z)e^{(n+1)u} > 0 \text{ in } D; \quad u = +\infty \text{ on } \partial D \tag{5.18}$$

Let $\rho_j(z) = -e^{-u_j(z)}$. If

$$\log f_1(z) - \log f_2(z) = O(\delta(z)^{n+1}) \tag{5.19}$$

Then ρ^1 and ρ^2 agree on ∂D up to order $n+1$.

6. Bottom of spectrum of Δ_g

In this section, we consider the spectrum of the Laplace-Beltrami operator on a complete Riemannian manifold (M^n, g). We only describe some works related to my recent works in this area on the Kähler case.

6.1. **Riemannian case.** Let (M, g) be an n-dimensional Riemaniann manifold. Let

$$\Delta_g u = -\frac{1}{\sqrt{\det(g_{ij})}} \sum_{j=1} \frac{\partial}{\partial x_j}\Big(\sqrt{\det(g_{ij})} g^{ij} \frac{\partial u}{\partial x_i}\Big) \tag{6.1}$$

be the Laplace-Beltrami operator with respect to the Riemannian metric g. Let

$$\lambda_1 = \inf\Big\{ \int_M \sum_{i,j=1}^n g^{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dv_g : u \in C_0^\infty(M) \text{ and } \int_M u^2 dv_g = 1\Big\}. \tag{6.2}$$

• When M is compact with boundary and Δ_g is uniformly elliptic, one has that λ_1 is the first positive eigenvalue of Δ_g with Dirichlet boundary condition (see [51, 52] and references therein).

• When M is a complete, non-compact manifold, λ_1 may not be eigenvalue of Δ_g. It is the bottom of the spectrum of Δ_g (see, [11], [53, 54], etc.).

• There are many works have been done on the eigenvalues related problem, I will mention a few of them, which give a direct introduction to some works of the author [63], a joint works with M-A. Tran [69] and a joint work with X-D Wang [71].

• On the upper bound estimate of λ_1, the following theorem was proved by S. Y. Cheng in [11].

Theorem 34. *Suppose that M is an n-dim complete noncompact Riemannian manifold and Ricci curvature of $M \geq -(n-1)k$. Then $\lambda_1(M) \leq (n-1)^2 k/4$*

• On the extremal case $\lambda_1 = (n-1)^2/4$, the following rigidity type theorem was proved by P. Li and J. Wang in [53, 54]. We will state the results only for the Kähler case here.

• **Conformally compact Einstein manifolds.** Let M^{n+1} be a compact manifold with boundary ∂M. Let r be a positive defining function for M ($M = \{x \in \overline{M} : r(x) < 0\}$, $\nabla r(z) \neq 0$ on ∂M. A Riemannian metric g on M is called conformally compact if $\overline{g} = r^2 g$ can extends as a smooth metric on $\overline{M}$ for smoothly positive defining function r. $\overline{g}|_{\partial M}$ gives a Riemannian metric for ∂M. If (M, g) is Einstein (Ric(g)+$ng = 0$) and g is conformally compact, we say that (M, g) is a conformally compact Einstein manifold.

• On the lower bound estimate, the following was proved by J. Lee in [44].

Theorem 35. *Let (M, g) be a conformally compact Einstein manifold. If its conformal infinity $(\partial M, \overline{g})$ has nonnegative Yamabe invariant, then $\lambda_1 = (n-1)^2/4$, i.e, the spectrum is $[(n-1)^2/4, \infty)$.*

6.2. **Kähler case.** Let (M^n, g) be a Kähler manifold of complex dimension n. Then the Laplace-Beltrami operator is defined as

$$\Delta_g = -4 \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \tag{6.4}$$

and

(6.5)
$$\lambda_1 =: 4 \inf \Big\{ \int_M \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial u}{\partial z_i} \frac{\partial u}{\partial \bar{z}_j} dv_g : u \in C_0^\infty(M) \text{ and } \int_M u^2 dv_g = 1 \Big\}.$$

6.2.1. *Upper bound estimates for λ_1.* As a generalization of Cheng's theorem, Munteanu [73] proved the following upper bound-estimate theorem.

Theorem 36. *(O. Munteanu, JDG, 2009) Let M^m, $m \geq 2$ be a complete noncompact Kähler manifold such that the Ricci curvature is bounded from below by*

$$Ric_M \geq -2(m+1) \quad (\textit{means that } R_{i\bar{j}} \geq -(m+1) g_{i\bar{j}}) \tag{6.6}$$

Then $\lambda_1(M) \leq m^2$.

• **Remark:** In fact, the above theorem was first proved by P. Li and J-P. Wang [55] under a stronger condition: Holomorphic bisectional curvature satisfying $\mathcal{K}_g \geq -1$.

6.2.2. *Lower bound estimates for λ_1.* **Question** When $\lambda_1 = n^2$?

EXAMPLE 3. *When $M = B_n$ is the unit ball in $\mathbb{C}^n$. If g is the Kähler-Einstein or the Bergman metric on M:*

$$g =: \frac{1}{(1-|z|^2)} (\delta_{ij} + \frac{\bar{z}_i z_j}{1-|z|^2}) dz_i \otimes d\bar{z}_j \tag{6.7}$$

then

$$\lambda_1(\Delta_g) = n^2. \tag{6.8}$$

Here, the Ricci curvature

$$R_{i\bar{j}} = -(n+1)g_{i\bar{j}}, \tag{6.9}$$

and curvature tensor:

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}}g_{k\bar{\ell}} + g_{i\bar{\ell}}g_{k\bar{j}}). \tag{6.10}$$

In particular, the holomorphic bisectional curvature $\mathcal{K}_g = -1$.

• **Constructing examples for exact $\lambda_1(\Delta_g)$.** Before our results in [48], (B_n, Δ_g) with the Kähler-Einstein metric g is the only known example with $\lambda_1(\Delta_g) = n^2$. Our work provides a way to construct many examples of the Kähler metric g on the Kähler manifold D such that $\lambda_1(\Delta_g) = n^2$. In [48], we proved the following theorem.

Theorem 37. *Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. Let $\rho \in C^2(\overline{D})$ be any strictly plurisubharmonic defining function for D. Let*

$$u(z) = -\log(-\rho(z)), \quad g =: \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j} dz_i \otimes d\overline{z}_j. \tag{6.11}$$

Then $\lambda_1(\Delta_g) = n^2$.

6.3. **Rigidity type theorems.** It is natural to consider the following questions.

Question 1: *Under the assumptions: Holomorphic bisectional curvature $\mathcal{K}_g \geq -1$ and $\lambda_1(\Delta_g) = n^2$. What can one say about M?*

Question 2: *Under the assumptions: Ricci curvature $R_{i\bar{j}} \geq -(n+1)g_{i\bar{j}}$ and $\lambda_1(\Delta_g) = n^2$. What can one say about M?*

Questions 1 and 2 for Riemannian case was studied by P. Li and J. Wang in [53, 54], they proved a very pretty splitting theorem. In [55, 56], Li and Wang considered Kähler manifolds and also obtained a similar splitting theorem. The following theorem is their results for the Kähler case.

Theorem 38. *Let (M^n, g) be a complete, non-compact Kähler manifold. Then*

(i) If the Ricci curvature $R_{i\bar{j}} \geq -(n+1)g_{i\bar{j}}$ and $\lambda_1 > \frac{n+1}{2}$, then M must have one infinite volume end;

(ii) If the holomorphic bisectional curvature $\mathcal{K}_g \geq -1$ and $\lambda_1 = n^2$, then either M has only one end or $M = \mathbb{R} \times N$ with N being a compact manifold. Moreover, the metric on M is of the form

$$ds_M^2 = dt^2 + e^{4t}\omega_2^2 + e^{2t}\sum_{i=3}^{2n} \omega_i^2, \tag{6.12}$$

where $\{\omega_2, \cdots, \omega_{2n}\}$ are orthonormal basis of N with $Jdt = \omega_2$.

In [73], O. Munteanu proved the same result under a weaker condition: $R_{i\bar{j}} \geq -(n+1)g_{i\bar{j}}$ and $\lambda_1 = n^2$. In [39], Kong, Li and Zhou considered a complete Quaternonic Kähler manifold (M^{4n}, g) and proved the same theorem under the condition: the scalar curvature $S_M \geq -16n(n+2)$ and $\lambda_1 \geq (2n+1)^2$.

Once again, if $M = B_n$ is the unit ball in $\mathbb{C}^n$ and g is the Kähler-Einstein metric then

$$\lambda_1(\Delta_g) = n^2, \quad R_{i\bar{j}} = -(n+1)g_{i\bar{j}}, \quad R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}}g_{k\bar{\ell}} + g_{k\bar{j}}g_{i\bar{\ell}}) \tag{6.13}$$

which means that the holomorphic bisectional curvature equals -1. Comparing Obata theorem and Cheng theorem for compact Riemannian manifolds (see [51, 52]). One may ask the following rigidity question:

Question 3: *Assume that D is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with a complete Kähler metric g satisfying either*

$$R_{i\bar{j}} = -(n+1)g_{i\bar{j}} \quad \text{or holomorphic bisectional curvature } \mathcal{K}_g \geq -1 \tag{6.14}$$

Assume that $\lambda_1(\Delta_g) = n^2$. Is D biholomorphic to the ball in $\mathbb{C}^n$?

Let

$$D(A) = \{z \in \mathbb{C}^n : \rho(z) =: |z|^2 + \mathrm{Re}\sum_{j=1}^n A_j z_j^2 - 1 < 0\} \tag{6.15}$$

Then $\partial D(A)$ is the real ellipsoid when $A_j \in (-1, 1)$, which is a strictly convex domain in $\mathbb{C}^n$. By linearly holomorphic changes of variables, we may assume that

$$0 \leq A_1 \leq A_2 \leq \cdots \leq A_n < 1. \tag{6.16}$$

The following theorem was proved by S. Webster [84].

Theorem 39. *$D(A)$ is biholomorphic to the unit ball in $\mathbb{C}^n$ if and only if $A = (A_1, \cdots, A_n) = 0$.*

- For $\mathcal{K}_g \geq -1$, case, the author [63] proved the following theorem.

Theorem 40. *For any $0 \leq A_1 \leq \cdots \leq A_n < 2/5$, there is a Kähler metric g^0 on $D(A)$ with $A_n \leq 2/5$ such that the holomorphic bisectional curvature $\mathcal{K}_{g^0} \geq -1$ and $\lambda_1(\Delta_{g_0}) = n^2$.*

By Theorem 6.6, one has that Theorem 6.7 answers Question 3 for the case $\mathcal{K}_g \geq -1$ negatively with the counter examples: $D(A)$ with $A \neq 0$ and $n > 1$.

- For Kähler-Einstein metric case, the author [63] proved the following theorem.

Theorem 41. *Let u be strictly plurisubharmonic, which is the potential function for Kähler-Einstein metric solving the Monge-Ampère equation:*

$$\det H(u) = e^{(n+1)u}, \quad z \in D(A) \ \text{ and } \ u = \infty \text{ on } \partial D(A). \tag{6.17}$$

Let

$$\rho(z) = -\exp(-u(z)), \quad z \in D(A). \tag{6.18}$$

Then $\rho(z)$ is strictly plurisubharmonic. In particular, $\lambda_1(\Delta_g) = n^2$.

• This also provides a counter example for Question 3 in the case of the Kähler-Einstein metric.

6.4. **Positive CR-Yamabe Invariant.** Consider a compact, integrable CR manifold (M, θ) of dimension $2n+1$ and CR dimension n with contact form or pseudo-Hermitian form θ (real, one-form on M). Let $H(M)$ be the holomorphic tangent bundle on M such that

$$\theta(X) = 0, \quad X \in H(M). \tag{6.19}$$

Let $H(M)^*$ be the holomorphic cotangent bundle on M. We say that M is strictly pseudoconvex if $L_\theta = -id\theta$ is positive definite on $H(M) \oplus \overline{H(M)}$. Choosing a local basis $\{\theta^1, \cdots, \theta^n\}$ for $H(M)^*$, one can write

$$d\theta = i \sum_{\alpha,\beta=1}^{n} h_{\alpha\overline{\beta}} \theta^\alpha \theta^{\overline{\beta}} \tag{6.20}$$

with $[h_{\alpha\overline{\beta}}]$ is a positive definite $n \times n$ matrix on M. It was proved by Webster [84] that there is a unique way to write

$$d\theta^\alpha = \sum_{\gamma=1}^{n} \theta^\gamma \wedge \omega_\gamma^\alpha + \theta \wedge \tau^\alpha \tag{6.21}$$

where τ^α is a $(0,1)$-form, which is a linear combination of $\theta^{\overline{\alpha}}$, and ω_α^β is 1-form so that

$$0 = dh_{\alpha\overline{\beta}} - h_{\gamma\overline{\beta}} \omega_\alpha^\gamma - h_{\alpha\overline{\gamma}} \omega_{\overline{\beta}}^{\overline{\gamma}}. \tag{6.22}$$

Using ω_β^α as a connection, Webster [84] introduced the pseudo Ricci curvature $\mathcal{R}_{\alpha\overline{\beta}}$ and pseudo scalar curvature $\mathcal{R}_\theta = h^{\alpha\overline{\beta}} \mathcal{R}_{\alpha\overline{\beta}}$.

• CR-Yamabe invariant is defined as:

$$\mathcal{Y}(M) = \inf \Big\{ \mathcal{Y}_M(\theta) \Big\}, \tag{6.23}$$

where

$$\mathcal{Y}_M(\theta) = \frac{\int_M R_\theta \theta \wedge (d\theta)^n}{(\int_M \theta \wedge (d\theta)^n)^{2+\frac{2}{n}}}. \tag{6.24}$$

• **Question:** Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^3 boundary. Let ρ be a defining function for D so that $u(z) = -\log(-\rho(z))$ is strictly plurisubharmonic in D with Kähler metric

$$g = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j} dz_i \otimes d\overline{z}_j. \tag{6.25}$$

If M is asymptotic Einstein and $R_{i\overline{j}} \geq -(n+1) g_{i\overline{j}}$. If $\mathcal{Y}(M) \geq 0$, can one conclude that $\lambda_1(\Delta_g) = n^2$?

A joint work with Xiao-Dong Wang [71], we prove the following theorem

Theorem 42. *If (D, g) is a Kähler manifold as above. If the Ricci curvature*

$$R_{i\overline{j}} \geq -(n+1) g_{i\overline{j}} \tag{6.26}$$

and scalar curvature $R = -n(n+1)$ *near* ∂D. *Let* $\theta = \frac{1}{2i}(\partial\rho - \overline{\partial}\rho)$ *be the pseudo-Hermitian form for* ∂D. *If the pseudo-scalar curvature* $R_\theta \geq 0$ *on* ∂D. *Then* $\lambda_1(D, g) = n^2$.

• **Question:** In general, we don't know how to replace $R_\theta \geq 0$ by non-negative Yamabe invariant.

In the theory of Kähler manifold, one knows the Ricci curvature:

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log \det[g_{k\bar{\ell}}].$$

However, for the pseudo Hermitian case, the problem becomes very complicated. The following explicit formula for pseudo Ricci curvature $R_{\alpha\overline{\beta}}$ was given by Li and Luk in [65]

Theorem 43. *Let* D *be a strictly pseudoconvex domain in* $\mathbb{C}^{n+1}$ *with a defining function* $\rho \in C^3(\overline{D}) \cap C^\infty(D)$ *such that* $u(z) = -\log(-\rho)$ *is strictly plurisubharmonic in* D. *Then* $J(\rho) > 0$ *in* D. *Let* $M = \partial D$ *and let* $\theta = \frac{1}{2i}(\partial\rho - \overline{\partial}\rho)$ *on* M. *Then*

$$Ric(w, \overline{v}) = -\sum_{k,j=1}^{n+1} \frac{\partial^2 \log J(\rho)}{\partial z_k \partial \overline{z}_j} w_k \overline{v}_j + (n+1)\frac{\det H(\rho)}{J(\rho)} L_\theta(w, \overline{v}), \tag{6.27}$$

for all $w, v \in H_z(M)$.

When $M = S^{2n+}$, the unit sphere in $\mathbb{C}^{n+1}$, if $\theta = \frac{1}{2i}(\partial\rho - \overline{\partial}\rho)$ with $\rho(z) = |z|^2 - 1$, then $R_\theta = \lambda(S^{2n+1}) = n(n+1)$ on M.

With the help of a formula (6.27), Li [59] proved the following theorem:

Theorem 44. *Let* D *be a smoothly strictly pseudoconvex domain in* $\mathbb{C}^{n+1}$ *with a defining function* ρ *such that* $u = -\log(-\rho)$ *is plurisubharmonic in* D. *Then*

(a) If $-\log J(\rho)$ *is harmonic in the metric* $u_{i\bar{j}} dz_i \otimes d\overline{z}_j$ *near* ∂D, *then*

$$R_\theta(z) = n(n+1)\frac{\det H(\rho)(z)}{J(\rho)(z)}, \quad z \in M = \partial D. \tag{6.28}$$

(b) If $-\log J(\rho)$ *is plurisubharmonic in* D *and it is harmonic in the metric* $u_{i\bar{j}} dz_i \otimes d\overline{z}_j$ *near* ∂D, *and if* $R_\theta = c$ *(a positive constant) on* ∂D, *then* D *is biholomorphic to the unit ball.*

(c) In particular, if u *the potential function of the Kähler-Einstein metric for* D *and* $R_\theta = c > 0$ *on* ∂D, *then* D *is biholomorphic to ball* B_{n+1}.

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