CHAPTER 6

CURRENTS

This chapter provides an introduction to the basic theory of currents, with particular emphasis on integer multiplicity rectifiable n-currents (briefly called integer multiplicity currents), which are essentially just integer n-varifolds equipped with an orientation.* The concept of such currents was introduced in the historic paper [FF] of Federer and Fleming; their advantage is that they are at once able to be represented as "generalized surfaces" (in terms of a countably n-rectifiable set with an integer multiplicity) and at the same time have nice compactness properties (see 27.3 below).

§25. PRELIMINARIES: VECTORS, CO-VECTORS, AND FORMS

$e_1,\ldots,e_P$ denote the standard orthonormal basis for $\mathbb{R}^P$ and $\omega^1,\ldots,\omega^P$ the dual basis for the dual space $\Lambda^1(\mathbb{R}^P)$ of $\mathbb{R}^P$. $\Lambda_n(\mathbb{R}^P)$, $\Lambda^n(\mathbb{R}^P)$ denote the spaces of n-vectors and n-covectors respectively. Thus $v \in \Lambda_n(\mathbb{R}^P)$ can be represented

$$v = \sum_{1\le i_1<\ldots<i_n\le P} a_{i_1\ldots i_n}\, e_{i_1}\wedge\ldots\wedge e_{i_n}$$

$$= \sum_{\alpha\in I_{n,P}} a_\alpha\, e_\alpha ,$$

using "multi-index" notation in which $\alpha = (i_1,\ldots,i_n) \in \mathbb{Z}_+^n \equiv \{(j_1,\ldots,j_n)$: each j_ℓ is a positive integer$\}$ and $I_{n,P} = \{\alpha = (i_1,\ldots,i_n) \in \mathbb{Z}_+^n$: $1 \le i_1<\ldots<i_n \le P\}$. Similarly any $w \in \Lambda^n(\mathbb{R}^P)$ can be represented as

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* These are precisely the currents called *locally rectifiable* in the literature (see [FF], [FH1]); we have adopted the present terminology both because it seems more natural and also because it is consistent with the varifold terminology of Allard (see Chapter 4, Chapter 8).

$$w = \sum_{\alpha \in I_{n,P}} a_\alpha \omega^\alpha ,$$

where $\omega^\alpha = \omega^{i_1} \wedge \ldots \wedge \omega^{i_n}$ if $\alpha = (i_1, \ldots, i_n) \in I_{n,P}$. Such a v (respectively w) is called *simple* if it can be expressed $v_1 \wedge \ldots \wedge v_n$ with $v_j \in \mathbb{R}^P$ (respectively $w_1 \wedge \ldots \wedge w_n$ with $w_j \in \Lambda^1(\mathbb{R}^P)$) . We assume $\Lambda_n(\mathbb{R}^P)$, $\Lambda^n(\mathbb{R}^P)$ are equipped with the inner products $<\ ,\ >$ naturally induced from $\mathbb{R}^P$ (making $\{e_\alpha\}_{\alpha \in I_{n,P}}$, $\{\omega^\alpha\}_{\alpha \in I_{n,P}}$ orthonormal bases) . Thus

$$\left\langle \sum_{\alpha \in I_{n,P}} a_\alpha e_\alpha , \sum_{\alpha \in I_{n,P}} b_\alpha e_\alpha \right\rangle = \sum_{\alpha \in I_{n,P}} a_\alpha b_\alpha$$

and

$$\left\langle \sum_{\alpha \in I_{n,P}} a_\alpha \omega^\alpha , \sum_{\alpha \in I_{n,P}} b_\alpha \omega^\alpha \right\rangle = \sum_{\alpha \in I_{n,P}} a_\alpha b_\alpha .$$

For open $U \subset \mathbb{R}^P$, $E^n(U)$ denotes the set of smooth n-forms $\omega = \sum_{\alpha \in I_{n,P}} a_\alpha dx^\alpha$ where $a_\alpha \in C^\infty(U)$ and $dx^\alpha = dx^{i_1} \wedge \ldots \wedge dx^{i_n}$ if $\alpha = (i_1, \ldots, i_n) \in I_{n,P}$. dx^j as usual denotes the 1-form given by

25.1 $$dx^j(f) = \frac{\partial f}{\partial x^j} , \ f \in C^\infty(U) .$$

If we make the usual identifications of $T_x \mathbb{R}^P$ and $\Lambda^1(T_x\mathbb{R}^P)$ with $\mathbb{R}^P$ and $\Lambda^1(\mathbb{R}^P)$, we are able to interpret $\omega \in E^n(U)$ as an element of $C^\infty(U;\Lambda^n\mathbb{R}^P)$; we shall do this frequently in the sequel.

The exterior derivative $E^n(U) \to E^{n+1}(U)$ is defined as usual by

25.2 $$d\omega = \sum_{j=1}^{P} \sum_{\alpha \in I_{n,P}} \frac{\partial a_\alpha}{\partial x^j} dx^j \wedge dx^\alpha$$

if $\omega = \sum_{\alpha \in I_{n,P}} a_\alpha dx^\alpha$. By direct computation (using $\dfrac{\partial^2 a_\alpha}{\partial x^i \partial x^j} = \dfrac{\partial^2 a_\alpha}{\partial x^j \partial x^i}$

and $dx^i \wedge dx^j = - dx^j \wedge dx^i$) one checks that

25.3 $$d^2\omega = 0 \quad \forall\, \omega \in \mathcal{E}^n(U)\ .$$

Given $\omega = \sum_{\alpha\in I_{n,Q}} a_\alpha(y)\,dy^\alpha \in \mathcal{E}^n(V)$, $V \subset \mathbb{R}^Q$ open, and a smooth map $f : U \to V$, we define the "pulled back" form $f^\#\omega \in \mathcal{E}^n(U)$ by

25.4 $$f^\#\omega = \sum_{\alpha=(i_1,\ldots,i_n)\in I_{n,Q}} a_\alpha \circ f\ df^{i_1} \wedge \ldots \wedge df^{i_n}\ ,$$

where df^j is $\sum_{i=1}^{P} \frac{\partial f^j}{\partial x^i}\, dx^i$, $j = 1,\ldots,Q$.

Notice that the exterior derivative commutes with pulling back:

25.5 $$df^\# = f^\# d\ .$$

We let $\mathcal{D}^n(U)$ denote the set of $\omega = \sum_{\alpha\in I_{n,P}} a_\alpha dx^\alpha \in \mathcal{E}^n(U)$ such that each a_α has compact support. We topologize $\mathcal{D}^n(U)$ with the usual locally convex topology, characterized by the assertion that $\omega^k = \sum_{\alpha\in I_{n,P}} a_\alpha^{(k)} dx^\alpha \to \omega = \sum_{\alpha\in I_{n,P}} a_\alpha\, dx^\alpha$ if there is a fixed compact $K \subset U$ such that $\operatorname{spt} a_\alpha^{(k)} \subset K$ $\forall\ \alpha \in I_{n,P}$, $k \geq 1$, and if $\lim D^\beta a_\alpha^{(k)} = D^\beta a_\alpha$ $\forall\ \alpha \in I_{n,P}$ and every multi-index β . For any $\omega \in \mathcal{D}^n(U)$, we define

25.6 $$|\omega| = \sup_{x\in U} \langle \omega(x), \omega(x)\rangle^{\frac{1}{2}}\ .$$

Notice that if $f : U \to V$ is smooth (U,V open in $\mathbb{R}^P$, $\mathbb{R}^Q$) and if f is *proper* (i.e. $f^{-1}(K)$ is a compact subset of U whenever K is a compact subset of V) then $f^\#\omega \in \mathcal{D}^n(U)$ whenever $\omega \in \mathcal{D}^n(V)$.

§26. GENERAL CURRENTS

Throughout this section U is an open subset of $\mathbb{R}^P$.

26.1 DEFINITION An n-dimensional current (briefly called an n-current) in U is a continuous linear functional on $\mathcal{D}^n(U)$. The set of such n-currents will be denoted $\mathcal{D}_n(U)$.

Note that in case $n=0$ the n-currents in U are just the Schwartz distributions on U. More importantly though, the n-currents, $n \geq 1$, can be interpreted as a generalization of the n-dimensional oriented submanifolds M having locally finite $\mathcal{H}^n$-measure in U. Indeed given such an $M \subset U$ with orientation ξ (thus $\xi(x)$ is continuous on M with $\xi(x) = \pm \tau_1 \wedge \ldots \wedge \tau_n$ $\forall\, x \in M$, where $\tau_1, \ldots, \tau_n$ is an orthonormal basis for $T_x M)^*$, then there is a corresponding n-current $[\![M]\!] \in \mathcal{D}_n(U)$ defined by

$$[\![M]\!](\omega) = \int_M \langle \omega(x), \xi(x) \rangle \, d\mathcal{H}^n(x) \ , \quad \omega \in \mathcal{D}^n(U) \ , \tag{26.2}$$

where $\langle\ ,\ \rangle$ denotes the dual pairing for $\Lambda^n(\mathbb{R}^P)$, $\Lambda_n(\mathbb{R}^P)$. (That is, the n-current $[\![M]\!]$ is obtained by integration of n-forms over M in the usual sense of differential geometry: $[\![M]\!](\omega) = \int_M \omega$ in the usual notation of differential geometry.)

Motivated by the classical Stokes' theorem ($\int_M d\omega = \int_{\partial M} \omega$ if M is a compact smooth manifold with smooth boundary) we are led (by 26.2) to quite generally define the boundary ∂T of an n-current $T \in \mathcal{D}_n(U)$ by

$$\partial T(\omega) = T(d\omega) \ , \quad \omega \in \mathcal{D}^n(U) \tag{26.3}$$

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* Thus $\xi(x) \in \Lambda_n(T_x M)$; notice this differs from the usual convention of differential geometry where we would take $\xi(x) \in \Lambda^n(T_x M)$.

(and $\partial T = 0$ if $n = 0$) ; thus $\partial T \in \mathcal{D}_{n-1}(U)$ if $T \in \mathcal{D}_n(U)$. Here and subsequently we define $\mathcal{D}_{n-1}(U) = 0$ in case $n = 0$.

Notice that $\partial^2 T = 0$ by 25.3

Again motivated by the special example $T = [\![M]\!]$ as in 26.2 we define the *mass* of T , $\underline{\underline{M}}(T)$, for $T \in \mathcal{D}_n(U)$ by

26.4 $$\underline{\underline{M}}(T) = \sup_{|\omega|\le 1, \omega\in\mathcal{D}^n(U)} T(\omega)$$

(so that $\underline{\underline{M}}(T) = \mathcal{H}^n(M)$ in case $T = [\![M]\!]$ as in 26.2). More generally for any open $W \subset U$ we define

26.5 $$\underline{\underline{M}}_W(T) = \sup_{\substack{|\omega|\le 1, \omega\in\mathcal{D}^n(U)\\ \mathrm{spt}\,\omega\subset W}} T(\omega)$$

26.6 REMARK Notice that there is some flexibility in the definition of $\underline{\underline{M}}$; we would still get the "correct" value $\mathcal{H}^n(M)$ for the case $T = [\![M]\!]$ if we were to make the definition $\underline{\underline{M}}(T) = \sup_{\substack{\|\omega(x)\|\le 1\\ \omega\in\mathcal{D}^n(U)}} T(\omega)$, provided only that $\| \ \|$ is a norm for $\Lambda^n(\mathbb{R}^P)$ with the properties:

(1) $\langle \omega,\xi\rangle \le \|\omega\| \ |\xi|$ whenever $\xi \in \Lambda_n(\mathbb{R}^P)$ is *simple*

and

(2) for each fixed simple $\xi \in \Lambda_n(\mathbb{R}^P)$, equality holds in (1) for some $\omega \ne 0$.

(Evidently $\| \ \| = | \ |$ is one such norm.) Notice that the *smallest* possible norm for $\Lambda^n(\mathbb{R}^P)$ having these properties is defined by

$$\|\omega\| = \sup_{\substack{\xi\in\Lambda_n(\mathbb{R}^P), |\xi|=1\\ \xi \text{ simple}}} \langle \omega,\xi \rangle$$

($\| \ \|$ is called the co-mass norm for $\Lambda^n(\mathbb{R}^P)$.) There is a good argument to say that one should adopt this norm in the definition of $\underline{\underline{M}}(T)$ (and indeed this is usually done - see e.g. [FF], [FH1]) since, by virtue of the consequent *maximality* of $\underline{\underline{M}}(T)$ it is more likely to yield *equality* in the general inequality $\underline{\underline{M}}(T) \le \liminf \underline{\underline{M}}([\![M_j]\!])$, if $\{M_j\}$ is a sequence of C^1 submanifolds with weak limit T (see 26.12 below). Nevertheless we will here stick to the definition 26.4, because it has certain advantages (e.g. the application of the Riesz representation theorem - see below - is cleaner, and 26.4 does yield the "correct" value in the most important case when T is an integer multiplicity current as in §27.)

Notice that by the Riesz Representation Theorem 4.1 we have that if $T \in \mathcal{D}_n(U)$ satisfies $\underline{\underline{M}}_W(T) < \infty \quad \forall\, W \subset\subset U$, then there is a Radon measure μ_T on U and μ_T-measurable function $\vec{T}$ with values in $\Lambda_n(\mathbb{R}^P)$, $|\vec{T}| = 1$ μ_T-a.e., such that

26.7 $$T(\omega) = \int \langle \omega(x), \vec{T}(x) \rangle \, d\mu_T(x) \, .$$

μ_T (the total variation measure associated with T) is characterized by

26.8 $$\mu_T(W) = \sup_{\substack{\omega \in \mathcal{D}^n(U),\, |\omega| \le 1 \\ \operatorname{spt}\omega \subset W}} T(\omega) \quad (\equiv \underline{\underline{M}}_W(T))$$

for any open $W \subset U$. In particular

$$\mu_T(U) = \underline{\underline{M}}(T) \, .$$

Notice that for such a T we can define, for any μ_T-measurable subset A of U (and in particular for any Borel set $A \subset U$), a new current $T \llcorner A \in \mathcal{D}_n(U)$ by

26.9 $$(T \llcorner A)(\omega) = \int_A \langle \omega, \vec{T} \rangle \, d\mu_T \, .$$

More generally, if ϕ is any locally μ_T-integrable function on U then we can define $T \llcorner \phi \in \mathcal{D}_n(U)$ by

26.10 $$(T \llcorner \phi)(\omega) = \int \phi \langle \omega, \xi \rangle \, d\mu_T \, .$$

Given $T \in \mathcal{D}_n(U)$ we define the *support* spt T of T to be the relatively closed subset of U defined by

26.11 $$\operatorname{spt} T = U \sim \cup W$$

where the union is over all open sets W such that $T(\omega) = 0$ whenever $\omega \in \mathcal{D}^n(U)$ with $\operatorname{spt} \omega \subset W$. Notice that if $\underline{\underline{M}}_W(T) < \infty$ for each $W \subset\subset U$ and if μ_T is the corresponding total variation measure (as in 26.7, 26.8) then

$$\operatorname{spt} T = \operatorname{spt} \mu_T$$

where $\operatorname{spt} \mu_T$ is the support of μ_T in the usual sense of Radon measures in U.

Given a sequence $\{T_q\} \subset \mathcal{D}_n(U)$, we write $T_q \rightharpoonup T$ in U ($T \in \mathcal{D}_n(U)$) if $\{T_q\}$ converges weakly to T in the usual sense of distributions:

26.12 $$T_q \rightharpoonup T \iff \lim T_q(\omega) = T(\omega) \quad \forall \, \omega \in \mathcal{D}^n(U) \, .$$

Notice that mass is trivially lower semi-continuous with respect to weak convergence: if $T_q \rightharpoonup T$ in U then

26.13 $$\underline{\underline{M}}_W(T) \le \liminf_{q \to \infty} \underline{\underline{M}}_W(T_q) \qquad \forall \text{ open } W \subset U \, .$$

Notice also that by applying the standard Banach-Alaoglu theorem [Roy] (in the Banach spaces $\mathcal{M}_n(W) = \{T \in \mathcal{D}_n(W) : \underline{\underline{M}}_W(T) < \infty\}$, $W \subset\subset U$) we deduce

26.14 LEMMA *If* $\{T_q\} \subset \mathcal{D}_n(U)$ *and* $\sup_{q \ge 1} \underline{\underline{M}}_W(T_q) < \infty$ *for each* $W \subset\subset U$, *then there is a subsequence* $\{T_{q'}\}$ *and a* $T \in \mathcal{D}_n(U)$ *such that* $T_{q'} \rightharpoonup T$ *in* U.

The following terminology will be used frequently:

26.15 TERMINOLOGY Given $T_1 \in \mathcal{D}_n(U_1)$, $T_2 \in \mathcal{D}_n(U_2)$ and an open $W \subset U_1 \cap U_2$, we say $T_1 = T_2$ in W if $T_1(\omega) = T_2(\omega)$ whenever ω is a smooth n-form in $\mathbb{R}^{n+k}$ with $\operatorname{spt}\omega \subset W$.

Next we want to describe the cartesian product of currents $T_1 \in \mathcal{D}_r(U_1)$, $T_2 \in \mathcal{D}_s(U_2)$, $U_1 \subset \mathbb{R}^{P_1}$, $U_2 \subset \mathbb{R}^{P_2}$ open. We are motivated by the case when $T_1 = [\![M_1]\!]$ and $T_2 = [\![M_2]\!]$ (Cf. 26.2) where M_1, M_2 are oriented submanifolds of dimension r, s respectively. We want to define $T_1 \times T_2 \in \mathcal{D}_{r+s}(U_1 \times U_2)$ in such a way that for this special case (when $T_j = [\![M_j]\!]$) we get $[\![M_1]\!] \times [\![M_2]\!] = [\![M_1 \times M_2]\!]$. We are thus inevitably led to the following

26.16 DEFINITION If $\omega \in \mathcal{D}^{r+s}(U_1 \times U_2)$ is written in the form

$$\omega = \sum_{\substack{(\alpha,\beta)\in I_{r',P_1}\times I_{s',P_2}\\ r'+s'=r+s}} a_{\alpha\beta}(x,y)\,dx^\alpha \wedge dy^\beta \quad \text{(using multi-index notation as in §26)}$$

then we define

$$T_1 \times T_2(\omega) = T_1\left(\sum_{\alpha\in I_{r,P_1}} T_2\left(\sum_{\beta\in I_{s,P_2}} a_{\alpha\beta}(x,y)\,dy^\beta\right) dx^\alpha\right).$$

(Notice in particular this gives $T_1 \times T_2(\omega_1 \wedge \omega_2) = 0$ if $\omega_1 \in \mathcal{D}^{r'}(U_1)$, $\omega_2 \in \mathcal{D}^{s'}(U_2)$ with $r' + s' = r + s$ but $(r',s') \neq (r,s)$.)

One readily checks, using this definition and the definition of ∂ (in 26.3) that

26.17
$$\partial(T_1 \times T_2) = (\partial T_1) \times T_2 + (-1)^r T_1 \times \partial T_2 .$$

(Notice this is valid also in case r or $s = 0$ if we interpret the appropriate terms as zero; e.g. if $r = 0$ then $\partial(T_1 \times T_2) = T_1 \times \partial T_2$.)

An important special case of 26.17 occurs when we take $T \in \mathcal{D}_n(U)$, $U \subset \mathbb{R}^P$, and we let $[\![(0,1)]\!]$ be the 1-current defined as in 26.3 with $M = (0,1) \subset \mathbb{R}$ ($(0,1)$ having its usual orientation). Then 26.17 gives

26.18 $$\partial([\![(0,1)]\!]\times T) = (\{1\}-\{0\})\times T - [\![(0,1)]\!]\times\partial T$$

$$\equiv \{1\}\times T - \{0\}\times T - [\![(0,1)]\!]\times\partial T \ .$$

Here and subsequently $\{p\}$, for a point $p \in U$, means the 0-current $\in \mathcal{D}_0(U)$ defined by

26.19 $$\{p\}(\omega) = \omega(p) \ , \quad \omega \in \mathcal{D}^0(U) \quad (\equiv C_c^\infty(U)) \ .$$

Next we want to discuss the notion of "pushing forward" a current T via a smooth map $f : U \to V$, $U \subset \mathbb{R}^P$, $V \subset \mathbb{R}^Q$ open. The main restriction needed is that $f|\operatorname{spt} T$ is *proper*; that is $f^{-1}(K) \cap \operatorname{spt} T$ is a compact subset of U whenever K is a compact subset of V . Assuming this, we can define

26.20 $$f_\# T(\omega) = T(\zeta f^\# \omega) \qquad \forall\ \omega \in \mathcal{D}^n(V) \ ,$$

where ζ is any function $\in C_c^\infty(U)$ such that $\zeta \equiv 1$ in a neighbourhood of $\operatorname{spt} T \cap \operatorname{spt} f^\#\omega$. One easily checks that the definition of $f_\# T$ in 26.20 is independent of ζ . (Of course such ζ exist and $\zeta f^\#\omega \in \mathcal{D}^n(U)$ because $f|\operatorname{spt} T$ is proper and $\operatorname{spt}\omega$ is a compact subset of V .)

26.21 REMARKS

(1) Notice that $\partial f_\# T = f_\# \partial T$ whenever f, T are as in 26.20.

(2) If $\underline{\underline{M}}_W(T) < \infty$ for each $W \subset\subset U$, so that T has a representation as in 26.7, then it is straightforward to check that $f_\# T$ is given explicitly by

$$f_{\#}T(\omega) = \int < f^{\#}\omega, \vec{T} > d\mu_T$$

$$= \int < \omega(f(x)), df_{x\#}\vec{T}(x) > d\mu_T(x) \ . \quad *$$

Notice that we can thus make sense of $f_{\#}T$ in case f is merely C^1 (with $f|\operatorname{spt} T$ proper).

(3) If $T = [\![M]\!]$ as in 26.2, then the above remark (2) tells us that if $f|(\bar{M} \cap U)$ is proper,

$$(*) \qquad f_{\#}T(\omega) = \int_M < \omega(x), df_{x\#}\xi(x) > d\mathcal{H}^n(x) \ ,$$

where ξ is the orientation for M. Notice that this makes sense if f is only Lipschitz (by virtue of Rademacher's Theorem 5.2). If f is 1:1 and if Jf is the Jacobian of f as in 8.3, then the area formula evidently tells us that (since $df_{x\#}\xi(x) = Jf(x)\tau(f(x))$, where τ is the orientation for $f(M_+)$, $M_+ = \{x \in M : Jf(x) > 0\}$, induced by f)

$$f_{\#}T(\omega) = \int_{f(M_+)} < \omega(y), \tau(y) > d\mathcal{H}^n(y) \ .$$

(Which confirms that our definition of $f_{\#}T$ is "correct".)

By using the above notions we can derive the important homotopy formula for currents as follows:

If $f, g : U \to V$ are smooth $(V \subset \mathbb{R}^Q)$ and $h : [0,1] \times U \to V$ is smooth

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* For a linear map $\ell : \mathbb{R}^P \to \mathbb{R}^Q$ and for $v = \sum_{\alpha \in I_{n,P}} a_\alpha e_\alpha \in \Lambda_n(\mathbb{R}^P)$ we define $\ell_{\#}v \in \Lambda_n(\mathbb{R}^Q)$ by $\ell_{\#}v = \sum_{\alpha \in I_{n,P}} a_\alpha \ell_{\#} e_\alpha = \sum_{\alpha=(i_1,\ldots,i_n) \in I_{n,P}} a_\alpha \ell(e_{i_1}) \wedge \ldots \wedge \ell(e_{i_n})$.

Then $<w, \ell_{\#}v> = <\ell^{\#}w, v>$, $w \in \Lambda^n(\mathbb{R}^Q)$.

with $h(0,x) \equiv f(x)$, $h(1,x) \equiv g(x)$, if $T \in \mathcal{D}_n(U)$, and if $h \mid [0,1] \times \operatorname{spt} T$ is proper, then (by the above discussion) $h_\#([\![(0,1)]\!] \times T)$ is well defined $(\in \mathcal{D}_{n+1}(V))$ and

$$\begin{aligned}\partial h_\#([\![(0,1)]\!] \times T) &= h_\# \partial([\![(0,1)]\!] \times T)\\ &= h_\#(\{1\} \times T - \{0\} \times T - [\![(0,1)]\!] \times \partial T)\\ &\equiv g_\# T - f_\# T - h_\#([\![(0,1)]\!] \times \partial T) .\end{aligned}$$

Thus we obtain the *homotopy formula*

26.22 $$g_\# T - f_\# T = \partial h_\#([\![(0,1)]\!] \times T) + h_\#([\![(0,1)]\!] \times \partial T) .$$

Notice that an important case of the above is given by

(*) $$h(t,x) = tg(x) + (1-t)f(x) = f(x) + t(g(x) - f(x))$$

(i.e. h is an "affine homotopy" from f to g). In this case we note that by using the integral representation 26.7 and Remark 26.21(2) above that

26.23 $$\underline{\underline{M}}(h_\#[\![(0,1)]\!] \times T)) \le \sup_{\operatorname{spt} T} |f-g| \cdot \sup_{x \in \operatorname{spt} T}(|df_x| + |dg_x|)\underline{\underline{M}}(T) .$$

(Indeed $\overrightarrow{[\![(0,1)]\!] \times T} = e_1 \wedge \vec{T}$ and $\mu_{[\![(0,1)]\!] \times T} = \mathcal{L}^1 \times \mu_T$, so by Remark 26.21(2) we have

$$\begin{aligned}h_\#([\![(0,1)]\!] \times T)(\omega) &= \int \langle \omega(h(t,x)), df_{(t,x)\#}\, e_1 \wedge \vec{T}(x) \rangle \, d\mu_T(x)\, dt\\ &= \int \langle \omega(h(t,x)), (g(x)-f(x)) \wedge (t df_x + (1-t) df_x)_\# \vec{T}(x) \rangle\\ &\qquad d\mu_T(x)\, dt\end{aligned}$$

and 26.23 follows immediately.)

We now give a couple of important applications of the above homotopy formula.

26.24 LEMMA *If* $T \in \mathcal{D}_n(U)$, $\underline{\underline{M}}_W(T)$, $\underline{\underline{M}}_W(\partial T) < \infty$ $\forall$ $W \subset\subset U$ *and if* $f, g : U \to V$ *are* C^1 *with* $f \mid \operatorname{spt} T = g \mid \operatorname{spt} T$ *proper , then* $f_\# T = g_\# T$. (Note that $f_\# T$, $g_\# T$ are well-defined by 26.21(2).)

Proof By the homotopy formula 26.22 we have, with $h(t,x) = tg(x) + (1-t)f(x)$,

$$g_\# T(\omega) - f_\# T(\omega) = \partial h_\# ([\![(0,1)]\!] \times T)(\omega) + h_\# ([\![(0,1)]\!] \times \partial T)(\omega)$$

$$= h_\# ([\![(0,1)]\!] \times T)(d\omega) + h_\# ([\![(0,1)]\!] \times \partial T)(\omega) ,$$

so that, by 26.23,

$$|f_\# T(\omega) - g_\# T(\omega)| \le c(\underline{\underline{M}}(T)|d\omega| + \underline{\underline{M}}(\partial T)|\omega|)\sup_{x \in \operatorname{spt} T}|f-g|$$

$$= 0 , \quad \text{since } f = g \text{ on } \operatorname{spt} T .$$

The homotopy formula also enables us to define $f_\# T$ in case f is merely Lipschitz, provided $f|\operatorname{spt} T$ is proper and $\underline{\underline{M}}_W(T)$, $\underline{\underline{M}}_W(\partial T) < \infty$ $\forall$ $W \subset\subset U$. In the following lemma we let $f^{(\sigma)} = f * \phi_\sigma$, $\phi_\sigma(x) = \sigma^{-n}\phi(\sigma^{-1}x)$, with ϕ a mollifier as in §6.

26.25 LEMMA *If* $T \in \mathcal{D}_n(U)$, $\underline{\underline{M}}_W(T)$, $\underline{\underline{M}}_W(\partial T) < \infty$ $\forall$ $W \subset\subset U$, *and if* $f : U \to V$ *is Lipschitz with* $f \mid \operatorname{spt} T$ *proper, then* $\lim_{\sigma \downarrow 0} f^{(\sigma)}_\# T(\omega)$ *exists for each* $\omega \in \mathcal{D}^n(V)$; $f_\# T(\omega)$ *is defined to be this limit; then* $\operatorname{spt} f_\# T \subset f(\operatorname{spt} T)$ *and* $\underline{\underline{M}}_W(f_\# T) \le (\operatorname{ess\,sup}_{f^{-1}(W)} |Df|)^n \underline{\underline{M}}_{f^{-1}(W)}(T)$ $\forall$ $W \subset\subset V$.

Proof If σ, τ are sufficiently small (depending on ω) then the homotopy formula gives

$$f_{\sigma\#}T(\omega) - f_{\tau\#}T(\omega) = h_{\#}([\![(0,1)]\!] \times T)(d\omega) + h_{\#}([\![(0,1)]\!] \times \partial T)(\omega)$$

where $h : [0,1] \times U \to V$ is defined by $h(t,x) = t\, f_\sigma(x) + (1-t) f_\tau(x)$. Then by 26.23, for sufficiently small σ, τ , we have

$$|f_{\sigma\#}T(\omega) - f_{\tau\#}T(\omega)| \le c \sup_{f^{-1}(K)\cap \operatorname{spt} T} |f_\sigma - f_\tau| \cdot \operatorname{Lip} f \ ,$$

where K is a compact subset of V with $\operatorname{spt}\omega \subset \operatorname{interior}(K)$. Since $f_\sigma \to f$ uniformly on compact subsets of U , the result now clearly follows.

Next we want to define the notion of the *cone* over a given current $T \in \mathcal{D}_n(U)$. We want to define this in such a way that if $T = [\![M]\!]$ where M is a submanifold of $S^{P-1} \subset \mathbb{R}^P$ then the cone over T is just $[\![C_M]\!]$, $C_M = \{\lambda x : x \in M, 0<\lambda\le 1\}$. We are thus led generally to make the definition that the cone over T , denoted $0 \times\!\!\!\!\times T$, is defined by

26.26 $$0 \times\!\!\!\!\times T = h_{\#}([\![(0,1)]\!] \times T)$$

whenever $T \in \mathcal{D}_n(U)$ with U star-shaped relative to 0 and $\operatorname{spt} T$ compact, where $h : \mathbb{R} \times \mathbb{R}^P \to \mathbb{R}^P$ is defined by $h(t,x) = tx$. Thus $0 \times\!\!\!\!\times T \in \mathcal{D}_{n+1}(U)$ and (by the homotopy formula)

$$\partial\, 0 \times\!\!\!\!\times T = T - 0 \times\!\!\!\!\times \partial T \ .$$

The following *Constancy Theorem* is very useful:

26.27 THEOREM *If* U *is open in* $\mathbb{R}^n$ *(i.e.* $P = n$*), if* U *is connected, if* $T \in \mathcal{D}_n(U)$ *and* $\partial T = 0$ *, then there is a constant* c *such that* $T = c[\![U]\!]$ *(using the notation of* 26.2 *in the special case* $n = P$ *,* $M = U$ *;* U *is of course equipped with the standard orientation* $e_1 \wedge \ldots \wedge e_n$*) .*

Proof We are given

(1) $$T(d\omega) = 0 \quad \text{whenever} \quad \omega \in \mathcal{D}^{n-1}(U) \ .$$

Let $\phi_\sigma(x) = \sigma^{-n}\,\phi(\sigma^{-1}x)$, with ϕ a mollifier as in §6, and define T_σ by

$$T_\sigma(\omega) = T(\phi_\sigma * \omega)$$

if $\operatorname{dist}(\operatorname{spt}\omega, \partial U) > \sigma$. ($\phi_\sigma * \omega$ means $(\phi_\sigma * a)\, dx^1 \wedge \ldots \wedge dx^n$ if $\omega = a\, dx^1 \wedge \ldots \wedge dx^n$, $a \in C_c^\infty(U)$; since $P = n$, any $\omega \in \mathcal{D}^n(U)$ has this form.)

Now if $W \subset\subset U$ and $\sigma < \operatorname{dist}(W, \partial U)$, we claim there is a constant $c = c(T, W, \sigma)$ such that

$$|T_\sigma(\omega)| \leq c \int_U |\omega|\, d\mathcal{L}^n . \tag{2}$$

Indeed this follows directly from the fact that for fixed σ, W the set $S = \{\phi_\sigma * \omega : \omega \in \mathcal{D}^n(U) , \operatorname{spt}\omega \subset W , \int_U |\omega|\, d\mathcal{L}^n \leq 1\}$ is compact in $\mathcal{D}^n(U)$, relative to the norm $|\;|$. By the Riesz Representation Theorem 4.1, we see that (1) implies

$$T_\sigma(\omega) = \int a\, \theta_\sigma\, d\mathcal{L}^n , \quad \omega = a\, dx^1 \wedge \ldots \wedge dx^n , \tag{3}$$

$a \in C_c^\infty(W)$.

On the other hand if $\operatorname{spt}\omega \subset W$, $\omega \in \mathcal{D}^{n-1}(U)$, then

$$T_\sigma(d\omega) = T(\phi_\sigma * d\omega) = T(d\phi_\sigma * \omega) = \partial T(\phi_\sigma * \omega) = 0$$

by (1) . In particular, taking $\omega = a\, dx^1 \wedge \ldots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \ldots \wedge dx^n$, so that $d\omega = \pm \partial a/\partial x^j dx^1 \wedge \ldots \wedge dx^n$, and using (3) we have

$$\int D_j a\, \theta_\sigma\, d\mathcal{L}^n = 0 , \quad j = 1, \ldots, n ,$$

for $a \in C_c^\infty(U)$ with $\operatorname{spt} a \subset W$. This evidently implies that θ_σ = constant (depending on σ) on each component of W . The required result now follows from (3) by letting $\sigma \downarrow 0$ and $W \uparrow U$.

26.28 REMARK Notice that if we merely have $\underline{\underline{M}}(\partial T) < \infty$ then the obvious modifications of the above argument (note that (3) still holds) give first that

$$\left| \int D_j a \, \theta_\sigma \, dL^n \right| \le c \sup|a| \, \underline{\underline{M}}(\partial T)$$

with c independent of σ, for $a \in C_c^\infty(U)$ such that dist(spt a, ∂U) $> \sigma$. Thus (see §6 and in particular Theorem 6.3) we deduce that $\theta_{\sigma_k} \to \theta$ in $L^1_{loc}(U)$ (for some sequence $\sigma_k \downarrow 0$), with $\theta \in BV_{loc}(U)$, and (from (3))

$$(*) \qquad T\omega = \int a \, \theta \, dL^n , \quad \omega = a \, dx^1 \wedge \ldots \wedge dx^n \in \mathcal{D}^n(U) .$$

Using the definition of $\underline{\underline{M}}(\partial T)$, we easily then check that $\underline{\underline{M}}_W(\partial T) = |D\theta|(W)$ for each open $W \subset U$ (and $\underline{\underline{M}}_W(T) = \int_W |\theta| \, dL^n$). Indeed in the present case $n = P$, any $\omega \in \mathcal{D}^{n-1}(U)$ can be written $\omega = \sum_{j=1}^n (-1)^j a_j dx^1 \wedge \ldots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \ldots \wedge dx^n$ for suitable $a_j \in C_c^\infty(U)$, and $d\omega = \operatorname{div} \underline{a} \, dx^1 \wedge \ldots \wedge dx^n$ for such ω ($\underline{a} = (a_1, \ldots, a_n)$). Therefore by (*) above we have

$$\partial T(\omega) = T(d\omega) = \int \operatorname{div} \underline{a} \, \theta \, dL^n$$

and the assertion $\underline{\underline{M}}_W(\partial T) = |D\theta|(W)$ then follows directly from the definition of $\underline{\underline{M}}_W(\partial T)$ and $|D\theta|$ (in §6).

In the following lemma, for $\alpha = (i_1, \ldots, i_n) \in \mathbb{Z}^n$ with $1 \le i_1 < i_2 < \ldots < i_n \le P$, we let p_α denote the orthogonal projection of $\mathbb{R}^P$ onto $\mathbb{R}^n$ given by

$$(x^1, \ldots, x^P) \mapsto (x^{i_1}, \ldots, x^{i_n}) .$$

26.29 LEMMA *Suppose* E *is a closed subset of* U, U *open in* $\mathbb{R}^P$, *with* $L^n(p_\alpha(E)) = 0$ *for each multi-index* $\alpha = (i_1, \ldots, i_n)$, $1 \le i_1 < i_2 < \ldots < i_n \le P$. *Then* $T \llcorner E = 0$ *whenever* $T \in \mathcal{D}_n(U)$ *with* $\underline{\underline{M}}_W(T)$, $\underline{\underline{M}}_W(\partial T) < \infty$ *for every* $W \subset\subset U$.

26.30 REMARK The hypothesis $\mathcal{L}^n(p_\alpha(E)) = 0$ is trivially satisfied if $\mathcal{H}^n(E) = 0$, so in particular we deduce $T \llcorner E = 0$ if $T \in \mathcal{D}_n(U)$ with $\underline{\underline{M}}_W(T)$, $\underline{\underline{M}}_W(\partial T) < \infty$ $\forall\, W \subset\subset U$ and $\mathcal{H}^n(E) = 0$.

Proof of 26.29 Let $\omega \in \mathcal{D}^n(U)$. Then we can write $\omega = \sum_{\alpha \in I_{n,P}} \omega_\alpha\, dx^\alpha$, $\omega_\alpha \in C_c^\infty(U)$, so that

$$T(\omega) = \sum_\alpha T(\omega_\alpha dx^\alpha) = \sum_\alpha (T \llcorner \omega_\alpha)(dx^\alpha)$$

$$= \sum_\alpha (T \llcorner \omega_\alpha) p_\alpha^\# \, dy$$

($dy = dy^1 \wedge \ldots \wedge dy^n$, $y^1, \ldots, y^n$ the standard coordinate functions in $\mathbb{R}^n$.)

Thus

$$(1) \qquad T(\omega) = \sum_\alpha p_{\alpha\#}(T \llcorner \omega_\alpha)(dy)$$

(which makes sense because $\operatorname{spt} T \llcorner \omega_\alpha \subset \operatorname{spt} \omega_\alpha$ = compact subset of U).

On the other hand

$$\underline{\underline{M}}(\partial p_{\alpha\#}(T \llcorner \omega_\alpha)) = \underline{\underline{M}}(p_{\alpha\#}\partial(T \llcorner \omega_\alpha))$$

$$\leq \underline{\underline{M}}(\partial(T \llcorner \omega_\alpha)) < \infty,$$

(because for any $\tau \in \mathcal{D}^{n-1}(U)$,

$$\partial(T \llcorner \omega_\alpha)(\tau) = (T \llcorner \omega_\alpha)(d\tau)$$

$$= T(\omega_\alpha d\tau)$$

$$= T(d(\omega_\alpha \tau)) - T(d\omega_\alpha \wedge \tau)$$

$$= \partial T(\omega_\alpha \tau) - T(d\omega_\alpha \wedge \tau);$$

thus in fact

$$\underline{\underline{M}}_W(\partial(T \llcorner \omega_\alpha)) \le \underline{\underline{M}}_W(\partial T)\,|\omega_\alpha|$$

$$+ \underline{\underline{M}}_W(T)\,|d\omega_\alpha| \ .)$$

Therefore by Remark 26.28 we have $\theta_\alpha \in BV(p_\alpha(U))$ such that $p_{\alpha\#}(T \llcorner \omega_\alpha)(\tau) = \int_{p_\alpha(U)} < \tau, e_1 \wedge \ldots \wedge e_n > \theta_\alpha dL^n$, and hence $p_{\alpha\#}(T \llcorner \omega_\alpha) \llcorner p_\alpha(E) = 0$ because $L^n(p_\alpha(E)) = 0$. Then, assuming without loss of generality that E is closed,

$$\begin{aligned}
(2) \qquad \underline{\underline{M}}(p_{\alpha\#}(T \llcorner \omega_\alpha)) &\le \underline{\underline{M}}(p_{\alpha\#}(T \llcorner \omega_\alpha) \llcorner (\mathbb{R}^n \sim p_\alpha(E))) \\
&= \underline{\underline{M}}(p_{\alpha\#}((T \llcorner \omega_\alpha) \llcorner (\mathbb{R}^P \sim p_\alpha^{-1} p_\alpha(E)))) \\
&\le \underline{\underline{M}}((T \llcorner \omega_\alpha) \llcorner (\mathbb{R}^P \sim p_\alpha^{-1} p_\alpha E)) \\
&\le \underline{\underline{M}}_W(T \llcorner (\mathbb{R}^P \sim p_\alpha^{-1} p_\alpha E)) \cdot |\omega_\alpha| \\
&\le \underline{\underline{M}}_W(T \llcorner (\mathbb{R}^P \sim E)) \cdot |\omega_\alpha|
\end{aligned}$$

for any W such that $\operatorname{spt} \omega \subset W \subset U$.

Combining (1) and (2) we then have

$$\underline{\underline{M}}_W(T) \le c\, \underline{\underline{M}}_W(T \llcorner (\mathbb{R}^P \sim E))$$

so that in particular

$$(3) \qquad \underline{\underline{M}}_W(T \llcorner E) \le c\, \underline{\underline{M}}_W(T \llcorner (\mathbb{R}^P \sim E)) \ .$$

Letting K be an arbitrary compact subset of E, we can choose $\{W_q\}$ so that $W_q \subset\subset U$, $W_{q+1} \subset W_q$, $\bigcap_{q=1}^{\infty} W_q = K$; using (3) with $W = W_q$ then gives $\underline{\underline{M}}(T \llcorner K) = 0$. Thus $\underline{\underline{M}}(T \llcorner E) = 0$ as required.

§27. INTEGER MULTIPLICITY RECTIFIABLE CURRENTS

In this section we want to develop the theory of integer multiplicity currents $T \in \mathcal{D}_n(U)$, which, roughly speaking are those currents obtained by assigning (in a $\mathcal{H}^n$-measurable fashion) an orientation to the tangent spaces T_xV of an integer multiplicity varifold V . (See Chapter 4 for terminology.)

These currents are precisely those called locally rectifiable by Federer and Fleming [FF], [FH1].

Throughout this section $n \geq 1$, $k \geq 1$ are integers and U is an open subset of $\mathbb{R}^{n+k}$.

27.1 DEFINITION If $T \in \mathcal{D}_n(U)$ we say that T is an integer multiplicity rectifiable n-current (briefly an integer multiplicity current) if it can be expressed

$$(*) \qquad T(\omega) = \int_M < \omega(x), \xi(x) > \theta(x)\, d\mathcal{H}^n(x) \ , \quad \omega \in \mathcal{D}^n(U) \ ,$$

where M is an $\mathcal{H}^n$-measurable countably n-rectifiable subset of U , θ is a locally $\mathcal{H}^n$-integrable positive integer-valued function, and $\xi : M \to \Lambda_n(\mathbb{R}^{n+k})$ is a $\mathcal{H}^n$-measurable function such that for $\mathcal{H}^n$ - a.e. point $x \in M$, $\xi(x)$ can be expressed in the form $\tau_1 \wedge \ldots \wedge \tau_n$, where $\tau_1, \ldots, \tau_n$ form an orthonormal basis for the approximate tangent space T_xM . (See Chapter 3,4.) Thus $\xi (= \vec{T})$ orients the approximate tangent spaces of M in an $\mathcal{H}^n$-measurable way. The function θ in (*) is called the *multiplicity* and ξ is called the orientation for T . If T is as in (*) we shall often write $T = \underline{\underline{\tau}}(M,\theta,\xi)$. Notice that there is associated with any such T the integer multiplicity varifold $V = \underline{\underline{v}}(M,\theta)$ in U .

27.2 REMARKS

(1) If $T_1, T_2 \in \mathcal{D}_n(U)$ are integer multiplicity, then so is $p_1T_1 + p_2T_2$, $p_1, p_2 \in \mathbb{Z}$.

(2) If $T_1 = \underline{\underline{\tau}}(M_1,\theta_1,\xi_1) \in \mathcal{D}_r(U)$, $T_2 = \underline{\underline{\tau}}(M_2,\theta_2,\xi_2) \in \mathcal{D}_s(V)$ ($V \subset \mathbb{R}^Q$ open), then $T_1 \times T_2 \in \mathcal{D}_{r+s}(U\times V)$ is also integer multiplicity, and in fact

$$T_1 \times T_2 = \underline{\underline{\tau}}(M_1 \times M_2,\ \theta_1\theta_2,\ \xi_1 \wedge \xi_2) .$$

(3) If $f : U \to V$ is Lipschitz, $T = \underline{\underline{\tau}}(M,\theta,\xi) \in \mathcal{D}_n(U)$ $(M \subset U)$ and $f \mid \operatorname{spt} T$ is proper, then we can define $f_\# T \in \mathcal{D}_n(V)$ by

$$(*) \qquad f_\# T(\omega) = \int_M \langle \omega(f(x)),\ d^M f_{x\#}\xi(x) \rangle\, \theta(x)\ d\mathcal{H}^n(x) .$$

Since $|d^M f_{x\#}\, \xi(x)| = J^M f(x)$ (as in §12) by the area formula this can be written

$$(**) \qquad f_\# T(\omega) = \int_{f(M)} \Big\langle \omega(y)\ ,\ \sum_{x\in f^{-1}(y)\cap M_+} \theta(x)\, \frac{d^M f_{x\#}\xi(x)}{|d^M f_{x\#}\xi(x)|} \Big\rangle\, d\mathcal{H}^n(y) ,$$

where $M_+ = \{x \in M : J_M f(x) > 0\}$. Furthermore at points y where the approximate tangent space $T_y(f(M))$ exists (which is $\mathcal{H}^n$ - a.e. y by virtue of the fact that $f(M)$ is countably n-rectifiable) and where T_xM, $d^M f_x$ exist $\forall\ x \in f^{-1}(y)$ (which is again for $\mathcal{H}^n$ - a.e. y because T_xM, $d^M f_x$ exist for $\mathcal{H}^n$ - a.e. $x \in M_+$), we have

$$(***) \qquad \frac{d^M f_{x\#}\xi(x)}{|d^M f_{x\#}\xi(x)|} = \pm\, \tau_1 \wedge \ldots \wedge \tau_n ,$$

where $\tau_1,\ldots,\tau_n$ is an orthonormal basis for $T_y(f(M))$. Hence (**) gives

$$f_\# T(\omega) = \int_{f(M)} \langle \omega(y),\ \eta(y) \rangle N(y)\ d\mathcal{H}^n(y)$$

where $\eta(y)$ is a suitable orientation for the approximate tangent space $T_y(f(M))$ and $N(y)$ is a non-negative integer. N, η in fact satisfy

$$\sum_{x\in f^{-1}(y)\cap M_+} \theta(x) \frac{d^M f_{x\#}\xi(x)}{|d^M f_{x\#}\xi(x)|} = N(y)\ \eta(y) \ ,$$

so that for $\mathcal{H}^n$-a.e. $y \in f(M)$ we have

$$N(y) \le \sum_{x\in f^{-1}(y)\cap M_+} \theta(x)$$

and

$$N(y) \equiv \sum_{x\in f^{-1}(y)\cap M_+} \theta(x) \pmod 2 \ .$$

Notice that, in case f is C^1, $f_\# T$ agrees with the previous definition in 26.20 (see also 26.21(2)). Notice also that if $f : U \to W$ is Lipschitz and if $V = \underline{\underline{v}}(M,\theta)$ is the varifold associated with $T = \underline{\underline{\tau}}(M,\theta,\xi)$, then

$$\mu_{f_\# T} \le \mu_{f_\# V}$$

(in the sense of measures) with equality if and only if, for $\mathcal{H}^n$-a.e. $y \in f(M)$, the sign in (***) above remains constant as x varies over $f^{-1}(y) \cap M_+$. In particular we have $\mu_{f_\# T} = \mu_{f_\# V}$ in case f is 1:1 .

A fact of central importance concerning integer multiplicity currents is the following compactness theorem, first proved by Federer and Fleming [FF].

27.3 THEOREM *If* $\{T_j\} \subset \mathcal{D}_n(U)$ *is a sequence of integer multiplicity currents with*

$$\sup_{j\geq 1}(\underline{\underline{M}}_W(T_j) + \underline{\underline{M}}_W(\partial T_j)) < \infty \quad \forall\, W \subset\subset U ,$$

then there is an integer multiplicity $T \in \mathcal{D}_n(U)$ *and a subsequence* $\{T_{j'}\}$ *such that* $T_{j'} \rightharpoonup T$ *in* U .

We shall give the proof of this in Chapter 8. Notice that the existence of a $T \in \mathcal{D}_n(U)$ and a subsequence $\{T_{j'}\}$ with $T_{j'} \rightharpoonup T$ is a consequence of the elementary lemma 26.14; only the fact that T is an integer multiplicity current is non-trivial.

27.4 REMARK Note that the proof of 27.3 in the codimension 1 case (when $P = n+1$) is a direct consequence of the Remark 26.28 and the compactness theorem 6.3 for BV functions.

In contrast to the difficulty in proving 27.3, it is quite straightforward to prove that if T_j converges to T in the strong sense that $\lim \underline{\underline{M}}_W(T_j - T) = 0 \quad \forall\, W \subset\subset U$, and if T_j are integer multiplicity $\forall j$, then T is integer multiplicity. Indeed we have the following lemma.

27.5 LEMMA *The set of integer multiplicity currents in* $\mathcal{D}_n(U)$ *is complete with respect to the topology given by the family* $\{\underline{\underline{M}}_W\}_{W\subset\subset U}$ *of semi-norms.*

Proof Let $\{T_Q\}$ be a sequence of integer multiplicity currents in $\mathcal{D}_n(U)$, and $\{T_Q\}$ is Cauchy with respect to the semi-norms $\underline{\underline{M}}_W$, $W \subset\subset U$. Suppose $T_Q = \underline{\underline{\tau}}(M_Q, \theta_Q, \xi_Q)$ (θ_Q positive integer-valued on M_Q , M_Q countably n-rectifiable, $\mathcal{H}^n(M_Q \cap W) < \infty$ for each $W \subset\subset U$.) Then

(1) $$\underline{\underline{M}}_W(T_Q - T_P) \equiv \int_W |\theta_P\xi_P - \theta_Q\xi_Q|\, d\mathcal{H}^n < \varepsilon_W(Q)$$

$\forall\ P \geq Q$, where $\varepsilon_W(Q) \downarrow 0$ as $Q \to \infty$ and where we adopt the convention $\xi_P = 0$, $\theta_P = 0$ on $U \sim M_P$. In particular, since $|\xi_P| = 1$ on M_P, we get

(2) $$\int_W |\theta_P - \theta_Q|\, d\mathcal{H}^n < \varepsilon_W(Q) \qquad \forall\ P \geq Q\ ,$$

and hence θ_P converges in $L^1(\mathcal{H}^n)$ locally in U to an integer-valued function θ. Of course (2) implies

(3) $$\mathcal{H}^n(((M_+ \sim M_Q) \cup (M_Q \sim M_+)) \cap W) \leq \varepsilon_W(Q)\ ,$$

where $M_+ = \{x \in U : \theta(x) > 0\}$. (1), (2) also imply

(4) $$\int_W \theta_P |\xi_P - \xi_Q|\, d\mathcal{H}^n \leq 2\varepsilon_W(Q) \qquad \forall\ P \geq Q\ ,$$

and hence by (3) ξ_P converges in $L^1(\mathcal{H}^n)$ locally in U to a function ξ with values in $\Lambda_n(\mathbb{R}^{n+k})$ with $|\xi| = 1$ and ξ simple on M_+.

Now $\xi_Q(x) \in \Lambda_n(T_x M_Q)$, $\mathcal{H}^n$-a.e. $x \in M_Q$, and (by (3)) $T_x M_+ = T_x M_Q$ except for a set of measure $\leq \varepsilon_W(Q)$ in $M_+ \cap W$. It follows that $\xi(x) \in \Lambda_n(T_x M_+)$ for $\mathcal{H}^n$-a.e. $x \in M_+$ and we have shown that $\underline{\underline{M}}_W(T_P - T) \to 0$, where $T = \underline{\underline{\tau}}(M_+, \theta, \xi)$ is an integer n-current in U.

Finally, we shall need the following useful *decomposition theorem* for codimension 1 integer multiplicity currents.

27.6 THEOREM *Suppose* $P = n+1$ *(i.e.* U *is open in* $\mathbb{R}^{n+1}$*) and* R *is an integer multiplicity current in* $\mathcal{D}_{n+1}(U)$ *with* $\underline{\underline{M}}_W(\partial R) < \infty \quad \forall\, W \subset\subset U$. *Then* $T = \partial R$ *is integer multiplicity, and in fact we can find a decreasing sequence of* $\mathcal{L}^{n+1}$*-measurable sets* $\{U_j\}_{j=-\infty}^{\infty}$ *of locally finite perimeter in* U *such that (in* U*)*

$$R = \sum_{j=1}^{\infty} [\![U_j]\!] - \sum_{j=-\infty}^{0} [\![V_j]\!] \,, \quad V_j = U \sim U_j \,, \; j \le 0 \,,$$

$$T = \sum_{j=-\infty}^{\infty} \partial[\![U_j]\!] \,,$$

and

$$\mu_T = \sum_{j=-\infty}^{\infty} \mu_{\partial[\![U_j]\!]} \,;$$

in particular

$$\underline{\underline{M}}_W(T) = \sum_{j=-\infty}^{\infty} \underline{\underline{M}}_W(\partial[\![U_j]\!]) \qquad \forall\, W \subset\subset U \,.$$

27.7 REMARK Let $* : C_c^{\infty}(U;\mathbb{R}^{n+1}) \to \mathcal{D}^n(U)$ be defined by $*g = \sum_{j=1}^{n+1} (-1)^{j-1} g_j dx^1 \wedge \ldots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \ldots \wedge dx^{n+1}$, so that $d*g = \operatorname{div} g \, dx^1 \wedge \ldots \wedge dx^{n+1}$. Then for any $\mathcal{L}^{n+1}$-measurable $A \subset U$ we have

$$\begin{aligned} \partial[\![A]\!]\,(*g) &= [\![A]\!]\,(d*g) \\ &= \int_U \chi_A \operatorname{div} g \; d\,{}^{n+1} \,, \end{aligned}$$

and hence by definition of $|D\chi_A|$ (in §6) and $\underline{\underline{M}}(T)$ (in §26) we see that

(*) A has locally finite perimeter in $U \Leftrightarrow \underline{\underline{M}}_W(\partial[\![A]\!]) < \infty \qquad \forall\, W \subset\subset U$,

and in this case

(**)
$$\begin{cases} \underline{\underline{M}}_W(\partial[\![A]\!]) = \int_W |D\chi_A| & \forall\ W \subset\subset U \\ \partial\overrightarrow{[\![A]\!]} = *\nu_A\ ,\ |D\chi_A|\ \text{a.e. in } U . \end{cases}$$

Here ν_A is the inward unit normal function for A (defined on the reduced boundary $\partial^* A$ as in 14.3).

Proof of 27.6 R must have the form

$$R = \underline{\underline{\tau}}(V,\theta,\xi) ,$$

where V is an $\mathcal{L}^{n+1}$-measurable subset of U and $\xi(x) = \pm e_1 \wedge \ldots \wedge e_{n+1}$ for each $x \in V$. Thus letting

$$\tilde{\theta}(x) = \begin{cases} \theta(x) & \text{when } x \in V \text{ and } \xi(x) = +e_1 \wedge \ldots \wedge e_{n+1} \\ -\theta(x) & \text{when } x \in V \text{ and } \xi(x) = -e_1 \wedge \ldots \wedge e_{n+1} \\ 0 & \text{when } x \notin V , \end{cases}$$

we have

(1)
$$R(\omega) = \int_V a\tilde{\theta}\ d\mathcal{L}^{n+1} ,$$

$\omega = a\ dx^1 \wedge \ldots \wedge dx^{n+1} \in \mathcal{D}^{n+1}(U)$ and (cf. 26.28)

(2)
$$\underline{\underline{M}}_W(R) = \int_W |\tilde{\theta}|\ d\mathcal{L}^{n+1}\ ,\ \underline{\underline{M}}_W(T) = \int_W |D\tilde{\theta}| \qquad \forall\ W \subset\subset U$$

(and $\tilde{\theta} \in BV_{loc}(U)$) .

Define

$$U_j = \{x \in U : \tilde{\theta}(x) \geq j\}\ ,\ j \in \mathbb{Z}$$
$$V_j = \{x \in U : \tilde{\theta}(x) \leq -1-j\}\ ,\ j \geq 0$$
$$(\equiv\ U \sim U_{-j}) .$$

Then one checks directly that

$$\tilde{\theta} = \sum_{j=1}^{\infty} \chi_{U_j} - \sum_{j=0}^{\infty} \chi_{V_j} \tag{3}$$

(χ_A = characteristic function of A , $A \subset U$) , and hence by (1)

$$R = \sum_{j=1}^{\infty} [\![U_j]\!] - \sum_{j=0}^{\infty} [\![V_j]\!] \quad \text{in} \quad U . \tag{4}$$

Since $T(\omega) = \partial R(\omega) = R(d\omega)$, $\omega \in \mathcal{D}^n(U)$, we then have

$$\begin{aligned} T = \partial R &= \sum_{j=1}^{\infty} \partial[\![U_j]\!] - \sum_{j=0}^{\infty} \partial[\![V_j]\!] \\ &= \sum_{j=-\infty}^{\infty} \partial[\![U_j]\!] , \end{aligned} \tag{5}$$

so we have the required decomposition, and it remains only to prove that each U_j has locally finite perimeter in U and that the corresponding measures add.

To check this, take $\psi_j \in C^1(\mathbb{R})$ with

$$\begin{cases} \psi_j(t) = 0 \quad \text{for} \quad t \le j-1+\varepsilon , \quad \psi_j(t) = 1 , \quad t \ge j-\varepsilon \\ 0 \le \psi_j \le 1 , \qquad \sup|\psi_j'| \le 1+3\varepsilon , \end{cases}$$

where $\varepsilon \in (0,\frac{1}{2})$. Then if $a \in C_c^{\infty}(U)$ and $g = (g^1,\dots,g^{n+1})$, $g^j \in C_c^{\infty}(U)$, with $|g| \le a$, we have (since $\chi_{U_j} = \psi_j \circ \tilde{\theta}\ \forall j$) that for any $M \le N$

$$\begin{aligned} \int_U \operatorname{div} g \sum_{j=M}^{N} \chi_{U_j} \, d\mathcal{L}^{n+1} &= \int_U \operatorname{div} g \sum_{j=M}^{N} \psi_j \circ \tilde{\theta} \, d\mathcal{L}^{n+1} \\ &= \lim_{\sigma \downarrow 0} \int_U \operatorname{div} g \sum_{j=M}^{N} \psi_j \circ \tilde{\theta}^{(\sigma)} d\mathcal{L}^{n+1} \\ &= -\lim_{\sigma \downarrow 0} \int_U g \cdot \operatorname{grad}\tilde{\theta}^{(\sigma)} \psi_j'(\tilde{\theta}^{(\sigma)}) \, d\mathcal{L}^{n+1} \\ &\le (1+3\varepsilon) \lim_{\sigma \downarrow 0} \int_U a|\operatorname{grad}\tilde{\theta}^{(\sigma)}| \, d\mathcal{L}^{n+1} \\ &= (1+3\varepsilon) \int_U a|D\tilde{\theta}| = (1+3\varepsilon) \int_U a \, d\mu_T \end{aligned} \tag{6}$$

by Lemma 6.2 and (2). (Here $\tilde{\theta}^{(\sigma)}$ are the mollified functions corresponding to $\tilde{\theta}$ as in 6.2.)

Then, taking $M = N$, we deduce that each U_j has locally finite perimeter in U. On the other hand taking $M = -N$ and defining $R_N = \sum_{j=1}^{N} [\![U_j]\!] - \sum_{j=0}^{N} [\![V_j]\!]$ we see that (with g as in 27.7) (6) implies

$$|R_N(d{*}g)| \le (1+3\varepsilon)\int_U a\,d\mu_T ,$$

and hence, with $T_N = \partial R_N$,

$$\text{(7)} \qquad \int_U a\,d\mu_{T_N} \le \int_U a\,d\mu_T \qquad \forall\ N \ge 1 ,$$

$a \ge 0$, $a \in C_c^\infty(U)$. On the other hand by 14.1 we have

$$\text{(8)} \qquad R_N(d{*}g) = \sum_{j=-N}^{N} \int_U \operatorname{div} g\, \chi_{U_j}\, dL^{n+1}$$

$$= -\sum_{j=-N}^{N} \int_{\partial^* U_j} \nu_j \cdot g\, dH^n ,$$

where ν_j is the inward unit normal for U_j and $\partial^* U_j$ is the reduced boundary for U_j (see §14 and in particular Lemma 14.3). By virtue of the fact that $U_{j+1} \subset U_j$ we see from 14.3(2) that $\nu_j \equiv \nu_k$ on $\partial^* U_j \cap \partial^* U_k$ $\forall\ j,k$. Hence (8) can be written

$$T_N({*}g) = -\int_U \nu \cdot g\, h_N\, dH^n ,$$

where $h_N = \sum_{j=-N}^{N} \chi_{\partial^* U_j}$ and where ν is defined on $\bigcup_{j=-\infty}^{\infty} \partial^* U_j$ by $\nu = \nu_j$ on $\partial^* U_j$. Since $|\nu| \equiv 1$ on $\bigcup_{j=-\infty}^{\infty} \partial^* U_j$ this evidently gives

$$\begin{aligned}\int a\, d\mu_{T_N} &= \int a\, h_N\, dH^n \\ &= \sum_{j=-N}^{N} \int_{\partial^* U_j} a\, dH^n \\ &= \sum_{j=-N}^{N} \int a\, d\mu_{\partial [\![U_j]\!]} .\end{aligned}$$

Letting $N \to \infty$ we thus have (by (7))

$$\mu_T \geq \sum_{j=-\infty}^{\infty} \mu_{\partial [\![U_j]\!]} .$$

Since the reverse inequality follows directly from (5), the proof is complete.

27.8 COROLLARY *Let* R *be integer multiplicity* $\in \mathcal{D}_{n+1}(U)$, $U \subset \mathbb{R}^P$, $P \geq n+1$, *and suppose there is an* $(n+1)$*-dimensional* C^1 *submanifold* N *of* $\mathbb{R}^P$ *with* $\operatorname{spt} R \subset N \cap U$. *Suppose further that* $T = \partial R$ *and* $\underset{=}{M}_W(T) < \infty$ $\forall\, W \subset\subset U$. *Then* T $(\in \mathcal{D}_n(U))$ *is integer multiplicity and for each point* $y \in N \cap U$ *there is* $W_y \subset\subset U$, $y \in W_y$, *and* $\mathcal{H}^{n+1}$ *measurable subsets* $\{U_j\}_{j=-\infty}^{\infty}$ *with* $U_{j+1} \subset U_j \subset N \cap U$, $\underset{=}{M}_{W_y}(\partial [\![U_j]\!]) < \infty$ $\forall j$, *and with the following identities holding in* W_y :

$$R = \sum_{j=1}^{\infty} [\![U_j]\!] - \sum_{j=0}^{\infty} [\![U \sim U_{-j}]\!]$$

$$T = \sum_{j=-\infty}^{\infty} \partial [\![U_j]\!]$$

$$\mu_T = \sum_{j=-\infty}^{\infty} \mu_{\partial [\![U_j]\!]} .$$

Proof The proof is an easy consequence of 27.6 using local coordinate representations for N .

§28. SLICING

We first want to define the notion of slice for integer multiplicity currents. Preparatory to this we have the following lemma:

28.1 LEMMA *If* M *is* $\mathcal{H}^n$*-measurable, countably* n*-rectifiable,* f *is Lipschitz on* $\mathbb{R}^{n+k}$ *and* $M_+ = \{x \in M : |\nabla^M f(x)| > 0\}$ *, then for* L^1*-almost all* $t \in \mathbb{R}$ *the following statements hold:*

(1) $M_t \equiv f^{-1}(t) \cap M_+$ *is countably* $\mathcal{H}^{n-1}$*-rectifiable*

(2) *For* $\mathcal{H}^{n-1}$-a.e. $x \in M_t$, $T_x M_t$ *and* $T_x M$ *both exist,* $T_x M_t$ *is an* (n-1)*-dimensional subspace of* $T_x M$ *, and in fact*

$$(*) \qquad T_x M = \{y + \lambda \nabla^M f(x) \;:\; y \in T_x M_t,\ \lambda \in \mathbb{R}\}.$$

Furthermore for any non-negative $\mathcal{H}^n$*-measurable function* g *on* M *we have*

$$\int_{-\infty}^{\infty} \Big(\int_{M_t} g \, d\mathcal{H}^{n-1} \Big) dt = \int_M |\nabla^M f| \, g \, d\mathcal{H}^n .$$

Proof In fact (1) is just a restatement of Remark 12.8(2), and (2) follows from 11.6 together with the facts that for L^1-a.e. $t \in \mathbb{R}$ and $\mathcal{H}^{n-1}$-a.e. $x \in M_t$

(1) $\nabla^M f(x) \in T_x M$ (by definition of $\nabla^M f$ in §12)

and

(2) $\langle \nabla^M f(x), \tau \rangle = 0 \qquad \forall\, \tau \in T_x M_t$.

(This last follows for example from the definition 12.1 of $\nabla^M f(x)$.)

The last part of the lemma is just a restatement of the appropriate version of the co-area formula (discussed in §12).

28.2 REMARK Note that by replacing g (in 28.1 above) by $g \times$ characteristic function of $\{x : f(x) < t\}$ we get the identity

$$\int_{M\cap\{f(x)<t\}} |\nabla^M f|\, g \; d\mathcal{H}^n = \int_{-\infty}^{t} \int_{M_s} g \; d\mathcal{H}^{n-1} ds$$

so that the left side as an absolutely continuous function of t and

$$\frac{d}{dt} \int_{M\cap\{f(x)<t\}} |\nabla^M f|\, g \; d\mathcal{H}^n = \int_{M_t} g \; d\mathcal{H}^{n-1} \;, \quad \text{a.e. } t \in \mathbb{R} \,.$$

Now let $T = \underline{\underline{\tau}}(M,\theta,\xi)$ be an integer multiplicity current in U (U open in $\mathbb{R}^{n+k}$, $M \subset U$), let f be Lipschitz in U and let θ_+ be defined $\mathcal{H}^n$-a.e. in M by

$$\theta_+(x) = \begin{cases} 0 & \text{if} \quad \nabla^M f(x) = 0 \\ \theta(x) & \text{if} \quad \nabla^M f(x) \neq 0 \,. \end{cases}$$

For the ($\mathcal{L}^1$-almost all) $t \in \mathbb{R}$ such that $T_x M$, $T_x M_t$ exist for $\mathcal{H}^{n-1}$-a.e. $x \in M_t$ and such that (*) of 28.1 holds, we have

28.3 $\quad \xi(x) \llcorner \nabla^M f(x)/|\nabla^M f(x)|$ *is simple* $\in \Lambda_{n-1}(T_x M_t) \subset \Lambda_{n-1}(T_x M)$

and has unit length (for $\mathcal{H}^{n-1}$-a.e. $x \in M_t$) . Here we use the notation that if $v \in \Lambda_n(T_x M)$ and $w \in T_x M$, then $v \llcorner w \in \Lambda_{n-1}(T_x M)$ is defined by

$$\langle v \llcorner w, a\rangle = \langle v, w \wedge a\rangle \;, \; a \in \Lambda_{n-1}(T_x M) \;.$$

Using this notation we can now define the notion of a slice of T by f ; we continue to assume $T \in \mathcal{D}_n(U)$ is given by $T = \underline{\underline{\tau}}(M,\theta,\xi)$ as above.

28.4 DEFINITION For the ($\mathcal{L}^1$-almost all) $t \in \mathbb{R}$ since that $T_x M$, $T_x M_t$ exist and 28.1(*) holds $\mathcal{H}^{n-1}$-a.e. $x \in M_t$, with the notation introduced above (and bearing in mind 28.3) we define the integer multiplicity current $\langle T,f,t\rangle \in \mathcal{D}_{n-1}(U)$ by

$$\langle T,f,t\rangle = \underline{\underline{\tau}}(M_t,\theta_t,\xi_t) \;,$$

where

$$\xi_t(x) = \xi(x) \llcorner \nabla^M f(x)/|\nabla^M f(x)| \ , \quad \theta_t = \theta_+|M_t \ .$$

So defined, $\langle T,f,t\rangle$ is called *the slice of* T *by* f *at* t .

The main facts concerning the slices $\langle T,f,t\rangle$ are given in the following lemma:

28.5 LEMMA

(1) *For each open* $W \subset U$

$$\int_{-\infty}^{\infty} \underline{\underline{M}}_W(\langle T,f,t\rangle)\,dt = \int_{M\cap W} |\nabla^M f|\theta \, d\mathcal{H}^n \le (\text{ess sup}_{M\cap W}|\nabla^M f|)\underline{\underline{M}}_W(T) \ .$$

(2) *If* $\underline{\underline{M}}_W(\partial T) < \infty \quad \forall\, W \subset\subset U$, *then for* $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$

$$\langle T,f,t\rangle = \partial[T \llcorner \{f<t\}] - (\partial T) \llcorner \{f<t\} \ .$$

(3) *If* ∂T *is integer multiplicity in* $\mathcal{D}_{n-1}(U)$, *then for* $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$

$$\langle \partial T,f,t\rangle = -\partial\langle T,f,t\rangle \ .$$

Proof (1) is a direct consequence of the last part of Lemma 28.1 (with $g = \theta_+$).

To prove (2) we first recall that, since M is countably n-rectifiable, we can write (see Remark 11.7)

$$M = \bigcup_{j=0}^{\infty} M_j \ , \tag{1}$$

where $M_i \cap M_j = \emptyset \quad \forall\, i \ne j$, $\mathcal{H}^n(M_0) = 0$, and $M_j \subset N_j \quad j \ge 1$, with N_j an embedded C^1 submanifold of $\mathbb{R}^{n+k}$. By virtue of this decomposition and the definition of ∇^M (in §12) it easily follows that if h is Lipschitz on $\mathbb{R}^{n+k}$ and if $h^{(\sigma)}$ are the mollified functions (as in §6) then, as $\sigma \downarrow 0$,

(2) $$v\cdot\nabla^M h^{(\sigma)} \to v\cdot\nabla^M h \qquad \text{(weak convergence in } L^2(\mu_T))$$

for any fixed bounded $\mathcal{H}^n$-measurable v with values in $\mathbb{R}^{n+k}$. (Indeed to check this, we have merely to check that (2) holds with N_j in place of M_j and with v vanishing on $\mathbb{R}^{n+k} \sim M_j$; since N_j is C^1 this follows fairly easily by approximating v by smooth functions and using the fact that $h^{(\sigma)}$ converges to h uniformly.)

Next let $\varepsilon > 0$ and let γ be the Lipschitz function on $\mathbb{R}$ defined by

$$\gamma(s) = \begin{cases} 1 , & s<t-\varepsilon \\ \text{linear} , & t-\varepsilon\le s\le t \\ 0 , & s>t \end{cases}$$

and apply the above to $h = \gamma\circ f$. Then letting $\omega \in \mathcal{D}^n(U)$ we have

$$\begin{aligned} \partial T(h^{(\sigma)}\omega) &= T(d(h^{(\sigma)}\omega)) \\ &= T(dh^{(\sigma)} \wedge \omega) + T(h^{(\sigma)} d\omega) . \end{aligned}$$

Then using the integral representations of the form 26.7 for ∂T we see that

(3) $$(\partial T \llcorner h)(\omega) = \lim_{\sigma\downarrow 0} T(dh^{(\sigma)} \wedge \omega) + (T\llcorner h)(d\omega) .$$

Since $\xi(x)$ orients T_xM, we have

(4) $$\begin{aligned} \langle\xi(x), dh^{(\sigma)} \wedge \omega\rangle &= \langle\xi(x), (dh^{(\sigma)}(x))^T \wedge \omega^T\rangle \\ &= \langle\xi(x), (dh^{(\sigma)}(x))^T \wedge \omega\rangle \end{aligned}$$

(where $(\ )^T$ denotes the orthogonal projection of $\Lambda^q(\mathbb{R}^{n+k})$ onto $\Lambda^q(T_xM)$). Thus

$$\begin{aligned} T(dh^{(\sigma)} \wedge \omega) &= \int_M \langle\xi(x), (dh^{(\sigma)}(x))^T \wedge \omega\rangle\theta\, d\mathcal{H}^n \\ &= \int_M \langle\xi(x) \llcorner \nabla^M h^{(\sigma)}(x), \omega\rangle\theta\, d\mathcal{H}^n \end{aligned}$$

so that by (2)

$$\lim_{\sigma\downarrow 0} T(dh^{(\sigma)} \wedge \omega) = \int_M \langle \xi(x) \llcorner \nabla^M h(x), \omega\rangle \, \theta \, d\mathcal{H}^n . \tag{5}$$

By definition 12.1 of $\nabla^M h$ and by the chain rule for the composition of Lipschitz functions we have

$$\nabla^M h = \gamma'(f)\nabla^M f \qquad \mathcal{H}^n\text{-a.e. on } M \tag{6}$$

(where we set $\gamma'(f) = 0$ when f takes the "bad" values t or $t-\varepsilon$; note that $\nabla^M h(x) = \nabla^M f(x) = 0$ for $\mathcal{H}^n$-a.e. in $\{x \in M : f(x) = c\}$, c any given constant).

Using (5), (6) in (3), we thus deduce

$$(\partial T \llcorner h)(\omega) = -\varepsilon^{-1} \int_{M \cap \{t-\varepsilon<f<t\}} \langle \xi \llcorner \nabla^M f, \omega\rangle \, \theta \, d\mathcal{H}^n$$

$$+ (T \llcorner h)(d\omega) .$$

Finally we let $\varepsilon \downarrow 0$ and we use Remark 28.2 with $g = \theta \langle \xi \llcorner \nabla^M f/|\nabla^M f|, \omega\rangle$ in order to complete the proof of (2); by considering a countable dense set of $\omega \in \mathcal{D}^n(U)$ one can of course show that 28.2 is applicable with $g = \theta \langle \xi \llcorner \nabla^M f/|\nabla^M f|, \omega\rangle$ except for a set F of t having $\mathcal{L}^1$-measure zero, with F independent of ω .

Finally to prove part (3) of the theorem, we first apply part (2) with ∂T in place of T . Since $\partial^2 T = 0$, this gives

$$\langle \partial T, f, t\rangle = \partial[(\partial T) \llcorner \{f<t\}] .$$

On the other hand, applying ∂ to each side of the original identity (for T) of (2), we get

$$\partial[(\partial T) \llcorner \{f<t\}] = -\partial\langle T,f,t\rangle$$

and hence (3) is established.

Motivated by the above discussion we are led to define slices for an arbitrary current $\in \mathcal{D}_n(U)$ which, together with its boundary, has locally finite mass in U. Specifically, suppose $\underline{\underline{M}}_W(T) + \underline{\underline{M}}_W(\partial T) < \infty \quad \forall\, W \subset\subset U$. Then we define "slices"

28.6 $$\langle T,f,t_-\rangle = \partial(T \llcorner \{f<t\}) - (\partial T) \llcorner \{f<t\}$$

and

28.7 $$\langle T,f,t_+\rangle = -\partial(T \llcorner \{f>t\}) + (\partial T) \llcorner \{f>t\} .$$

Of course $\langle T,f,t_+\rangle = \langle T,f,t_-\rangle$ (and the common value is denoted $\langle T,f,t\rangle$) for all but the countably many values of t such that $\underline{\underline{M}}(T \llcorner \{f=t\}) + \underline{\underline{M}}((\partial T) \llcorner \{f=t\}) > 0$.

The important properties of the above slices are that if f is Lipschitz on U (and if we continue to assume $\underline{\underline{M}}_W(T) + \underline{\underline{M}}_W(\partial T) < \infty \quad \forall\, W \subset\subset U$), then

28.8 $$\operatorname{spt}\langle T,f,t_\pm\rangle \subset \operatorname{spt} T \cap \{x : f(x)=t\}$$

and, $\forall$ open $W \subset U$,

28.9 $$\begin{cases} \underline{\underline{M}}_W(\langle T,f,t_+\rangle) \le \operatorname{ess\,sup}_W |Df| \liminf\limits_{h\downarrow 0} h^{-1}\, \underline{\underline{M}}_W(T \llcorner \{t<f<t+h\}) \\[2ex] \underline{\underline{M}}_W(\langle T,f,t_-\rangle) \le \operatorname{ess\,sup}_W |Df| \liminf\limits_{h\downarrow 0} h^{-1}\, \underline{\underline{M}}_W(T \llcorner \{t-h<f<t\}) . \end{cases}$$

Notice that $\underline{\underline{M}}_W(T \llcorner \{f<t\})$ is increasing in t, hence is differentiable for $\mathcal{L}^1$- a.e. $t \in \mathbb{R}$ and $\int_a^b \frac{d}{dt}\underline{\underline{M}}_W(T \llcorner \{f<t\})\,dt \le \underline{\underline{M}}_W(T \llcorner \{a<f<b\})$. Thus 28.9 gives

28.10 $$\int_a^{*b} \underline{\underline{M}}_W(\langle T,f,t_\pm\rangle)\,dt \le \text{ess}\ \sup_W|Df|\ \underline{\underline{M}}_W(T \llcorner \{a<f<b\})$$

for every open $W \subset U$.

To prove 28.8 and 28.9 we consider first the case when f is C^1 and take any smooth increasing function $\gamma : \mathbb{R} \to \mathbb{R}_+$ and note that

$$
\begin{aligned}
(*) \qquad & \partial(T \llcorner \gamma\circ f)(\omega) - ((\partial T) \llcorner \gamma\circ f)(\omega) \\
&= (T \llcorner \gamma\circ f)(d\omega) - ((\partial T) \llcorner \gamma\circ f)(\omega) \\
&= T(\gamma\circ f d\omega) - T(d(\gamma\circ f\omega)) \\
&= -T(\gamma'(f)\,df \wedge \omega) \ .
\end{aligned}
$$

Now let $\varepsilon > 0$ be arbitrary and choose γ such that

$$\gamma(t) = 0 \quad \text{for} \quad t < a \ , \ \gamma(t) = 1 \quad \text{for} \quad t > b \ , \ 0 \le \gamma'(t) \le \frac{1+\varepsilon}{b-a} \quad \text{for} \quad a < t < b \ .$$

Then the left side of (*) converges to $\langle T,f,a_+\rangle$ if we let $b \downarrow a$, and hence 28.8 follows because $\operatorname{spt} \gamma' \subset [a,b]$. Furthermore the right side R of (*) evidently satisfies

$$|R| \le (\sup_W|Df|)\left(\frac{1+\varepsilon}{b-a}\right) \underline{\underline{M}}_W(T \llcorner \{a<f<b\})\,|\omega| \qquad (\operatorname{spt} \omega \subset W)$$

and so we also conclude the first part of 28.9 for $f \in C^1$. We similarly establish the second part for $f \in C^1$. To handle general Lipschitz f we simply use $f^{(\sigma)}$ in place of f in 28.6, 28.7 and in the above proof, then let $\sigma \downarrow 0$ where appropriate.

§29. THE DEFORMATION THEOREM

The deformation theorem, given below in Theorem 29.1 and Corollary 29.3 is a central result in the theory of currents, and was first proved by Federer and Fleming [FF].

The special notation for this section is as follows:

$1 \le n$, $1 \le k$,

$C = [0,1] \times \ldots \times [0,1]$ (Standard unit cube in $\mathbb{R}^{n+k}$)

$\mathbb{Z}^{n+k} = \{z = (z^1, \ldots, z^{n+k}) : z^j \in \mathbb{Z}\}$ $(\subset \mathbb{R}^{n+k})$

L_j = j-skeleton of the decomposition $\bigcup_{z \in \mathbb{Z}^{n+k}} (z+C)$

of $\mathbb{R}^{n+k}$

$\mathcal{L}_j$ = collection of j-faces in L_j

$= \{z + F : z \in \mathbb{Z}^{n+k}$, F is a closed j-face of $C\}$

$\mathcal{L}_j(\rho) = \{\rho F : F \in \mathcal{L}_j\}$, $\rho > 0$

$S_1, \ldots, S_N$ $\left(N = \binom{n+k}{n+1} = \binom{n+k}{k-1}\right)$ denote the

(n+1)-dimensional subspaces of $\mathbb{R}^{n+k}$ which contain an (n+1)-face of the standard cube C .

p_j denotes the orthogonal projection of $\mathbb{R}^{n+k}$ onto S_j , $j = 1, \ldots, n$.

29.1 THEOREM (Deformation Theorem, unscaled version)

Suppose T *is an* n-*current in* $\mathbb{R}^{n+k}$ (i.e. $T \in \mathcal{D}_n(\mathbb{R}^{n+k})$) *with* $\underline{\underline{M}}(T) + \underline{\underline{M}}(\partial T) < \infty$. *Then we can write*

$$T - P = \partial R + S ,$$

where P, R, S *satisfy*

$$P = \sum_{F \in \mathcal{L}_n} \beta_F [\![F]\!] \qquad (\beta_F \in \mathbb{R}) ,$$

with

$$\underline{\underline{M}}(P) \le c\underline{\underline{M}}(T) \ , \ \underline{\underline{M}}(\partial P) \le c\underline{\underline{M}}(\partial T)$$

$$\underline{\underline{M}}(R) \le c\underline{\underline{M}}(T) \ , \ \underline{\underline{M}}(S) \le c\underline{\underline{M}}(\partial T)$$

$(c = c(n,k))$, *and*

$$\operatorname{spt} P \cup \operatorname{spt} R \subset \{x : \operatorname{dist}(x, \operatorname{spt} T) < 2\sqrt{n+k}\}$$

$$\operatorname{spt} \partial P \cup \operatorname{spt} S \subset \{x : \operatorname{dist}(x, \operatorname{spt} \partial T) < 2\sqrt{n+k}\} .$$

In case T *is an integer multiplicity current, then* P , R *can be chosen to be integer multiplicity currents (and the* β_F *appearing in the definition of* P *are integers). If in addition* ∂T *is integer multiplicity*, then* S *can be chosen to be integer multiplicity.*

29.2 REMARKS

(1) Note that this is slightly sharper than the corresponding theorem in [FF], [FH1], because there is no term involving $\underline{\underline{M}}(\partial T)$ in the bound for $\underline{\underline{M}}(P)$.

(2) It follows automatically from the other conclusions of the theorem that $\underline{\underline{M}}(\partial S) \le c\underline{\underline{M}}(\partial T)$. Also, it follows from the inequalities $\underline{\underline{M}}(\partial P), \underline{\underline{M}}(S) \le c\underline{\underline{M}}(\partial T)$ that $S = 0$ and $\partial P = 0$ when $\partial T = 0$.

The following "scaled version" of 29.1 is obtained from the above by first changing scale $x \to \rho^{-1}x$, then applying 29.1, then changing scale back by $x \to \rho x$.

* Actually ∂T *automatically* is integer multiplicity if T is integer multiplicity and $\underline{\underline{M}}(\partial T) < \infty$, see Theorem 30.3.

29.3 COROLLARY (Deformation Theorem, scaled version)

Suppose T, ∂T *are as in* 29.1, *and* $\rho > 0$. *Then*

$$T - P = \partial R + S ,$$

where P, R, S *satisfy*

$$P = \sum_{F \in \mathcal{L}_j(\rho)} \beta_F [\![F]\!] \qquad (\beta_F \in \mathbb{R})$$

$$\underline{\underline{M}}(P) \le c\underline{\underline{M}}(T) \ , \ \underline{\underline{M}}(\partial P) \le c\underline{\underline{M}}(\partial T)$$

$$\underline{\underline{M}}(R) \le c\rho\underline{\underline{M}}(T) \ , \ \underline{\underline{M}}(S) \le c\rho\underline{\underline{M}}(\partial T) \ ,$$

and

$$\operatorname{spt} P \cup \operatorname{spt} R \subset \{x : \operatorname{dist}(x, \operatorname{spt} T) < 2\sqrt{n+k}\ \rho\}$$

$$\operatorname{spt} \partial P \cup \operatorname{spt} S \subset \{x : \operatorname{dist}(x, \operatorname{spt} \partial T) < 2\sqrt{n+k}\ \rho\} .$$

As in 29.1, *in case* T *is integer multiplicity, so are* P, R ; *if* ∂T *is integer multiplicity then so is* S.

The main step in the proof of the deformation theorem will involve "pushing" T onto the n-skeleton L_n via a certain retraction map ψ. We first have to establish the existence of a suitable class of retraction maps. This is done in the following lemma, in which we use the notation:

$$q = \text{centre point of } C = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}) \ ,$$

$$L_{k-1}(a) = a + L_{k-1} \quad (a \text{ a given point in } B_{\frac{1}{4}}(q)) \ ,$$

$$L_{k-1}(a;\rho) = \{x \in \mathbb{R}^{n+k} : \operatorname{dist}(x, L_{k-1}(a)) < \rho\} \quad (\rho \in (0, \tfrac{1}{4})) \ .$$

Note that $\operatorname{dist}(L_{k-1}(a), L_n) \ge \frac{1}{4}$ for any point $a \in B_{\frac{1}{4}}(q)$.

29.4 LEMMA *For every* $a \in B_{\frac14}(q)$ *there is a locally Lipschitz map*

$$\psi : \mathbb{R}^{n+k} \sim L_{k-1}(a) \to \mathbb{R}^{n+k} \sim L_{k-1}(a)$$

such that

$$\psi(C \sim L_{k-1}(a)) = C \cap L_n \ , \ \psi | C \cap L_n = \underset{\approx}{1}_{C\cap L_n} \ ,$$

$$|D\psi(x)| \le c/\rho \ , \ L^{n+k}\text{-a.e. } x \in C \sim L_{k-1}(a;\rho) \ , \ \rho \in (0,\tfrac14) \ ,$$

$(c = c(n,k))$, *and such that*

$$\psi(z+x) = z+\psi(x) \ , \ x \in \mathbb{R}^{n+k} \sim L_{k-1}(a) \ , \ z \in \mathbb{Z}^{n+k} \ .$$

Proof We first construct a locally Lipschitz retraction $\psi_0 : C \sim L_{k-1}(a)$ onto the n-faces of C . This is done as follows:

Firstly for each j-face F of C , $j \ge n+1$, let $a_F \in F$ denote the orthogonal projection of a onto F , and let ψ_F denote the retraction of $\bar{F} \sim \{a_F\}$ onto ∂F which takes a point $x \in \bar{F} \sim \{a_F\}$ to the point $y \in \partial F$ such that $x \in \{a_F + \lambda(y-a_F) : \lambda \in (0,1]\}$. (Thus ψ_F is the "radial retraction" of F with a_F as origin.) Of course $\psi_F|\partial F = \underset{\approx}{1}_{\partial F}$. Notice also that for any j-face F of C , $j \ge n+1$, the line segment $\overline{aa_F}$ is contained in $L_{k-1}(a)$; in fact if J_F denotes the set of ℓ such that S_ℓ (see notation prior to 29.1) is parallel to an (n+1)-face of F , then (because $\overline{aa_F}$ is orthogonal to F , hence orthogonal to each S_ℓ , $\ell \in J_F$) we have

$$\text{(1)} \qquad \overline{aa_F} \subset \bigcap_{\ell \in J_F} p_\ell^{-1}(p_\ell(a)) \ ,$$

and this is contained in $L_{k-1}(a)$, because (by definition)

$$\text{(2)} \qquad L_{k-1}(a) = \bigcup_{\ell=1}^{N} \ \bigcup_{z\in\mathbb{Z}^{n+k}} (z+p_\ell^{-1}(p_\ell(a))) \ .$$

Next, for each $j \geq n+1$, define

$$\psi^{(j)} : \cup\{\bar{F} \sim \{a_F\} : F \text{ is a } j\text{-face of } C\}$$

$$\to \cup\{\bar{G} : G \text{ is a } (j-1)\text{-face of } C\}$$

by setting

$$\psi^{(j)} | \bar{F} \sim \{a_F\} = \psi_F .$$

(Notice that then $\psi^{(j)}$ is locally Lipschitz on its domain by virtue of the fact that each ψ_F is the identity on ∂F, F any j-face of C.)

Then the composite $\psi^{(n+1)} \circ \psi^{(n+2)} \circ \ldots \circ \psi^{(n+k)}$ makes sense on $C \sim L_{k-1}(a)$ (by (1)), so we can set

$$\psi_0 = \psi^{(n+1)} \circ \psi^{(n+2)} \circ \ldots \circ \psi^{(n+k)} \mid C \sim L_{k-1}(a) .$$

Notice that ψ_0 has the additional property that if

$$z \in \mathbb{Z}^{n+k} \text{ and } x, z+x \in C, \text{ then } \psi_0(z+x) = z+\psi_0(x) .$$

(Indeed $x, z+x \in C$ means that either $x, z+x$ are in L_n (where ψ_0 is the identity) or else lie in the interior of parallel j-faces $F_1, F_2 = z+F_1$ ($j \geq n+1$) of C with z orthogonal to F_1 and $a_{F_2} = z+a_{F_1}$.) It follows that we can then define a retraction ψ of all of $C \sim L_{k-1}(a)$ onto L_n by setting

$$\psi(z+x) = z+\psi_0(x) , \; x \in C \sim L_{k-1}(a) , \; z \in \mathbb{Z}^{n+k} .$$

We now claim that

$$\text{(3)} \qquad \sup|D\psi| \leq c/\rho \text{ on } \mathbb{R}^{n+k} \sim L_{k-1}(a,\rho) , \; c = c(n,k) .$$

(This will evidently complete the proof of the lemma.)

We can prove (3) by induction on k as follows. First note that (3) is evident from construction in case $k = 1$. Hence assume $k \geq 2$ and assume (3) holds in case $k-1$ replaces k in the above construction. Let x be any point of interior $(C) \sim L_{k-1}(a;\rho)$, let $y = \psi^{n+k}(x)$ (ψ^{n+k} is the radial retraction of $C \sim \{a\}$ onto ∂C), and let F be any closed $(n+k-1)$-face of C which contains y.

Suppose now new coordinates are selected so that $F \subset \mathbb{R}^{n+k-1} \times \{0\} \subset \mathbb{R}^{n+k}$, and also let $\tilde{L}_{k-2}(a) = L_{k-1}(a) \cap \mathbb{R}^{n+k-1} \times \{0\})$. By virtue of (1) we have $a_F \in L_{k-1}(a)$, hence

$$|y-a_F| \geq \operatorname{dist}(y, L_{k-1}(a)) . \tag{4}$$

Let p_F be the orthogonal projection of $\mathbb{R}^{n+k}$ onto $\mathbb{R}^{n+k-1} \times \{0\}$ $(\supset F)$, so that $a_F = p_F(a)$. Evidently $|p_F(x) - a_F| \geq \operatorname{dist}(x, p_F^{-1}(p_F(a)))$ and hence by (2) we deduce

$$|p_F(x) - a_F| \geq \operatorname{dist}(x, L_{k-1}(a)) . \tag{5}$$

Furthermore by definition of y we know that $y-a = \dfrac{|y-a|}{|x-a|}(x-a)$ and hence, applying p_F, we have

$$y-a_F = \frac{|y-a|}{|x-a|}\, p_F(x-a) .$$

Hence since $|y-a| \geq 3/4$, we have

$$|y-a_F| \geq (3/4)\, |p_F(x-a)| / |x-a| . \tag{6}$$

Now let $\tilde{\psi}$ be the retraction of $F \sim \tilde{L}_{k-2}(a)$ onto the n-faces of F ($\tilde{\psi}$ defined as for ψ but with $(k-1)$ in place of k, a_F in place of a, $\mathbb{R}^{n+k-1}$ in place of $\mathbb{R}^{n+k}$ and $\tilde{L}_{k-2}(a) = L_{k-2}(a_F)$ in place of $L_{k-1}(a)$). By the inductive hypothesis, together with (4), (5), (6) we have

$$(7) \qquad |\bar{D}\tilde{\psi}(y)| \le \frac{c}{\operatorname{dist}(y,\tilde{L}_{k-2}(a))}, \quad \left(|\bar{D}\tilde{\psi}(y)| = \limsup_{z\to y}\frac{|\tilde{\psi}(z)-\tilde{\psi}(y)|}{|z-y|}\right)$$

$$\le \frac{c}{|y-a_F|}\,(4/3)\,c\,\frac{|x-a|}{|p_F(x-a)|}$$

$$\le (4/3)c\frac{|x-a|}{\operatorname{dist}(x,L_{k-1}(a))}.$$

Also, by the definition of ψ^{n+k} we have that

$$(8) \qquad |\bar{D}\psi^{n+k}(x)| \le \frac{c}{|x-a|}, \quad |\bar{D}\psi^{n+k}(x)| = \limsup_{y\to x}\frac{|\psi^{n+k}(y)-\psi^{n+k}(x)|}{|y-x|}.$$

Since $\psi(x) = \tilde{\psi}\circ\psi^{n+k}(x)$, we have by (7), (8) and the chain rule that

$$|\bar{D}\psi(x)| \le |\bar{D}\tilde{\psi}(y)|\;|\bar{D}\psi^{n+k}(x)| \le \frac{c}{|x-a|}\;\frac{|x-a|}{\operatorname{dist}(x,L_{k-1}(a))}$$

$$= \frac{c}{\operatorname{dist}(x,L_{k-1}(a))}.$$

Proof of Deformation Theorem

We use the subspaces $S_1,\dots,S_N$ and projections $p_1,\dots,p_N$ introduced at the beginning of the section. Let $F_j = C\cap S_j$ (so that F_j is a closed $(n+1)$-dimensional face of C), let x_j be the central point of F_j, and for each $j=1,\dots,N$ define a "good" subset $G_j \subset F_j \cap B_{\frac14}(x_j)$ by $g\in G_j \Leftrightarrow g\in F_j\cap B_{\frac14}(x_j)$ and

$$(1) \qquad \underline{\underline{M}}\Big(T \llcorner \bigcup_{z\in\mathbb{Z}^{n+k}\cap S_j} p_j^{-1}(B_\rho(g+z))\Big) \le \beta\rho^{n+1}\underline{\underline{M}}(T) \qquad \forall\,\rho\in(0,\tfrac14)$$

(β to be chosen, $G_j = G_j(\beta)$).

We now claim that the "bad" set $B_j = F_j \cap B_{\frac14}(x_j) \sim G_j$ in fact has L^{n+1}-measure (taken in S_j) small; in fact we claim

$$(2) \qquad L^{n+1}(B_j) \le 20^{n+1}\beta^{-1}\omega_{n+1}(\tfrac14)^{n+1} \quad (\omega_{n+1} = L^{n+1}(B_1(0))) ,$$

which is indeed small if we choose large β. To see (2), we argue as follows. For each $b \in B_j$ there is (by definition) a $\rho_b \in (0,\frac14)$ such that

$$(3) \qquad \underline{\underline{M}}(T \llcorner \bigcup_{z\in\mathbb{Z}^{n+k}\cap S_j} p_j^{-1}(B_{\rho_b}(b+z))) \ge \beta\rho_b^{n+1}\underline{\underline{M}}(T) ,$$

and by the covering theorem 3.3 there is a pairwise disjoint subcollection $\{B_{\rho_\ell}(b_\ell)\}_{\ell=1,2\ldots}(\rho_\ell=\rho_{b_\ell})$ of the collection $\{B_{\rho_b}(b)\}_{b\in B_j}$ such that

$$(4) \qquad B_j \subset \bigcup_\ell B_{5\rho_\ell}(b_\ell) .$$

But then, setting $b = b_\ell$ in (3) and summing, we get

$$\beta(\sum_\ell \rho_\ell^{n+1})\ \underline{\underline{M}}(T) \le \underline{\underline{M}}(T) \quad (\text{i.e. } \sum_\ell \rho_\ell^{n+1} \le \beta^{-1}) ,^{(*)}$$

(using the fact that $\{p_j^{-1}B_{\rho_\ell}(b_\ell+z)\}_{\substack{\ell=1,2,\ldots\\ z\in\mathbb{Z}^{n+k}\cap S_j}}$ is a pairwise disjoint collection for fixed j). Thus by (4) we conclude

$$L^{n+1}(B_j) \le \beta^{-1}\ 5^{n+1}\omega_{n+1} ,$$

which after trivial re-arrangement gives (2) as required. Thus we have

$$L^{n+1}(G_j) \ge (1-20^{n+1}\beta^{-1})\omega_{n+1}(\tfrac14)^{n+1} ,$$

and it follows that

$$(5) \qquad L^{n+k}(p_j^{-1}(G_j)\cap B_{\frac14}(q)) \ge \left(1-\frac{\omega_{n+1}}{\omega_{n+k}}\,20^{n+1}\beta^{-1}\right)\omega_{n+k}(\tfrac14)^{n+k} ,$$

where q is the centre point $(\frac12,\ldots,\frac12)$ of C. (So $p_j(q) = x_j$.)

(*) We of course assume $T \ne 0$.

Then selecting β large enough so that $20^{n+1}\omega_{n+1}N\beta^{-1} < \omega_{n+k}/(n+k)$, we see from (5) that we can choose a point $a \in \bigcap_{j=1}^{N} p_j^{-1}(G_j) \cap B_{\frac{1}{4}}(q)$. Next let $L_{k-1}(a) = a + L_{k-1}$, $L_{k-1}(a;\rho) = \{x \in \mathbb{R}^{n+k}: \operatorname{dist}(x, L_{k-1}(a)) < \rho\}$ (as in the proof of 29.4) and note that in fact

$$L_{k-1}(a;\rho) = \bigcup_{j=1}^{N} \bigcup_{z\in\mathbb{Z}^{n+k}\cap S_j} p_j^{-1}(B_\rho(p_j(a)+z)) .$$

Then since $p_j(a) \in G_j$ we have (by definition of G_j)

$$\text{(6)} \qquad \underline{\underline{M}}(T \llcorner L_{k-1}(a;\rho)) \le N\beta\rho^{n+1}\underline{\underline{M}}(T) \qquad \forall\, \rho \in (0,\tfrac{1}{4}) .$$

Indeed let us suppose that we take β such that $20^{n+1}\omega_{n+1}N\beta^{-1} < \omega_{n+k}/2(n+k)$. Then more than half the ball $B_{\frac{1}{4}}(q)$ is in the set $\bigcap_{j=1}^{N} p_j^{-1}(G_j)$ and hence, repeating the whole argument above with ∂T in place of T, we can actually select a so that, *in addition to* (6), we also have

$$\text{(7)} \qquad \underline{\underline{M}}(\partial T \llcorner L_{k-1}(a;\rho)) \le N\beta\rho^{n+1}\underline{\underline{M}}(\partial T) \qquad \forall\, \rho \in (0,\tfrac{1}{4}) .$$

Now let ψ be the retraction of $\mathbb{R}^{n+k} \sim L_{k-1}(a)$ onto L_n given in Lemma 29.4, and let

$$\text{(8)} \qquad T_\rho = T \llcorner L_{k-1}(a;\rho) , \quad (\partial T)_\rho = T \llcorner L_{k-1}(a;\rho) ,$$

so that by (6), (7)

$$\text{(9)} \qquad \underline{\underline{M}}(T_\rho) \le c\rho^{n+1}\underline{\underline{M}}(T) , \quad \underline{\underline{M}}((\partial T)_\rho) \le c\rho^{n+1}\underline{\underline{M}}(\partial T) .$$

Furthermore by 28.10 we know that for each $\rho \in (0,\frac{1}{4})$ we can find $\rho^* \in (\rho/2, \rho)$ such that

$$\text{(10)} \qquad \underline{\underline{M}}(\langle T, d, \rho^*\rangle) \le \frac{c}{\rho}\,\underline{\underline{M}}(T_\rho - T_{\rho/2}) \le c\rho^{n}\underline{\underline{M}}(T) ,$$

where d is the (Lipschitz) distance function to $L_{k-1}(a)$ $(d(x) = \operatorname{dist}(x, L_{k-1}(a)),\ \operatorname{Lip}(d) = 1)$ and $\langle T, d, \rho^* \rangle$ is the slice of T by d at ρ^*. (Notice that we can choose ρ^* such that (10) holds and such that $\langle T, d, \rho^* \rangle$ is integer multiplicity in case T is integer multiplicity - see Lemma 28.5 and the following discussion.)

We now want to apply the homotopy formula 26.22 to the case when $h(x,t) = x + t(\psi(x) - x)$, $x \in \mathbb{R}^{n+k} \sim L_{k-1}(a;\sigma)$, $\sigma > 0$. Notice that h is only Lipschitz on $\mathbb{R}^{n+k} \sim L_{k-1}(a;\sigma)$, so we define $h_\#$, $\psi_\#$ as in Lemma 26.25. (We shall apply $h_\#$, $\psi_\#$ only to currents supported away from $[0,1] \times L_{k-1}(a)$ and $L_{k-1}(a)$ respectively.)

Keeping this in mind we note that by 29.4, (6) and (7) we have

$$\text{(11)} \qquad \underline{\underline{M}}(\psi_\#(T_\rho - T_{\rho/2})) \le \frac{c}{\rho^n} \rho^{n+1} \underline{\underline{M}}(T) \le c\rho \underline{\underline{M}}(T)$$

and

$$\text{(12)} \qquad \underline{\underline{M}}(\psi_\#((\partial T)_\rho - (\partial T)_{\rho/2})) \le \frac{c}{\rho^{n-1}} \rho^{n+1} \underline{\underline{M}}(\partial T) \le c\rho \underline{\underline{M}}(\partial T) .$$

Similarly by the homotopy formula 26.22, together with 26.23 and (6), (7) above, we have

$$\text{(13)} \qquad \underline{\underline{M}}(h_\#([\![(0,1)]\!] \times (T_\rho - T_{\rho/2}))) \le c\rho \underline{\underline{M}}(T)$$

and

$$\text{(14)} \qquad \underline{\underline{M}}(h_\#([\![(0,1)]\!] \times ((\partial T)_\rho - (\partial T)_{\rho/2})) \le c\rho \underline{\underline{M}}(\partial T) .$$

Notice also that by (6), (10) and 26.23 we have

$$\text{(15)} \qquad \underline{\underline{M}}(\psi_\# \langle T, d, \rho^* \rangle) \le c\rho \underline{\underline{M}}(T)$$

and

(16) $$\underline{\underline{M}}(h_{\#}([\![(0,1)]\!] \times \langle T,d,\rho^*\rangle)) \le c\rho\underline{\underline{M}}(T) \ .$$

Next note that by iteration (11), (12) imply

(17) $$\begin{cases} \underline{\underline{M}}(\psi_{\#}(T_\rho - T_{\rho/2^\nu})) \le 2c\rho\underline{\underline{M}}(T) \\ \underline{\underline{M}}(\psi_{\#}((\partial T)_\rho - (\partial T)_{\rho/2^\nu})) \le 2c\rho\underline{\underline{M}}(\partial T) \end{cases}$$

for each integer $\nu \ge 1$, where c is as in (11), (12) (c independent of ν). Selecting $\rho = \frac{1}{4}$ and using the arbitrariness of ν , it follows that

(18) $$\begin{cases} \underline{\underline{M}}(\psi_{\#}(T - T_\sigma)) \le c\underline{\underline{M}}(T) \\ \underline{\underline{M}}(\psi_{\#}(\partial T - (\partial T)_\sigma)) \le c\underline{\underline{M}}(\partial T) \end{cases}$$

for each $\sigma \in (0,1)$ (with c independent of σ) .

Now select $\rho = \rho_\nu \equiv 2^{-\nu}$ and $\rho_\nu^* \in [2^{-\nu-1}, 2^{-\nu}]$ such that (10), (15), (16) hold with ρ_ν^* in place of ρ^* ; then by (15), (16), (17), (18) we have that

$$\psi_{\#}(T - T_{\rho_\nu^*}) \ , \ h_{\#}([\![(0,1)]\!] \times (T - T_{\rho_\nu^*})) \ ,$$

$$\psi_{\#}(\partial T - \partial T_{\rho_\nu^*}) \ , \ h_{\#}([\![0,1)]\!] \times \partial(T - T_{\rho_\nu^*}))$$

are Cauchy sequences relative to $\underline{\underline{M}}$, and $\underline{\underline{M}}(\langle T,d,\rho_\nu^*\rangle) + \underline{\underline{M}}(\psi_{\#}\langle T,d,\rho_\nu^*\rangle) \to 0$. Hence there are currents Q , $S_1 \in \mathcal{D}_n(\mathbb{R}^{n+k})$ and $R_1 \in \mathcal{D}_{n+1}(\mathbb{R}^{n+k})$ such that

(19) $$\begin{cases} \lim \underline{\underline{M}}(Q - \psi_{\#}(T - T_{\rho_\nu^*})) = 0 \\ \lim \underline{\underline{M}}(S_1 - h_{\#}([\![(0,1)]\!] \times \partial(T - T_{\rho_\nu^*}))) = 0 \\ \lim \underline{\underline{M}}(R_1 - h_{\#}([\![(0,1)]\!] \times (T - T_{\rho_\nu^*})) = 0 \ . \end{cases}$$

Furthermore by the homotopy formula and 26.23 we have for each ν

$$(20)\qquad T - T_{\rho_\nu^*} - \psi_\#(T-T_{\rho_\nu^*})$$
$$= \partial(h_\#([\![(0,1)]\!] \times (T-T_{\rho_\nu^*}))$$
$$- h_\#([\![(0,1)]\!] \times \partial(T-T_{\rho_\nu^*})) \ .$$

Since $\partial T_{\rho_\nu^*} = (\partial T)_{\rho_\nu^*} - \langle T, d, \rho_\nu^* \rangle$ (by the definition 28.6, 28.7 of slice) we thus get that

$$(21)\qquad T - Q = \partial R_1 + S_1 \ .$$

(Notice that Q, R_1 are integer multiplicity by (19), 28.4, 28.5 and 27.5 in case T is integer multiplicity; similarly S_1 is integer multiplicity if ∂T is.)

Using the fact that ψ retracts $\mathbb{R}^{n+k} \sim L_{k-1}(a)$ onto L_n we know (by 26.23) that $\operatorname{spt} \psi_\#(T-T_{\rho_\nu^*}) \subset L_n$, and hence

$$(22)\qquad \operatorname{spt} Q \subset L_n \ .$$

We also have (since $\psi(z+C) \subset z+C \quad \forall\, z \in \mathbb{Z}^{n+k}$) that

$$(23)\qquad \begin{cases} \operatorname{spt} R_1 \cup \operatorname{spt} Q \subset \{x : \operatorname{dist}(x, \operatorname{spt} T) < \sqrt{n+k}\ \} \\[2ex] \operatorname{spt} S_1 \subset \{x : \operatorname{dist}(x, \operatorname{spt} \partial T) < \sqrt{n+k}\ \} \end{cases}$$

and, by (18), (19), we have

$$(24)\qquad \begin{cases} \underline{\underline{M}}(Q) \le c\underline{\underline{M}}(T) \ , \ \underline{\underline{M}}(R_1) \le c\underline{\underline{M}}(T) \\[2ex] \underline{\underline{M}}(S_1) \le c\underline{\underline{M}}(\partial T) \ . \end{cases}$$

Also by (18) and the semi-continuity of $\underline{\underline{M}}$ under weak convergence, we have

(25) $$\underline{\underline{M}}(\partial Q) \le \liminf \underline{\underline{M}}(\partial\psi_{\#}(T-T_{\rho_\nu^*}))$$
$$= \liminf \underline{\underline{M}}(\psi_{\#}\partial(T-T_{\rho_\nu^*}))$$
$$\le c\underline{\underline{M}}(\partial T) .$$

Now let F be a given face of L_n (i.e. $F \in \mathcal{L}_n$) and let $\mathring{F}$ = interior of F. Assume for the moment that $F \subset \mathbb{R}^n \times \{0\}$ $(\subset \mathbb{R}^{n+k})$, and let p be the orthogonal projection onto $\mathbb{R}^n \times \{0\}$. By construction of ψ we know that $p \circ \psi = \psi$ in a neighbourhood of any point $y \in \mathring{F}$. We therefore have (since Q is given by (18)) that

(26) $$p_{\#}(Q \llcorner \mathring{F}) = Q \llcorner \mathring{F} .$$

It then follows, by the obvious modifications of the arguments in the proof of the constancy theorem (Theorem 26.27) and in Remark 26.28, that

(27) $$(Q \llcorner \mathring{F})(\omega) = \int_{\mathring{F}} \langle e_1 \wedge \ldots \wedge e_n, \omega(x) \rangle \, \theta_F(x) \, d\mathcal{L}^n(x)$$

$\forall\, \omega \in \mathcal{D}^n(\mathbb{R}^{n+k})$, for some $BV_{loc}(\mathbb{R}^n)$ function θ_F, and

(28) $$\underline{\underline{M}}(Q \llcorner \mathring{F}) = \int_{\mathring{F}} |\theta_F| \, d\mathcal{L}^n , \quad \underline{\underline{M}}((\partial Q) \llcorner \mathring{F}) = \int_{\mathring{F}} |D\theta_F| .$$

Furthermore, since

$$(Q \llcorner \mathring{F} - \beta[\![F]\!])(\omega) = \int_{\mathring{F}} (\theta_F - \beta) \langle e_1 \wedge \ldots \wedge e_n, \omega(x) \rangle \, d\mathcal{L}^n(x)$$

(by (27)), we have (again using the reasoning of 26.28)

(29) $$\begin{cases} \underline{\underline{M}}(Q \llcorner \mathring{F} - \beta[\![F]\!]) = \displaystyle\int_{\mathring{F}} |\theta_F - \beta| \, d\mathcal{L}^n \\ \underline{\underline{M}}(\partial(Q \llcorner \mathring{F} - \beta[\![F]\!])) = \displaystyle\int_{\mathbb{R}^n} |D(\chi_{\mathring{F}}(\theta_F - \beta))| , \end{cases}$$

where $\chi_{\mathring{F}}$ = characteristic function of $\mathring{F}$.

Thus taking $\beta = \beta_F$ such that

$$(30)\qquad \min\{L^n\{x \in \mathring{F} : \theta_F \geq \beta\}, L^n\{x \in \mathring{F} : \theta_F(x) \leq \beta\} \geq \frac{1}{2}$$

(which we can do because $L^n(\mathring{F}) = 1$; notice that we can take $\beta_F \in \mathbb{Z}$ if θ_F is integer-valued), we have by 6.4, 6.6, (28) and (29) that

$$(31)\qquad \begin{cases} \underline{\underline{M}}(Q \llcorner \mathring{F} - \beta[\![F]\!]) \leq c\int_{\mathring{F}} |D\theta_F| = c\underline{\underline{M}}(Q \llcorner \mathring{F}) \\ \underline{\underline{M}}(\partial(Q \llcorner \mathring{F} - \beta[\![F]\!])) \leq c\int_{\mathring{F}} |D\theta_F| = c\underline{\underline{M}}(Q \llcorner \mathring{F}) . \end{cases}$$

We also have by 26.30

$$(32)\qquad Q \llcorner \partial F = 0 .$$

Then summing over $F \in L_n$ and using (31), (32) we have, with $P = \sum_{F \in L_n} \beta_F[\![F]\!]$, that

$$(33)\qquad \begin{cases} \underline{\underline{M}}(Q-P) \leq c\underline{\underline{M}}(\partial Q) \\ \underline{\underline{M}}(\partial Q - \partial P) \leq c\underline{\underline{M}}(\partial Q) . \end{cases}$$

Actually by (30) we have

$$(34)\qquad |\beta_F| \leq 2\int_{\mathring{F}} |\theta_F| \, dL^n ,$$

and hence (using again the first part of (28)), since $\underline{\underline{M}}(P) = \sum_F |\beta_F|$,

$$(35)\qquad \underline{\underline{M}}(P) \leq c\underline{\underline{M}}(Q) .$$

Notice that the second part of (33) gives

$$(36)\qquad \underline{\underline{M}}(\partial P) \leq c\underline{\underline{M}}(\partial Q) .$$

Finally we note that (21) can be written

$$\text{(37)} \qquad T-P = \partial R_1 + (S_1+(Q-P)) \ .$$

Setting $R = R_1$, $S = S_1 + (Q-P)$, the theorem now follows immediately from (23), (24), (25) and (33), (35), (36), (37); the fact that P , R are integer multiplicity if T is should be evident from the remarks during the course of the above proof, as should be the fact that S is integer multiplicity if T , ∂T are.

§30. APPLICATIONS OF THE DEFORMATION THEOREM

We here establish a couple of simple (but very important) applications of the deformation theorem, namely the isoperimetric theorem and the weak polyhedral approximation theorem. This latter theorem, when combined with the compactness theorem 27.3 implies the important "boundary rectifiability theorem" (30.3 below), which asserts that if T is an integer multiplicity current in $\mathcal{D}_n(U)$ and if $\underline{\underline{M}}_W(\partial T) < \infty \quad \forall\, W \subset\subset U$, then $\partial T (\in \mathcal{D}_{n-1}(U))$ is integer mutiplicity. (Notice that in the case $k=0$, this has already been established in 27.6.)

30.1 THEOREM (Isoperimetric Theorem)

Suppose $T \in \mathcal{D}_{n-1}(\mathbb{R}^{n+k})$ *is integer multiplicity,* $n \geq 2$, $\operatorname{spt} T$ *is compact and* $\partial T = 0$. *Then there is an integer multiplicity current* $R \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *with* $\operatorname{spt} R$ *compact,* $\partial R = T$, *and*

$$\underline{\underline{M}}(R)^{\frac{n-1}{n}} \leq c\, \underline{\underline{M}}(T) \ ,$$

where $c = c(n,k)$.

Proof The case $T = 0$ is trivial, so assume $T \neq 0$. Let P, R, S be integer multiplicity currents as in 29.3, where for the moment $\rho > 0$ is arbitrary, and note that $S = 0$ because $\partial T = 0$. Evidently (since $\mathcal{H}^{n-1}(F) = \rho^{n-1} \quad \forall\, F \in \mathcal{F}_{n-1}(\rho)$) we have

$$(*) \qquad \underline{\underline{M}}(P) = N(\rho)\rho^{n-1}$$

for some non-negative *integer* $N(\rho)$. But since $\underline{\underline{M}}(P) \leq c\, \underline{\underline{M}}(T)$ (from 29.3) we see that necessarily $N(\rho) = 0$ in (*) if we choose $\rho = (2c\underline{\underline{M}}(T))^{\frac{1}{n-1}}$. Then $P = 0$, and 29.3 gives $T = \partial R$ for some integer multiplicity current R with $\operatorname{spt} R$ compact and $\underline{\underline{M}}(R) \leq c\rho\underline{\underline{M}}(T) = c'(\underline{\underline{M}}(T))^{\frac{1}{n-1}}$.

30.2 THEOREM (Weak polyhedral approximation theorem)

Given any integer multiplicity $T \in \mathcal{D}_n(U)$ *with* $\underline{\underline{M}}_W(\partial T) < \infty \quad \forall\, W \subset\subset U$, *there is a sequence* $\{P_k\}$ *of currents of the form*

$$(**) \qquad P_k = \sum_{F \in \mathcal{F}_n(\rho_k)} \beta_F^{(k)} [\![F]\!] , \qquad (\beta_F^{(k)} \in \mathbb{Z}) , \quad \rho_k \downarrow 0 ,$$

such that $P_k \rightharpoonup T$ *(and hence also* $\partial P_k \rightharpoonup \partial T$*) in* U *(in the sense of* 26.12).

Proof First consider the case $U = \mathbb{R}^{n+k}$ and $\underline{\underline{M}}(T)$, $\underline{\underline{M}}(\partial T) < \infty$. In this case we simply use the deformation theorem: for any sequence $\rho_k \downarrow 0$, the scaled version of the deformation theorem (with $\rho = \rho_k$) gives P_k as in (**) such that

$$(1) \qquad T - P_k = \partial R_k + S_k$$

for some R_k, S_k such that

$$(2) \qquad \begin{cases} \underline{\underline{M}}(R_k) \leq c\rho_k\underline{\underline{M}}(T) \to 0 \\ \\ \underline{\underline{M}}(S_k) \leq c\rho_k\underline{\underline{M}}(\partial T) \to 0 \end{cases}$$

and

$$\underline{\underline{M}}(P_k) \le c\underline{\underline{M}}(T) \ , \ \underline{\underline{M}}(\partial P_k) \le c\underline{\underline{M}}(\partial T) \ .$$

Evidently (1), (2) give $P_k(\omega) \to T_k(\omega) \quad \forall\ \omega \in \mathcal{D}^n(\mathbb{R}^{n+k})$, and $\partial P_k = 0$ if $\partial T = 0$, so the theorem is proved in case $U = \mathbb{R}^{n+k}$ and $T, \partial T$ are of finite mass.

In the general case we take any Lipschitz function ϕ on $\mathbb{R}^{n+k}$ such that $\phi > 0$ in U, $\phi \equiv 0$ in $\mathbb{R}^{n+k} \sim U$ and such that $\{x = \phi(x) > \lambda\} \subset\subset U$ $\forall\ \lambda > 0$. For $\mathcal{L}^1$-a.e. $\lambda > 0$, 28.5 implies that $T_\lambda \equiv T \,\llcorner\, \{x : \phi(x) > \lambda\}$ is such that $\underline{\underline{M}}(\partial T_\lambda) < \infty$. Since $\operatorname{spt} T_\lambda \subset\subset U$, we can apply the argument above to approximate T_λ for any such λ . Taking a suitable sequence $\lambda_j \downarrow 0$, the required approximation then immediately follows.

30.3 THEOREM (Boundary rectifiability theorem)

Suppose T *is an integer multiplicity current in* $\mathcal{D}_n(U)$ *with* $\underline{\underline{M}}_W(\partial T) < \infty \quad \forall\ W \subset\subset U$. *Then* $\partial T (\in \mathcal{D}_{n-1}(U))$ *is an integer multiplicity current.*

Proof A direct consequence of 30.2 above and the compactness theorem 27.3.

30.4 REMARK Notice that only the case $\partial T_j = 0 \quad \forall j$ of 27.3 is needed in the above proof.

§31. THE FLAT METRIC(*) TOPOLOGY

The main result to be proved here is the equivalence of weak convergence and "flat metric" convergence (see below for terminology) for a sequence of

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(*) Note that the word "flat" here has *no* physical or geometric significance, but relates rather to Whitney's use of the symbol ♭ (the "flat" symbol in musical notation) in his work. We mention this because it is often a source of confusion.

integer multiplicity currents $\{T_j\} \subset \mathcal{D}_n(U)$ such that $\sup_{j\geq 1}(\underline{\underline{M}}_W(T_j) + \underline{\underline{M}}_W(\partial T_j)) < \infty \quad \forall\, W \subset\subset U$.

We let U denote (as usual) an arbitrary open subset of $\mathbb{R}^{n+k}$,

$$\mathcal{I} = \{T \in \mathcal{D}_n(U) : T \text{ is integer multiplicity and } \underline{\underline{M}}_W(\partial T) < \infty \quad \forall\, W \subset\subset U\},$$

and

$$\mathcal{I}_{M,W} = \{T \in \mathcal{I} : \operatorname{spt} T \subset \overline{W},\ \underline{\underline{M}}(T) + \underline{\underline{M}}(\partial T) \leq M\}$$

for any $M > 0$ and $W \subset\subset U$.

On $\mathcal{I}$ we define a family of pseudometrics $\{d_W\}_{W\subset\subset U}$ by

31.1 $$d_W(T_1,T_2) = \inf\{\underline{\underline{M}}_W(S) + \underline{\underline{M}}_W(R) : T_1 - T_2 = \partial R + S, \text{ where } R \in \mathcal{D}_{n+1}(U),\ S \in \mathcal{D}_n(U) \text{ are integer multiplicity}\}$$

We henceforth assume $\mathcal{I}$ is equipped with the topology given (in the usual way) by the family $\{d_W\}_{W\subset\subset U}$ of pseudometrics. This topology is called the "flat metric topology" for $\mathcal{I}$: there is a countable base of neighbourhoods at each point, and $T_j \to T$ in this topology if and only if $d_W(T_j,T) \to 0 \quad \forall\, W \subset\subset U$.

31.2 THEOREM *Let* T, $\{T_j\} \subset \mathcal{D}_n(U)$ *be integer multiplicity with* $\sup_{j\geq 1}\{\underline{\underline{M}}_W(T_j) + \underline{\underline{M}}_W(\partial T_j)\} < \infty \quad \forall\, W \subset\subset U$. *Then* $T_j \to T$ *(in the sense of* 26.12) *if and only if* $d_W(T_j,T) \to 0$ *for each* $W \subset\subset U$.

31.3 REMARK Notice that no use is made of the compactness theorem 27.3 in this theorem; however if we combine the compactness theorem with it, then we get the statement that for any family of positive (finite) constants

$\{c(W)\}_{W\subset\subset U}$ the set $\{T\in \mathcal{I} : \underline{\underline{M}}_W(T)+\underline{\underline{M}}_W(\partial T)\le c(W) \quad \forall\, W\subset\subset U\}$ is sequentially compact when equipped with the flat metric topology.

Proof of 31.2 First note that the "if" part of the theorem is trivial (indeed for a given $W\subset\subset U$, the statement $d_W(T_j,T)\to 0$ evidently implies $(T_j-T)(\omega)\to 0$ for any fixed $\omega\in\mathcal{D}^n(U)$ with $\operatorname{spt}\omega\subset W$).

For the "only if" part of the theorem, the main difficulty is to establish the appropriate "total boundedness" property; specifically we show that for any given $\varepsilon>0$ and $W\subset\subset \tilde W\subset\subset U$, we can find $N=N(\varepsilon,W,\tilde W,M)$ and integer multiplicity currents $P_1,\ldots,P_N\in\mathcal{D}_n(U)$ such that

$$\mathcal{I}_{M,W}\subset\sum_{j=1}^{N}B_{\varepsilon,\tilde W}(P_j)\,, \tag{1}$$

where, for any $P\in\mathcal{I}$, $B_{\varepsilon,\tilde W}(P)=\{S\in\mathcal{I}: d_{\tilde W}(S,P)<\varepsilon\}$. This is an easy consequence of the deformation theorem: in fact for any $\rho>0$, 29.3 guarantees that for $T\in\mathcal{I}_{M,W}$ we can find integer multiplicity P, R, S such that

$$T-P=\partial R+S \tag{2}$$

$$P=\sum_{F\in\mathcal{F}_n(\rho)}\beta_F[\![F]\!]\,,\quad \beta_F\in\mathbb{Z} \tag{3}$$

$$\operatorname{spt}P\subset\{x:\operatorname{dist}(x,\operatorname{spt}T)<2\sqrt{n+k}\,\rho\} \tag{4}$$

$$\underline{\underline{M}}(P)\Bigl(\equiv\sum_{F\in\mathcal{F}_n(\rho)}|\beta_F|\rho^n\Bigr)\le c\underline{\underline{M}}(T)\le cM \tag{5}$$

$$\left\{\begin{aligned}&\operatorname{spt}R\cup\operatorname{spt}S\subset\{x:\operatorname{dist}(x,\operatorname{spt}T)<2\sqrt{n+k}\,\rho\}\\ &\underline{\underline{M}}(R)+\underline{\underline{M}}(S)\le c\rho\underline{\underline{M}}(T)\le c\rho M\,.\end{aligned}\right. \tag{6}$$

Then for ρ small enough to ensure $2\sqrt{n+k}\,\rho<\operatorname{dist}(W,\partial\tilde W)$, we see from (2), (6) that

$$d_{\tilde{W}}(T,P) \le c\rho M .$$

Hence, since there are only finitely many $P_1,\ldots,P_N$ currents P as in (3), (4), (5) (N depends only on M,W,n,k,ρ), we have (1) as required.

Next note that (by 28.5 (1), (2) and an argument as in 10.7(2)) we can find a subsequence $\{T_{j'}\} \subset \{T_j\}$ and a sequence $\{W_i\}$, $W_i \subset\subset W_{i+1} \subset\subset U$, $\bigcup_{i=1}^{\infty} W_i = U$, such that $\sup_{j'\ge 1} \underline{\underline{M}}(\partial(T_{j'} \llcorner W_i)) < \infty \quad \forall i$. Thus from now on we can assume without loss of generality that $W \subset\subset U$ and

$$\text{(7)} \qquad \operatorname{spt} T_j \subset \bar{W} \quad \forall j .$$

Then take any $\tilde{W}$ such that $W \subset\subset \tilde{W} \subset\subset U$ and apply (1) with $\varepsilon = 1, \frac{1}{2}, \frac{1}{4}$ etc. to extract a subsequence $\left\{T_{j_r}\right\}_{r=1,2,\ldots}$ from $\{T_j\}$ such that

$$d_{\tilde{W}}(T_{j_{r+1}}, T_{j_r}) < 2^{-r}$$

and hence

$$\text{(8)} \qquad T_{j_{r+1}} - T_{j_r} = \partial R_r + S_r$$

where R_r, S_r are integer multiplicity,

$$\operatorname{spt} R_r \cup \operatorname{spt} S_r \subset \tilde{W}$$

$$\underline{\underline{M}}(R_r) + \underline{\underline{M}}(S_r) \le \frac{1}{2^r} .$$

Therefore by 27.5 we can define integer multiplicity $R^{(\ell)}$, $S^{(\ell)}$ by the $\underline{\underline{M}}$- absolutely convergent series

$$R^{(\ell)} = \sum_{r=\ell}^{\infty} R_r , \quad S^{(\ell)} = \sum_{r=\ell}^{\infty} S_r ;$$

then

$$\underline{\underline{M}}(R^{(\ell)}) + \underline{\underline{M}}(S^{(\ell)}) \le 2^{-\ell+1}$$

and (from (8))

$$T - T_{j_\ell} = \partial R^{(\ell)} + S^{(\ell)} .$$

Thus we have a subsequence $\{T_{j_\ell}\}$ of $\{T_j\}$ such that $d_{\tilde W}(T,T_{j_\ell}) \to 0$. Since we can thus extract a subsequence converging relative to $d_{\tilde W}$ from any given subsequence of $\{T_j\}$, we then have $d_{\tilde W}(T,T_j) \to 0$; since this can be repeated with $W = W_i$, $\tilde W = W_{i+1}$ $\forall i$ (W_i as above), the required result evidently follows.

§32. RECTIFIABILITY THEOREM, AND PROOF OF THE COMPACTNESS THEOREM.

Here we prove the important rectifiability theorem for currents T which, together with ∂T , have locally finite mass and which have the additional property that $\Theta^{*n}(\mu_T,x) > 0$ for μ_T - a.e. x . The main tool of the proof is the structure theorem 13.2. Having established the rectifiability theorem, we show (in 32.2, 32.3) that it is then straightforward to establish the compactness theorem 27.3. Although this proof of compactness theorem has the advantage of being conceptually straightforward, it is rather lengthy if one takes into account the effort needed to prove the structure theorem. Recently B. Solomon [SB] showed that it is possible to prove the compactness theorem (and to develop the whole theory of integer multiplicity currents) without use of the structure theorm.

32.1 THEOREM (Rectifiability Theorem)

Suppose $T \in \mathcal{D}_n(U)$ *is such that* $\underline{\underline{M}}_W(T) + \underline{\underline{M}}_W(\partial T) < \infty$ $\forall\, W \subset\subset U$, *and* $\Theta^{*n}(\mu_T,x) > 0$ *for* μ_T - a.e. $x \in U$. *Then* T *is* *<u>rectifiable</u>* ; *that is*

$$T = \underline{\underline{\tau}}(M,\theta,\xi) \ ,^{(*)}$$

where M *is countably* n*-rectifiable,* $\mathcal{H}^n$*-measurable,* θ *is a positive locally* $\mathcal{H}^n$*-integrable function on* M *, and* $\xi(x)$ *orients the approximate tangent space* T_xM *of* M *for* $\mathcal{H}^n$ - a.e. $x \in M$.

Proof First note that (by Theorem 3.2(1))

$$\text{(1)} \qquad \mathcal{H}^n\{x \in W : \Theta^{*n}(\mu_T, x) > K\} \le K^{-1}\underline{\underline{M}}_W(T)$$

for $W \subset\subset U$, and hence

$$\text{(2)} \qquad \mathcal{H}^n\{x \in U : \Theta^{*n}(\mu_T, x) = \infty\} = 0 \ .$$

Notice that the same argument applies with ∂T in place of T in order to give

$$\text{(3)} \qquad \mathcal{H}^n\{x \in U : \Theta^{*n}(\mu_{\partial T}, x) = \infty\} = 0 \ .$$

(Notice we could also conclude $\mathcal{H}^d\{x \in U : \Theta^{*d}(\mu_{\partial T}, x) = \infty\} = 0$ for any $d > 0$ by 3.2(1).)

Next notice that, because $\underline{\underline{M}}_W(T) + \underline{\underline{M}}_W(\partial T) < \infty \quad \forall\, W \subset\subset U$, we know from 26.29 (see in particular Remark 26.30) that (by (2))

$$\text{(4)} \qquad \mu_T\{x \in U : \Theta^{*n}(\mu_T, x) = \infty\} = 0 \ ,$$

and (by (3))

$$\text{(5)} \qquad \mu_T\{x \in U : \Theta^{*n}(\mu_{\partial T}, x) = \infty\} = 0 \ .$$

(*) The notation here is as for integer multiplicity rectifiable currents (§27):

$$\underline{\underline{\tau}}(M,\theta,\xi)(\omega) = \int_M \langle \xi, \omega \rangle \theta \, d\mathcal{H}^n \ ,$$

although of course θ is not assumed to be integer-valued here.

Now let

$$M = \{x \in U : \Theta^{*n}(\mu_T, x) > 0\}$$

and note by (1) that M is the countable union of sets of finite $\mathcal{H}^n$-measure. Furthermore by 26.29 we know that $\mu_T(P) = 0$ for each purely unrectifiable subset of M, and hence

(6) $$\mathcal{H}^n(P) = 0 \quad \forall \text{ purely unrectifiable } P \subset M$$

by virtue of 3.2(1) and the fact that $\Theta^{*n}(\mu_T, x) > 0$ for every $x \in M$ (by definition of M). Then by the structure theorem 13.2 we deduce that

(7) M *is countably* n-*rectifiable.*

Furthermore (since $\Theta^{*n}(\mu_T, x) > 0$ for μ_T - a.e. $x \in U$ by assumption), we have

(8) $$T = T \llcorner M .$$

Next we note that μ_T is absolutely continuous with respect to $\mathcal{H}^n$ (by (4) and 3.2(2)), and hence by the differentiation theorem 4.7 we have

$$\mu_T = \mathcal{H}^n \llcorner \theta$$

where θ is a positive locally $\mathcal{H}^n$-integrable function on M and $\theta \equiv 0$ on $U \sim M$. Then by the Riesz representation theorem 4.1 we have

(9) $$T(\omega) = \int_U \langle \xi, \omega \rangle \theta \, d\mathcal{H}^n ,$$

for some $\mathcal{H}^n$-measurable, $\Lambda_n(\mathbb{R}^{n+k})$-valued function ξ, $|\xi| = 1$.

It thus remains only to prove that $\xi(x)$ orients T_xM for $\mathcal{H}^n$ - a.e. $x \in M$. (i.e. $\xi(x) = \pm\tau_1 \wedge \ldots \wedge \tau_n$ for $\mathcal{H}^n$ - a.e. $x \in M$, where $\tau_1, \ldots, \tau_n$ is any orthonormal basis for the approximate tangent space T_xM of M.) To see this, write $M = \bigcup_{j=0}^{\infty} M_j$, M_j pairwise disjoint, $\mathcal{H}^n(M_0) = 0$, $M_j \subset N_j$, N_j a C^1 submanifold of $\mathbb{R}^{n+k}$, $j \geq 1$. Now, by 3.5, if $j \geq 1$ we have, for $\mathcal{H}^n$-a.e. $x \in M_j$,

$$\Theta^{*n}(\mu, \bigcup_{r \neq j} M_r, x) = 0 . \tag{10}$$

Hence, writing as usual $\eta_{x,\lambda}(y) = \lambda^{-1}(y-x)$, we have for any $\omega \in \mathcal{D}^n(\mathbb{R}^{n+k})$ that, for all $x \in M_j$ such that (10) holds, and for λ small enough to ensure that $\operatorname{spt} \omega \subset \eta_{x,\lambda}(U)$,

$$\begin{aligned}\eta_{x,\lambda\#}T(\omega) &= T(\eta_{x,\lambda}^{\#}\omega) \\ &= \int_{N_j} \langle \xi, \eta_{x,\lambda}^{\#}\omega \rangle \, \theta d\mathcal{H}^n + \varepsilon(\lambda) ,\end{aligned}$$

where $\varepsilon(\lambda) \to 0$ as $\lambda \downarrow 0$. ($\varepsilon(\lambda)$ depending on x and ω .) That is

$$\eta_{x,\lambda\#}T(\omega) = \int_{\eta_{x,\lambda}(N_j)} \langle \xi(x+\lambda z), \omega(z) \rangle \theta(x+\lambda z) \, d\mathcal{H}^n(z) + \varepsilon(\lambda)$$

for all $x \in M_j$ such that (10) holds. Since N_j is C^1 , this gives

$$\lim_{\lambda \downarrow 0} \eta_{x,\lambda\#}T(\omega) = \theta(x) \int_P \langle \xi(x) , \omega(z) \rangle \, d\mathcal{H}^n(z) \tag{11}$$

for $\mathcal{H}^n$-a.e. $x \in M_j$ (independent of ω) , where P is the tangent space T_xN_j of N_j at x . Thus (by definition of T_xM - see §12) we have (11) with $P = T_xM$ for $\mathcal{H}^n$-a.e. $x \in M_j$. On the other hand by (5) we have

$$\begin{aligned}\partial\eta_{x,\lambda\#}T(\omega) &= \eta_{x,\lambda\#}\partial T(\omega) = \partial T(\eta_{x,\lambda}^{\#}\omega) \\ &= o(\lambda) \quad \text{as} \quad \lambda \downarrow 0\end{aligned}$$

for $\mathcal{H}^n$-a.e. $x \in M_j$ (independent of ω) . Thus for such x

(12) $$\lim_{\lambda\downarrow 0}(\partial\eta_{x,\lambda\#}T)(\omega) = 0 .$$

On the other hand for μ_T - a.e. $x \in U$, for any $W \subset\subset \mathbb{R}^{n+k}$, we have by (4) that

(13) $$\limsup_{\lambda\downarrow 0} \underline{\underline{M}}_W(\eta_{x,\lambda\#}T) < \infty .$$

Thus (by (11), (12), (13)), for $\mathcal{H}^n$ - a.e. $x \in M$, we can find a sequence $\lambda_\ell \downarrow 0$ such that

$$\eta_{x,\lambda_\ell\#}T \rightharpoonup S_x , \quad \partial S_x = 0 ,$$

where $S_x \in \mathcal{D}_n(\mathbb{R}^{n+k})$ is defined by

(14) $$S_x(\omega) = \theta(x) \int_P \langle\xi(x) , \omega(z)\rangle \, d\mathcal{H}^n(z) ,$$

$\omega \in \mathcal{D}^n(\mathbb{R}^{n+k})$, $P = T_xM$. We now claim that (14), taken together with the fact that $\partial S_x = 0$, implies that $\xi(x)$ orients P. To see this, assume (without loss of generality) that $P = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+k}$ and select $\omega \in \mathcal{D}^{n-1}(\mathbb{R}^{n+k})$ so that $\omega(y) = y^j\phi(y)\,dy^{i_1} \wedge \ldots \wedge dy^{i_{n-1}}$, where $y = (y^1,\ldots,y^{n+k})$, $j \geq n+1$, $\{i_1,\ldots,i_{n-1}\} \subset \{1,\ldots,n+k\}$, and $\phi \in C_c^\infty(\mathbb{R}^{n+k})$. Then since $y_j \equiv 0$ on $\mathbb{R}^n \times \{0\}$ we deduce, from (14) and the fact that $\partial S_x = 0$,

$$0 = \partial S_x(\omega) = S_x(d\omega) = \theta(x) \int_P \phi(y)\langle\xi(x), dy^j \wedge dy^{i_1} \wedge \ldots \wedge dy^{i_{n-1}}\rangle$$

$$= \theta(x) \int_P \phi(y)\xi(x)\cdot(e_j \wedge e_{i_1} \wedge \ldots \wedge e_{i_{n-1}})\,d\mathcal{H}^n(y) .$$

That is, since $\phi \in C_c^\infty(\mathbb{R}^{n+k})$ is arbitrary, we deduce that $\xi(x)\cdot(e_j \wedge e_{i_1} \wedge \ldots \wedge e_{i_{n-1}}) = 0$ whenever $j \geq n+1$ and $\{i_1,\ldots,i_{n-1}\} \subset \{1,\ldots,n+k\}$. Thus we must have (since $|\xi(x)| = 1$), $\xi(x) = \pm e_1 \wedge \ldots \wedge e_n$ as required.

We can now give the proof of the compactness theorem 27.3. For convenience we first re-state the theorem in a slightly weaker form. (See the remark (2) following the statement for the proof that the previous version 27.3 follows.)

32.2 THEOREM *Suppose* $\{T_j\} \subset \mathcal{D}_n(U)$, *suppose* T_j , ∂T_j *are integer multiplicity for each* j ,

$$(*) \qquad \sup_{j\geq 1}(\underline{\underline{M}}_W(T_j) + \underline{\underline{M}}_W(\partial T_j)) < \infty \qquad \forall\, W \subset\subset U ,$$

and suppose $T_j \rightharpoonup T \in \mathcal{D}_n(U)$. *Then* T *is an integer multiplicity current.*

32.3 REMARKS

(1) Note that the general case of the theorem follows from the special case when $U = \mathbb{R}^P$ and $\operatorname{spt} T_j \subset K$ for some fixed compact K ; in fact if T_j are as in the theorem and if $\xi \in U$, then by 28.5 (1), (2) and an argument like that in Remark 10.7(2) we know that, for $\mathcal{L}^1$-a.e. $r > 0$, $\partial(T_j \llcorner B_r(\xi))$ are integer multiplicity and (*) holds with $T_j \llcorner B_r(\xi)$ in place of T_j for some subsequence $\{j'\} \subset \{j\}$ (depending on r) .

(2) The previous (formally slightly stronger) version 27.3 of the above theorem follows by using 30.3. (Note that the proof of 30.3 needed only the weaker version of the compactness theorem given above in 32.2; indeed, as mentioned in Remark 30.4, it used only the case $\partial T_j = 0$ of 27.3.

Proof of 32.2 We shall use induction on n with $U \subset \mathbb{R}^P$ (U,P fixed independent of n) . First note that the theorem is trivial in case $n = 0$. Then assume $n \geq 1$ and suppose the theorem is true with $n-1$ in place of n .

By the above remark (1) we shall assume without loss of generality that $\operatorname{spt} T_j \subset K$ for some fixed compact K , and that $U = \mathbb{R}^P$. Furthermore, by remark (1) in combination with the inductive hypothesis, for each $\xi \in \mathbb{R}^P$ we have

(1) $\partial(T \llcorner B_r(\xi))$ *is an integer multiplicity current*

(in $\mathcal{D}_{n-1}(\mathbb{R}^P)$) for $\mathcal{L}^1$-a.e. $r > 0$.

From the above assumptions $U = \mathbb{R}^P$, spt $T_j \subset K$ we know that $0 \times\!\!\!\!\times \partial T - T$ zero boundary and is the weak limit of $0 \times\!\!\!\!\times \partial T_j - T_j$; since $0 \times\!\!\!\!\times \partial T$ is integer multiplicity (by the inductive hypothesis) we thus see that the general case of the theorem follows from the special case when $\partial T = 0$. We shall therefore henceforth also assume $\partial T = 0$.

Next, define (for $\xi \in \mathbb{R}^P$ fixed)

$$f(r) = \underline{\underline{M}}(T \llcorner B_r(\xi)), \quad r > 0 .$$

By virtue of 28.9 we have (since $\partial T = 0$)

(2) $$\underline{\underline{M}}(\partial(T \llcorner B_r(\xi))) \le f'(r), \quad \mathcal{L}^1\text{-a.e. } r > 0 .$$

(Notice that $f'(r)$ exists a.e. $r > 0$ because $f(r)$ is increasing.) On the other hand if $\Theta^{*n}(\mu_T, \xi) < \eta$ ($\eta > 0$ a given constant), then $\limsup_{\rho \downarrow 0} \frac{f(\rho)}{\omega_n \rho^n} < \eta$, and hence for each $\delta > 0$ we can arrange

(3) $$\frac{d}{dr}(f^{1/n}(r)) \le 2\omega_n^{1/n}\eta$$

for a set of $r \in (0,\delta)$ of positive $\mathcal{L}^1$-measure. (Because $\delta^{-1}\int_0^\delta \frac{d}{dr}(f^{1/n}(r))\,dr \le \delta^{-1} f^{1/n}(\delta) \le \omega_n^{1/n}\eta$ for all sufficiently small $\delta > 0$.)

Now by (1) and the isoperimetric theorem, we can find an integer multiplicity $S_r \in \mathcal{D}_n(\mathbb{R}^P)$ such that $\partial S_r = \partial(T \llcorner B_r(\xi))$ and

$$(4)\qquad \underline{\underline{M}}(S_r)^{\frac{n-1}{n}} \le c\underline{\underline{M}}(\partial(T \llcorner B_r(\xi))) \le c\eta\underline{\underline{M}}(T \llcorner B_r(\xi))^{\frac{n-1}{n}} \qquad \text{(by (2),(3))}$$

for a set of r of positive L^1-measure in $(0,\delta)$.* Since δ was arbitrary we then have both (1), (4) for a sequence of $r \downarrow 0$. But then (since we can repeat this for any ξ such that $\Theta^{*n}(\mu_T,\xi) < \eta$) if C is any compact subset of $\{x \in \mathbb{R}^P : \Theta^{*n}(\mu_T,x) < \eta\}$, by Remark 4.5(2) we get for each given $\rho > 0$ a pairwise disjoint family $B_j = \bar{B}_{r_j}(\xi_j)$ of closed balls covering μ_T-almost all of C, with

$$(5)\qquad \bigcup_j B_j \subset \{x : \text{dist}(x,C) < \rho\}$$

and with

$$(6)\qquad \underline{\underline{M}}(S_j^{(\rho)}) \le c\eta\underline{\underline{M}}(T \llcorner B_j)$$

for some integer multiplicity $S_j^{(\rho)}$ with

$$(7)\qquad \partial S_j^{(\rho)} = \partial(T \llcorner B_j) .$$

Now because of (7) we have $S_j^{(\rho)} - T \llcorner B_j = \partial(\{\xi_j\} \times\!\!\!\!\times (S_j^{(\rho)} - T \llcorner B_j))$, and hence (by 26.23, 26.26) we have for $\omega \in \mathcal{D}^n(\mathbb{R}^P)$

$$(8)\qquad |(S_j^{(\rho)} - T \llcorner B_j)(\omega)| \le c\rho\underline{\underline{M}}(S_j^{(\rho)} - T \llcorner B_j)|d\omega| \le c\rho\underline{\underline{M}}(T \llcorner B_j)|d\omega| \qquad \text{(by (6))}.$$

Therefore we have $\sum_j (S_j^{(\rho)} - T \llcorner B_j) \rightharpoonup 0$ as $\rho \downarrow 0$, and hence

$$(9)\qquad T + \sum_j (S_j^{(\rho)} - T \llcorner B_j) \rightharpoonup T$$

as $\rho \downarrow 0$. However since the series $\sum_j S_j^{(\rho)}$ and $\sum_j T \llcorner B_j$ are $\underline{\underline{M}}$-absolutely convergent (by (6) and the fact that the B_j are disjoint), we deduce that the left side in (9) can be written $T \llcorner (\mathbb{R}^P \sim \bigcup_j B_j) + \sum_j S_j^{(\rho)}$ and hence

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* In case $n = 1$, (1), (2), (3) (for $\eta < \frac{1}{2}$) imply $\partial(T \llcorner B_r(\xi)) = 0$, hence we get, in place of (4), $\underline{\underline{M}}(S_r) \le \underline{\underline{M}}(T \llcorner B_r(\xi))$ trivially by taking $S_r = 0$.

(using (6) again, together with the lower-semicontinuity of $\underline{\underline{M}}_W$ (W open) under weak convergence)

$$\mu_T(\{x : \mathrm{dist}(x,C) < \rho\}) \le \mu_T(\{x : \mathrm{dist}(x,C) < \rho\} \sim C) +$$

$$c\eta\mu_T(\{x : \mathrm{dist}(x,C) < \rho\}) .$$

Choosing η such that $c\eta \le \frac{1}{2}$, this gives

$$\mu_T(\{x : \mathrm{dist}(x,C) < \rho) \le 2\mu_T(\{x : \mathrm{dist}(x,C) < \rho\} \sim C\} .$$

Letting $\rho \downarrow 0$, we get $\mu_T(C) = 0$.

Thus we have shown that $\Theta^{*n}(\mu_T,x) > 0$ for μ_T - a.e. $x \in \mathbb{R}^P$. We can therefore apply 32.1 in order to conclude that $T = \underline{\underline{\tau}}(M,\theta,\xi)$ as in 32.1. It thus remains only to prove that θ is integer-valued. This is achieved as follows:

First note that for L^n - a.e. $x \in M$ we have (cf. the argument leading to (11) in the proof of 32.1)

$$\text{(10)} \qquad \eta_{x,\lambda\#}T \rightharpoonup \theta(x)[\![T_xM]\!] \quad \text{as} \quad \lambda \downarrow 0 ,$$

where $[\![T_xM]\!]$ is oriented by $\xi(x)$. Assuming without loss of generality that $T_xM = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^P$ and setting $d(y) = \mathrm{dist}(y, \mathbb{R}^n \times \{0\})$, by 28.5(1) we can find a sequence $\lambda_j \downarrow 0$ and a $\rho > 0$ such that $\langle \eta_{x,\lambda_j\#}T, d, \rho\rangle$ is integer multiplicity with

$$\underline{\underline{M}}_\Omega(\langle \eta_{x,\lambda_j\#}T, d, \rho\rangle) \le c \quad (\text{independent of } j)$$

where $\Omega = B_1^n(0) \times \mathbb{R}^{P-n} \subset \mathbb{R}^P$. Then by 28.5(2) we have $S_j \equiv (\eta_{x,\lambda_j\#}T) \llcorner \{y : d(y) < \rho\}$ is such that, writing $\Omega = B_1^n(0) \times \mathbb{R}^{P-n} \subset \mathbb{R}^P$,

$$\text{(11)} \qquad \sup_{j \ge 1}(\underline{\underline{M}}_\Omega(S_j) + \underline{\underline{M}}_\Omega(\partial S_j)) < \infty .$$

Now let p denote the restriction to Ω of the orthogonal projection of $\mathbb{R}^P$ onto $\mathbb{R}^n$; and let $\tilde{S}_j$ be the current in $\mathcal{D}_n(\Omega)$ obtained by setting $\tilde{S}_j(\omega) = S_j(\tilde{\omega})$, $\omega \in \mathcal{D}^n(\Omega)$, $\tilde{\omega} \in \mathcal{D}^n(\mathbb{R}^P)$ such that $\tilde{\omega} = \omega$ in Ω and $\tilde{\omega} \equiv 0$ on $\mathbb{R}^P \sim \Omega$. Then we have $p_{\#}\tilde{S}_j \in \mathcal{D}_n(B_1^n(0))$, and hence, by 26.28 and (11) above,

$$p_{\#}\tilde{S}_j(\omega) = \int_{B_1^n(0)} a\theta_j \, d\mathcal{L}^n \,, \ \omega = a dx^1 \wedge \ldots \wedge dx^n \,, \ a \in C_c^\infty(\mathbb{R}^n) \,,$$

for some integer-valued $BV_{loc}(B_1^n(0))$ function θ_j with

$$(12) \qquad \begin{cases} \underline{\underline{M}}_{B_1^n(0)}(p_{\#}\tilde{S}_j) = \displaystyle\int_{B_1^n(0)} |\theta_j| \, d\mathcal{L}^n \\[2ex] \underline{\underline{M}}_{B_1^n(0)}(\partial p_{\#}\tilde{S}_j) = \displaystyle\int_{B_1^n(0)} |D\theta_j| \,. \end{cases}$$

Then by (11), (12) we deduce $\int_{B_1^n(0)} |D\theta_j| + \int_{B_1^n(0)} |\theta_j| \, d\mathcal{L}^n \le c$, c independent of j, and hence by the compactness theorem 6.3 we know θ_j converges strongly in L^1 in $B_1^n(0)$ to an integer-valued BV function θ_*. On the other hand $S_j \rightharpoonup \theta(x)[\![\mathbb{R}^n \times \{0\}]\!]$ by (10), and hence $p_{\#}\tilde{S}_j \rightharpoonup \theta(x) p_{\#}[\![\mathbb{R}^n \times \{0\}]\!] = \theta(x)[\![\mathbb{R}^n]\!]$ in $B_1^n(0)$. We thus deduce that $\theta_* \equiv \theta(x)$ in $B_1^n(0)$; thus $\theta(x) \in \mathbb{Z}$ as required.