APPENDIX A

A GENERAL REGULARITY THEOREM

We here prove a useful general regularity theorem, which is essentially an abstraction of the "dimension reducing" argument of Federer [FH2]. There are a number of important applications of this general theorem in the text.

Let $P \geq n \geq 2$ and let F be a collection of functions $\phi = (\phi^1,\ldots,\phi^Q) : \mathbb{R}^P \to \mathbb{R}^Q$ ($Q = 1$ is an important case) such that each ϕ^j is locally H^n-integrable on $\mathbb{R}^P$. For $\phi \in F$, $y \in \mathbb{R}^P$ and $\lambda > 0$ we let $\phi_{y,\lambda}$ be defined by

$$\phi_{y,\lambda}(x) = \phi(y+\lambda x)\ ,\ x \in \mathbb{R}^P\ .$$

Also, for $\phi \in F$ and a given sequence $\{\phi_k\} \subset F$ we write $\phi_k \rightharpoonup \phi$ if $\int \phi_k f \, dH^n \to \int \phi f \, dH^n$ (in $\mathbb{R}^Q$) for each given $f \in C^0_c(\mathbb{R}^P)$.

We subsequently make the following 3 special assumptions concerning F:

A.1 (Closure under appropriate scaling and translation): *If* $|y| \leq 1-\lambda$, $0 < \lambda < 1$, *and if* $\phi \in F$, *then* $\phi_{y,\lambda} \in F$.

A.2 (Existence of homogeneous degree zero "tangent functions"): *If* $|y| < 1$, *if* $\{\lambda_k\} \downarrow 0$ *and if* $\phi \in F$, *then there is a subsequence* $\{\lambda_{k'}\}$ *and* $\psi \in F$ *such that* $\phi_{y,\lambda_{k'}} \rightharpoonup \psi$ *and* $\psi_{0,\lambda} = \psi$ *for each* $\lambda > 0$.

A.3 ("Singular set" hypotheses): *We assume there is a map*

$$\text{sing} : F \to C \ (\ = \text{set of closed subsets of } \mathbb{R}^P)$$

such that

(1) $\text{sing}\,\phi = \emptyset$ *if* $\phi \in F$ *is a constant multiple of the characteristic function of an* n*-dimensional subspace of* $\mathbb{R}^P$,

(2) *if* $|y| \le 1-\lambda$, $0 < \lambda < 1$, *then* $\operatorname{sing} \phi_{y,\lambda} = \lambda^{-1}(\operatorname{sing}\phi - y)$,

(3) *if* $\phi, \phi_k \in F$ *with* $\phi_k \to \phi$, *then for each* $\varepsilon > 0$ *there is a* $k(\varepsilon)$ *such that*

$$B_1(0) \cap \operatorname{sing} \phi_k \subset \{x \in \mathbb{R}^P : \operatorname{dist}(\operatorname{sing}\phi, x) < \varepsilon\} \quad \forall\, k \ge k(\varepsilon) .$$

We can now state the main result of this section:

A.4 THEOREM *Subject to the notation and assumptions* A.1, A.2, A.3 *above, we have*

$$(*) \qquad \dim B_1(0) \cap \operatorname{sing} \phi \le n-1 \qquad \forall\, \phi \in F .$$

(Here "dim" is Hausdorff dimension, so that (*) means $H^{n-1+\alpha}(\operatorname{sing} \phi) = 0$ $\forall\, \alpha > 0$.)

In fact either $\operatorname{sing} \phi \cap B_1(0) = \emptyset$ *for every* $\phi \in F$ *or else there is an integer* $d \in [0,n-1]$ *such that*

$$\dim \operatorname{sing} \phi \cap B_1(0) \le d \qquad \forall\, \phi \in F$$

and such that there is some $\psi \in F$ *and a* d*-dimensional subspace* $L \subset \mathbb{R}^P$ *with*

$$(**) \qquad \psi_{y,\lambda} = \psi \qquad \forall\, y \in L , \quad \lambda > 0$$

and

$$\operatorname{sing} \psi = L .$$

If $d = 0$ *then* $\operatorname{sing} \phi \cap B_\rho(0)$ *is finite for each* $\phi \in F$ and each $\rho < 1$.

A.5 REMARK One readily checks that if L is an n-dimensional subspace of $\mathbb{R}^P$ and $\psi \in F$ satisfies (**), then ψ is exactly a constant multiple of the characteristic function of L (hence $\operatorname{sing} \psi = \emptyset$ by A.3(1)); otherwise we would have $P > n$ and $\psi \equiv \text{const.} \ne 0$ on some (n+1)-dimensional half-space, thus contradicting the fact that ψ is locally $\mathcal{H}^n$-integrable on $\mathbb{R}^P$.

Proof of A.4 Assume $\operatorname{sing}\phi \cap B_1(0) \neq \emptyset$ for some $\phi \in \mathcal{F}$, and let $d = \sup\{\dim L : L$ is a d-dimensional subspace of $\mathbb{R}^P$ and there is $\phi \in \mathcal{F}$ with $\operatorname{sing}\phi \neq \emptyset$ and $\phi_{y,\lambda} = \phi \;\; \forall y \in L,\ \lambda > 0\}$. Then by Remark A.5 we have $d \leq n-1$.

For a given $\phi \in \mathcal{F}$ and $y \in B_1(0)$ we let $T(\phi,y)$ be the set of $\psi \in \mathcal{F}$ with $\psi_{0,\lambda} = \psi \;\forall\lambda > 0$ and with $\lim \phi_{y,\lambda_k} = \psi$ for some sequence $\lambda_k \downarrow 0$. ($T(\phi,y) \neq \emptyset$ by assumption A.2).

Let $\ell \geq 0$ and let

(1) $$\mathcal{F}^\ell = \{\phi \in \mathcal{F} : \mathcal{H}^\ell(\operatorname{sing}\phi \cap B_1(0)) > 0\}.$$

Our first task is to prove the implication

(2) $$\phi \in \mathcal{F}^\ell \Rightarrow \exists\ \psi \in T(\phi,x) \cap \mathcal{F}^\ell$$

for $\mathcal{H}^\ell$-a.e. $x \in \operatorname{sing}\phi \cap B_1(0)$.

To see this, let $\mathcal{H}^\ell_\delta$ be the "size δ approximation" of $\mathcal{H}^\ell$ as described in §2 and recall that $\mathcal{H}^\ell(A) > 0 \Leftrightarrow \mathcal{H}^\ell_\infty(A) > 0$, so that $\mathcal{F}^\ell = \{\phi \in \mathcal{F} : \mathcal{H}^\ell_\infty(\operatorname{sing}\phi \cap B_1(0)) > 0\}$. Also note that (by 3.6(2)), for any bounded subset A of $\mathbb{R}^P$,

(3) $$\mathcal{H}^\ell_\infty(A) > 0 \Rightarrow \Theta^{*n}(\mathcal{H}^\ell_\infty \llcorner A, x) > 0 \quad \text{for } \mathcal{H}^\ell\text{-a.e. } x \in A.$$

Thus we see that if $\phi \in \mathcal{F}^\ell$ then for $\mathcal{H}^\ell$-a.e. $x \in \operatorname{sing}\phi \cap B_1(0)$ we have $\Theta^{*\ell}(\mathcal{H}^\ell_\infty \llcorner \operatorname{sing}\phi, x) > 0$. For such x we thus have a sequence $\lambda_k \downarrow 0$ such that

$$(4) \qquad \lim_{k\to\infty} \frac{\mathcal{H}^{\ell}_{\infty}(\operatorname{sing}\phi \cap B_{\lambda_k}(x))}{\lambda_k^{\ell}} > 0 \ ,$$

and by assumption A.2 there is a subsequence $\{\lambda_{k'}\}$ such that $\phi_{x,\lambda_{k'}} \rightharpoonup \psi \in T(\phi,x)$. If now $\mathcal{H}^{\ell}_{\infty}(\operatorname{sing}\psi) = 0$, then for any $\varepsilon > 0$ we could find open balls $\{B_{\rho_j}(x_j)\}$ such that

$$(5) \qquad \operatorname{sing}\psi \subset \bigcup_j B_{\rho_j}(x_j)$$

and

$$(6) \qquad \sum_j \omega_{\ell}\, \rho_j^{\ell} < \varepsilon$$

(by definition of $\mathcal{H}^{\ell}_{\infty}$) . Now (5) in particular implies that $K \equiv \overline{B_1}(0) \sim \bigcup_j B_{\rho_j}(x_j)$ is a compact set with positive distance from $\operatorname{sing}\psi$. Hence by assumption A.3(3) we have

$$(7) \qquad \operatorname{sing}\phi_{x,\lambda_{k'}} \cap B_1(0) \subset \bigcup_j B_{\rho_j}(x_j)$$

for all sufficiently large k , and hence by (6)

$$\mathcal{H}^{\ell}_{\infty}(\operatorname{sing}\phi_{x,\lambda_{k'}} \cap B_1(0)) < \varepsilon \ , \ k \geq k(\varepsilon) \ .$$

Thus since $\lambda_k^{-1}(\operatorname{sing}\phi - x) = \operatorname{sing}\phi_{x,\lambda_k}$ (by A.3(2)) we have

$$\lambda_{k'}^{-\ell}\, \mathcal{H}^{\ell}_{\infty}(\operatorname{sing}\phi \cap B_{\lambda_{k'}}(x)) < \varepsilon$$

for all sufficiently large k , thus a contradiction for

$\varepsilon < \lim_{k\to\infty} \lambda_k^{-\ell}\, \mathcal{H}^{\ell}_{\infty}(\operatorname{sing}\phi \cap B_{\lambda_k}(x))$. (Such ε can be chosen by (4).)

We have therefore established the general implication (2). From now on take $\ell > d-1$ so that $F^{\ell} \neq \emptyset$ (which is automatic for $\ell \leq d$ by definition of d). By (2) there is $\phi \in F^{\ell}$ with $\phi_{0,\lambda} = \phi \quad \forall \lambda > 0$. Suppose also that there is a k-dimensional subspace $(k \geq 0)$ S of $\mathbb{R}^P$ such that $\phi_{y,\lambda} = \phi$ $\forall y \in S$, $\lambda > 0$. (Notice of course this is no additional restriction for ϕ in case $k = 0$.) Now if $k \geq d+1$ then, by definition of d, we can assert $\operatorname{sing} \phi = \emptyset$, thus contradicting the fact that $\phi \in F^{\ell}$. Therefore $0 \leq k \leq d$, and if $k \leq d-1$ $(< \ell)$, then $H^{\ell}(S) = 0$ and in particular

(8) $$\exists\, x \in B_1(0) \cap \operatorname{sing} \phi \sim S .$$

But by A.2 we can choose $\psi \in T(\phi,x)$. Since $\psi = \lim \phi_{x,\lambda_j}$ for some sequence $\lambda_j \downarrow 0$, we evidently have (since $\phi_{y+x,\lambda} = \phi_{x,\lambda} \quad \forall y \in S$, $\lambda > 0$)

(9) $$\psi_{y,1} = \lim \phi_{y+x,\lambda_j} = \lim \phi_{x,\lambda_j} = \psi \qquad \forall y \in S$$

and

(10) $$\psi_{\beta x,1} = \lim \phi_{x+\lambda_j \beta x,\lambda_j} = \psi \qquad \forall \beta \in \mathbb{R} .$$

(All limits in the weak sense described at the beginning of the section.) Thus $\psi_{z,\lambda} = \psi$ for each $\lambda > 0$ and each z in the $(k+1)$-dimensional subspace T of $\mathbb{R}^P$ spanned by S and x. $\operatorname{Sing} \psi \neq \emptyset$ (by A.3(3)), hence by induction on k we can take $k = d-1$; i.e. $\dim T = d$, and hence $\operatorname{sing} \psi \supset T$ by A.3(2). On the other hand if $\exists\, \tilde{x} \in \operatorname{sing} \psi \sim T$ then we can repeat the above argument (beginning at (8)) with T in place of S and ψ in place of ϕ. This would then give a $(d+1)$-dimensional subspace $\tilde{T}$ and a $\tilde{\psi} \in F$ with $\operatorname{sing} \tilde{\psi} \supset \tilde{T}$, thus contradicting the definition of d. Therefore $\operatorname{sing} \phi = T$. Furthermore if $\ell > d$ then the above induction works up to $k = d$ and again therefore we would have a contradiction. Thus $\dim(B_1(0) \cap \operatorname{sing}\phi) \leq d \ \forall \phi \in F$.

Finally to prove the last claim of the theorem, we suppose that $d=0$. Then we have already established that

$$(11) \qquad \mathcal{H}^{\alpha}(\operatorname{sing}\phi \cap B_1(0)) = 0 \quad \forall\ \alpha > 0\ ,\ \phi \in F\ .$$

If $\operatorname{sing}\phi \cap B_\rho(0)$ is not finite, then we select $x \in \overline{B_\rho}(0)$ such that $x = \lim x_k$ for some sequence $x_k \in \operatorname{sing}\phi \cap B_1(0) \sim \{x\}$. Then letting $\lambda_k = 2|x_k - x|$ we see from A.3(2) that there is a subsequence $\{\lambda_{k'}\}$ with $\phi_{x,\lambda_{k'}} \rightharpoonup \psi \in T(\phi,x)$ and $(x_{k'}-x)/|x_{k'}-x| \to \xi \in \partial B_1(0)$. Now by A.3(2), (3) we know that $\{\xi/2\} \cap \{0\} \subset \operatorname{sing}\psi$ and, since $\psi_{0,\lambda} = \psi$, this (together with A.3(2)) gives $L_\xi \subset \operatorname{sing}\psi$ where L_ξ is the ray determined by 0 and ξ. Then $\mathcal{H}^1(\operatorname{sing}\psi \cap B_1(0)) > 0$, thus contracting (11), because $\psi \in F$.

APPENDIX B

NON-EXISTENCE OF STABLE MINIMAL CONES, $2 \le n \le 6$.

Here we describe J. Simons [SJ] result on non-existence of n-dimensional stable minimal cones (previously established in case $n = 2,3$ by Fleming [F] and Almgren [A4] respectively). The proof here follows essentially Schoen-Simon-Yau [SSY], and is slightly cleaner than the original proof in [SJ].

Suppose to begin that $C \in \mathcal{D}_n(\mathbb{R}^{n+1})$ is a cone $(\eta_{0,\lambda\#}C = C)$ and C is integer multiplicity with $\partial C = 0$. If $\operatorname{sing} C \subset \{0\}$ and if C is minimizing in $\mathbb{R}^{n+1}$ then, writing $M = \operatorname{spt} C \sim \{0\}$ and taking M_t as in §9, we have $\frac{d}{dt} \mathcal{H}^n(M_t)\Big|_{t=0} = 0$ and $\frac{d^2}{dt^2} \mathcal{H}^n(M_t)\Big|_{t=0} \ge 0$. (This is clear because in fact $\mathcal{H}^n(M_t)$ takes its minimum value at $t = 0$, by virtue of our assumption that C is minimizing.) Notice that M is orientable, with orientation induced from C, and hence in particular we can deduce from 9.8 that

B.1
$$\int_M (|\nabla^M \zeta|^2 - \zeta^2 |A|^2)\, d\mathcal{H}^n \ge 0$$

for any $\zeta \in C^1_c(M)$ (notice $0 \notin M$, so such ζ vanish in a neighbourhood of 0). Here A is the second fundamental form of M and $|A|$ is its length, as described in §7 and in 9.8.

The main result we need is given in the following theorem.

B.2 THEOREM *Suppose* $2 \le n \le 6$ *and* M *is an* n*-dimensional cone embedded in* $\mathbb{R}^{n+1}$ *with zero mean curvature* (*see* §7) *and with* $\bar{M} \sim M = \{0\}$, *and suppose that* M *is stable in the sense that* B.1 *holds. Then* $\bar{M}$ *is a hyperplane.*

(As explained above, the hypotheses are in particular satisfied if $M = \operatorname{spt} C \sim \{0\}$, with $C \in \mathcal{D}_n(\mathbb{R}^{n+1})$ a minimizing cone with $\partial C = 0$ and $\operatorname{sing} C \subset \{0\}$.)

B.3 REMARK Theorem B.2 is false for $n=7$; J. Simons [SJ] was the first to point out that the cone $M=\{(x^1,\ldots,x^8)\in\mathbb{R}^8 : \sum_{i=1}^{4}(x^i)^2 = \sum_{i=5}^{8}(x^i)^2\}$ is a stable minimal cone. (Notice that M is the cone over the compact manifold $\left(\frac{1}{\sqrt{2}}S^3\right)\times\left(\frac{1}{\sqrt{2}}S^3\right)\subset S^7\subset\mathbb{R}^8$.) The fact that the mean curvature of M is zero is checked by direct computation. The fact that M is actually *stable* is checked as follows. First, by direct computation one checks that the second fundamental form A of M satisfies $|A|^2 = 6/|x|^2$.

On the other hand for a stationary hypersurface $M\subset\mathbb{R}^{n+1}$ the first variation formula 9.3 says $\int \operatorname{div}_M X\,d\mathcal{H}^n = 0$ if $\operatorname{spt}|X|$ is a compact subset of M . Taking $X_x = (\zeta^2/r^2)x$, $\zeta\in C_c^\infty(M)$, $r=|x|$, and computing as in §17, we get

$$(n-2)\int_M (\zeta^2/r^2)\,d\mathcal{H}^n = -2\int_M \zeta r^{-2}x\cdot\nabla^M\zeta\,d\mathcal{H}^n .$$

Using the Schwartz inequality on the right we get

$$\frac{(n-2)^2}{4}\int_M (\zeta^2/r^2)\,d\mathcal{H}^n \le \int_M |\nabla^M\zeta|^2 d\mathcal{H}^n .$$

Thus we have stability for M (in the sense of B.1) whenever A satisfies $|x|^2|A|^2 \le (n-2)^2/4$.

For the example above we have $n=7$ and $|x|^2\,|A|^2 = 6$, so that this inequality is satisfied, and the cone over $S^3\times S^3$ is stable as claimed. (Similarly the cone over $S^q\times S^q$ is stable for $q\ge 3$; i.e. when the dimension of the cone is ≥ 7.)

Before giving the proof of B.2 we need to derive the identity of J. Simons for the Laplacian of the length of the second fundamental form of a hypersurface (Lemma B.8 below).

The simple derivation here assumes the reader's familiarity with basic Riemannian geometry. (A completely elementary derivation, assuming no such background, is described in [G].)

For the moment let M be an arbitrary hypersurface in $\mathbb{R}^{n+1}$ (M not necessarily a cone, and not necessarily having zero mean curvature).

Let $\tau_1,\ldots,\tau_n$ be a locally defined family of smooth vector fields which, together with the unit normal ν of M, define an orthonormal basis for $\mathbb{R}^{n+1}$ at all points in some region of M.

The second fundamental form of M relative to the unit normal ν is the tensor $A = h_{ij}\tau_i \otimes \tau_j$, where $h_{ij} = \langle D_{\tau_j}\nu, \tau_i\rangle$. (Cf. §7.) Recall that we have

B.4 $$h_{ij} = h_{ji} ,$$

and, since the Riemann tensor of $\mathbb{R}^{n+1}$ is zero, we have the *Codazzi equations*

B.5 $$h_{ij,k} = h_{ik,j} , \quad i,j,k \in \{1,\ldots,n\} .$$

Here $h_{ij,k}$ denotes the covariant derivative of A with respect to τ_k; that is, $h_{ij,k}$ are such that $\nabla_{\tau_k} A = h_{ij,k}\tau_i \otimes \tau_j$.

We also have the *Gauss curvature* equations

B.6 $$R_{ijk\ell} = h_{i\ell}h_{jk} - h_{ik}h_{j\ell} ,$$

where $R = R_{ijk\ell}\,\tau_i \otimes \tau_j \otimes \tau_k \otimes \tau_\ell$ is the Riemann curvature tensor of M, and where we use the sign convention such that R_{ijji} $(i \neq j)$ are sectional curvatures of M ($= +1$ if $M = S^n$).

From the properties of R (in fact essentially by definition of R) we also have, for any 2-tensor $a_{ij}\,\tau_i \otimes \tau_j$,

$$a_{ij,k\ell} = a_{ij,\ell k} + a_{im}R_{mj\ell k} + a_{mj}R_{mi\ell k}$$

(where $a_{ij,k\ell}$ means $a_{ij,k,\ell}$ - i.e. the covariant derivative with respect to τ_ℓ of the tensor $a_{ij,k}\,\tau_i \otimes \tau_j \otimes \tau_k$) . In particular

B.7 $$h_{ij,k\ell} = h_{ij,\ell k} + h_{im}R_{mj\ell k} + h_{mj}R_{mi\ell k}$$

$$= h_{ij,\ell k} + h_{im}[h_{m\ell}h_{jk}-h_{mk}h_{j\ell}] - h_{mj}[h_{i\ell}h_{mk}-h_{ik}h_{m\ell}]$$

by B.6.

B.8 LEMMA *In the notation above*

$$\Delta_M(\tfrac{1}{2}|A|^2) = \sum_{i,j,k} h^2_{ij,k} - |A|^4 + h_{ij}H_{,ij} + Hh_{mi}h_{mj}h_{ij} ,$$

where $H = h_{kk}$ = trace A .

Proof We first compute $h_{ij,kk}$:

$$h_{ij,kk} = h_{ik,jk} \qquad \text{(by B.5)}$$

$$= h_{ki,jk} \qquad \text{(by B.4)}$$

$$= h_{ki,kj} + h_{km}[h_{mj}h_{ik}-h_{mk}h_{ij}]$$

$$- h_{mi}[h_{kj}h_{mk}-h_{kk}h_{mj}] \qquad \text{(by B.7)}$$

$$= h_{ki,kj} - \left(\sum_{m,k} h^2_{mk}\right)h_{ij} + h_{kk}h_{mi}h_{mj}$$

$$= h_{kk,ij} - \left(\sum_{m,k} h^2_{mk}\right)h_{ij} + h_{kk}h_{mi}h_{mj} \qquad \text{(by B.5)}$$

Now multiplying by h_{ij} we then get (since $h_{ij}h_{ij,kk} = \frac{1}{2}\left(\sum_{i,j} h^2_{ij}\right)_{,kk} - \sum_{i,j,k} h^2_{ij,k}$)

$$\frac{1}{2}\Big(\sum_{i,j} h_{ij}^2\Big)_{,kk} = \sum_{i,j,k} h_{ij,k}^2 - \Big(\sum_{i,j} h_{ij}^2\Big)^2 + h_{ij}H_{,ij} + Hh_{mi}h_{mj}h_{ij} ,$$

which is the required identity.

We now want to examine carefully the term $\sum_{i,j,k} h_{ij,k}^2$ appearing in the identity of B.8 in case M is a cone with vertex at 0 (i.e. $\eta_{0,\lambda}M=M$ $\forall\lambda > 0$) . In particular we want to compare $\sum_{i,j,k} h_{ij,k}^2$ with $|\nabla^M|A||^2$ in this case. Since $|\nabla^M|A||^2 = \sum_{k=1}^n |A|^{-2}(h_{ij}h_{ij,k})^2$, we look at the difference

$$(*) \qquad D \equiv \sum_{i,j,k} h_{ij,k}^2 - \sum_{k=1}^n |A|^{-2}(h_{ij}h_{ij,k})^2 .$$

B.9 LEMMA *If* M *is a cone (not necessarily minimal) the quantity* D *defined in* (*) *satisfies*

$$D(x) \geq 2|x|^{-2}|A(x)|^2 , \ x \in M .$$

Proof Let $x \in M$ and select the frame $\tau_1,\ldots,\tau_n$ so that τ_n is radial $(x/|x|)$ along the ray ℓ_x through x , and so that (as vectors in $\mathbb{R}^{n+1}$) $\tau_1,\ldots,\tau_n$ are constant along ℓ_x . Then

$$(1) \qquad h_{nj} = h_{jn} = 0 \quad \text{along } \ell_x , \ j=1,\ldots,n ,$$

and (since $h_{ij}(\lambda x) = \lambda^{-1}h_{ij}(x)$, $\lambda > 0$)

$$(2) \qquad h_{ij,n} = -r^{-1}h_{ij} \quad \text{along } \ell_x .$$

Rearranging the expression for D , we have

$$D = \frac{1}{2}\sum_{k=1}^n \sum_{i,j,r,s=1}^n |A|^{-2}(h_{rs}h_{ij,k} - h_{ij}h_{rs,k})^2 ,$$

as one easily checks by expanding the square on the right. Now since

$$\sum_{i,j,r,s=1}^{n} (\quad)^2 \geq 4 \sum_{\substack{i,j,r=1\\ s=n}}^{n-1} (\quad)^2 ,$$

we thus have

$$D \geq 2|A|^{-2} \sum_{k=1}^{n} \sum_{i,j,r=1}^{n-1} (h_{ij}h_{rn,k})^2 .$$

By the Codazzi equations B.5 and (2) this gives

$$D \geq 2r^{-2}|A|^{-2} \sum_{k=1}^{n} \sum_{i,j,r=1}^{n-1} h_{ij}^2 h_{rk}^2$$

$$= 2r^{-2}|A|^{-2}|A|^4 \qquad \text{(by (1))}$$

$$= 2r^{-2}|A|^2 ,$$

as required.

Proof of B.2 Notice that so far we have not used the minimality of M (i.e. we have not used $H(=h_{kk}) = 0$) . We now do set $H=0$ in the above computations, thus giving (by B.8, B.9)

(1) $$\Delta_M(\tfrac{1}{2}|A|^2) + |A|^4 \geq 2r^{-2}|A|^2 + |\nabla|A||^2$$

for the minimal cone M . (Notice that $|A|$ is Lipschitz, and hence $|\nabla|A||$ makes sense $\mathcal{H}^n$ - a.e. in M.)

Our aim now is to use (1) in combination with the stability inequality B.1 to get a contradiction in case $2 \leq n \leq 6$.

Specifically, replace ζ by $\zeta|A|$ in B.1. This gives

(2) $$\int_m \zeta^2|A|^4 \leq \int_M |\nabla(\zeta|A|)|^2$$

$$= \int_M (|\nabla\zeta|^2|A|^2 + \zeta^2|\nabla|A||^2)$$

$$+ 2\int_M \zeta|A|\nabla\zeta\cdot\nabla|A| .$$

Now

$$2\int_M \zeta|A|\nabla\zeta\cdot\nabla|A| = 2\int_M \zeta\nabla\zeta\cdot\nabla(\tfrac{1}{2}|A|^2)$$
$$= \int_M (\nabla\zeta^2)\cdot\nabla(\tfrac{1}{2}|A|^2)$$
$$= -\int_M \zeta^2\Delta_M(\tfrac{1}{2}|A|^2)$$
$$\leq \int_M (|A|^4\zeta^2 - 2r^{-2}\zeta^2|A|^2 + \zeta^2|\nabla|A||^2) \qquad \text{by (1)},$$

and hence (2) gives

(3) $$2\int_M r^{-2}\zeta^2|A|^2 \leq \int_M |A|^2|\nabla\zeta|^2 \qquad \forall\ \zeta \in C_c^1(M).$$

Now we claim that (3) is valid even if ζ does not have compact support on M, provided that ζ is locally Lipschitz and

(4) $$\int_M r^{-2}\zeta^2|A|^2 < \infty .$$

(This is proved by applying (3) with $\zeta\gamma_\varepsilon$ in place of ζ, where γ_ε is such that $\gamma_\varepsilon(x) \equiv 1$ for $|x| \in (\varepsilon,\varepsilon^{-1})$, $|\nabla\gamma_\varepsilon(x)| \leq 3/|x|$ for all x, $\gamma_\varepsilon(x) = 0$ for $|x| < \varepsilon/2$ or $|x| > 2\varepsilon^{-1}$, and $0 \leq \gamma_\varepsilon \leq 1$ everywhere, then letting $\varepsilon \downarrow 0$ and using (4).)

Since M is a cone we can write

(5) $$\int_M \phi(x)\,d\mathcal{H}^n(x) = \int_0^\infty r^{n-1}\int_\Sigma \phi(r\omega)\,d\mathcal{H}^{n-1}(\omega)\,dr$$

for any non-negative continuous ϕ on M, where $\Sigma = M \cap S^n$ is a compact $(n-1)$-dimensional submanifold. Since $|A(x)|^2 = r^{-2}|A(x/|x|)|^2$, we can now use (5) to check that $\zeta = r^{1+\varepsilon}r_1^{1-n/2-2\varepsilon}$, $r_1 = \max\{1,r\}$, is a valid choice to ensure (4), hence we may use this choice in (3). This is easily seen to give

$$(6) \qquad 2\int_M r^{2\varepsilon} r_1^{2-n-4\varepsilon}|A|^2 \le ((n/2)-2+\varepsilon)^2 \int_{M\cap\{r>1\}} |A|^2 r^{2-n-2\varepsilon} + (1+\varepsilon)^2 \int_{M\cap\{r<1\}} |A|^2 r^{2\varepsilon} < \infty .$$

For $2 \le n \le 6$ we can choose ε such that $((n/2)-2+\varepsilon)^2 < 2$ and $(1+\varepsilon)^2 < 2$, hence (6) gives $|A|^2 \equiv 0$ on M as required.