CHAPTER 1

PRELIMINARY MEASURE THEORY

In this chapter we briefly review the basic theory of outer measure (with Caratheodory's definition of measurability). Hausdorff measure is discussed, including the main results concerning n-dimensional densities and the way in which they relate more general measures to Hausdorff measures. The final section of the chapter gives the basic theory of Radon measures (including the Riesz representation theorem and the differentiation theory).

Throughout the chapter X will denote a metric space with metric d. In the last section X satisfies the additional requirements of being locally compact and separable.

§1. BASIC NOTIONS

Recall that an *outer measure* (henceforth simply called a *measure*) on X is a *monotone subadditive* function $\mu : 2^X \to [0,\infty]$ with $\mu(\phi) = 0$. Thus $\mu(\phi) = 0$ and

$$\mu(A) \le \sum_{j=1}^{\infty} \mu(A_j) \quad \text{whenever} \quad A \subset \bigcup_{j=1}^{\infty} A_j$$

with $A, A_1, A_2, \ldots$ any countable collection of subsets of X. Of course this in particular implies $\mu(A) \le \mu(B)$ whenever $A \subset B$.

We adopt Caratheodory's notion of *measurability* :

A subset $A \subset X$ is said to be μ-measurable if

$$\mu(S) = \mu(S \sim A) + \mu(S \cap A)$$

for each subset $S \subset X$. Of course by subadditivity of μ we only actually have to check that

1.1 $$\mu(S) \geq \mu(S \sim A) + \mu(S \cap A)$$

for each subset $S \subset X$ with $\mu(S) < \infty$. One readily checks (see for example [M] or [FH1]) that the collection $\mathcal{S}$ of all measurable subsets forms a *σ-algebra*; that is

(1) ϕ, $X \in \mathcal{S}$

(2) If $A_1, A_2, \ldots \in \mathcal{S}$ then $\bigcup_{j=1}^{\infty} A_j$ and $\bigcap_{j=1}^{\infty} A_j \in \mathcal{S}$

(3) If $A \in \mathcal{S}$ then $X \sim A \in \mathcal{S}$.

Furthermore all sets of μ-measure zero are trivially μ-measurable (because 1.1 holds trivially in case $\mu(A) = 0$). If $A_1, A_2, \ldots$ are pairwise disjoint μ-measurable subsets of X, then $\mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$. If $A_1 \subset A_2 \subset \ldots$ are μ-measurable then $\lim_{i\to\infty} \mu(A_i) = \mu(\bigcup_{i=1}^{\infty} A_i)$ and if $A_1 \supset A_2 \supset \ldots$ are μ-measurable then $\lim_{i\to\infty} \mu(A_i) = \mu(\bigcap_{i=1}^{\infty} A_i)$ provided $\mu(A_1) < \infty$.

A measure μ on X is said to be *regular* if for each subset $A \subset X$ there is a μ-measurable subset $B \supset A$ with $\mu(B) = \mu(A)$. One readily checks that for a regular measure μ the relation $\lim_{i\to\infty} \mu(A_i) = \mu(\bigcup_{i=1}^{\infty} A_i)$ is valid provided $A_i \subset A_{i+1}\ \forall i$, *even if the* A_i *are not assumed to be* μ-*measurable*.

A measure X is said to be *Borel-regular* if all Borel sets are μ-measurable and if for each subset $A \subset X$ there is a Borel set $B \supset A$ such that $\mu(B) = \mu(A)$. (Notice that this does *not* imply $\mu(B \sim A) = 0$ unless A

is μ-measurable and $\mu(A) < \infty$.)

Given any subset $A \subset X$ and any measure μ on X, we can define a new measure $\mu \llcorner A$ on X by

$$(\mu \llcorner A)(Z) = \mu(A \cap Z) \ , \ Z \subset X \ .$$

One readily checks that all μ-measurable subsets are also $(\mu \llcorner A)$-measurable (even if A is *not* μ-measurable). It is also easy to check that $\mu \llcorner A$ is Borel regular whenever μ is, provided A is μ-measurable.

The following theorem, due to Caratheodory, is particularly useful. In the statement we use the notation

$$d(A,B) = \operatorname{dist}(A,B) = \inf\{d(a,b) : a \in A \ , \ b \in B\} \ .$$

1.2 THEOREM (Caratheodory's Criterion) *If* μ *is a measure on* X *such that*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

whenever A, B *are subsets of* X *with* $d(A,B) > 0$, *then all Borel sets are* μ*-measurable.*

Proof Since the measurable sets form a σ-algebra, it is enough to prove that all closed sets are μ-measurable, so that by 1.1 we have only to check that

$$(1) \qquad \mu(S) \geq \mu(S \sim C) + \mu(S \cap C)$$

whenever $\mu(S) < \infty$ and C is closed.

Let $C_j = \{x \in X : \operatorname{dist}(x,C) \leq 1/j\}$. Then $d(S \sim C_j, S \cap C) > 0$, hence

$$\mu(S) \geq \mu((S \sim C_j) \cup (S \cap C)) = \mu(S \sim C_j) + \mu(S \cap C) \ ,$$

and we will have (1) if we can show $\lim_{j\to\infty} \mu(S\sim C_j) = \mu(S\sim C)$. To check this, note that since C is closed we can write

$$S \sim C = (S\sim C_j) \cup \left(\bigcup_{k=j}^{\infty} R_k\right)$$

where $R_k = \{x \in S : \frac{1}{k+1} < \operatorname{dist}(x,C) \le \frac{1}{k}\}$. But then by subadditivity of μ we have

$$\mu(S\sim C_j) \le \mu(S\sim C) \le \mu(S\sim C_j) + \sum_{k=j}^{\infty} \mu(R_k) ,$$

and hence we will have $\lim_{j\to\infty} \mu(S\sim C_j) = \mu(S\sim C)$ as required, provided only that $\sum_{k=1}^{\infty} \mu(R_k) < \infty$.

To check this we note that $d(R_i, R_j) > 0$ if $j \ge i+2$, and hence by the hypothesis of the theorem and induction on N we have for each integer $N \ge 1$

$$\sum_{k=1}^{N} \mu(R_{2k}) = \mu\left(\bigcup_{k=1}^{N} R_{2k}\right) \le \mu(S) < \infty$$

and

$$\sum_{k=1}^{N} \mu(R_{2k-1}) = \mu\left(\bigcup_{k=1}^{N} R_{2k-1}\right) \le \mu(S) < \infty .$$

The following regularity properties of Borel-regular measures are of basic importance.

1.3 THEOREM *Suppose* μ *is a Borel-regular measure on* X *and* $X = \bigcup_{j=1}^{\infty} V_j$, *where* $\mu(V_j) < \infty$ *and* V_j *is open for each* $j = 1,2,\ldots$. *Then*

(1) $$\mu(A) = \inf_{U \text{ open}, U\supset A} \mu(U)$$

for each subset $A \subset X$, *and*

(2) $$\mu(A) = \sup_{C \text{ closed}, C \subset A} \mu(C)$$

for each μ*-measurable subset* $A \subset X$.

1.4 REMARK In case the metric space X is *locally compact* and *separable, the condition* $X = \bigcup_{j=1}^{\infty} V_j$ *with* V_j *open and* $\mu(V_j) < \infty$ *is automatically satisfied provided* $\mu(K) < \infty$ *for each compact* K . Furthermore in this case we have from 1.3(2) that

$$\mu(A) = \sup_{K \text{ compact}, K \subset A} \mu(K)$$

for each μ-measurable subset $A \subset X$ with $\mu(A) < \infty$, because under the above conditions on X any closed set C can be written $C = \bigcup_{i=1}^{\infty} K_i$, compact.

Proof of Theorem 1.3 First note that 1.3(2) follows quite easily from 1.3(1). To prove 1.3(1), we assume first that $\mu(X) < \infty$. By Borel regularity of the measure μ , it is enough to prove (1) in case A is a Borel set. Then let

$$\mathcal{A} = \{\text{Borel sets } A : 1.3(1) \text{ holds}\} .$$

Trivially $\mathcal{A}$ contains all open sets and one readily checks that $\mathcal{A}$ is closed under both countable unions and intersections; in particular $\mathcal{A}$ must also contain the *closed sets,* because any closed set in X can be written as a countable intersection of open sets. Thus if we let $\tilde{\mathcal{A}} = \{A \in \mathcal{A} : X \sim A \in \mathcal{A}\}$ then $\tilde{\mathcal{A}}$ is a σ-algebra containing all the closed sets, and hence $\tilde{\mathcal{A}}$ contains all the Borel sets. Thus $\mathcal{A}$ contains all the Borel sets and 1.3(1) is proved in case $\mu(X) < \infty$.

In the general case $(\mu(X) \le \infty)$ it still suffices to prove 1.3(1) when A is a Borel set. For each $j = 1,2,\ldots$ apply the previous case to the measure $\mu \llcorner V_j$, $j = 1,2,\ldots$. Then for each $\varepsilon > 0$ we can select an open

$U_j \supset A$ such that

$$\mu(U_j \cap V_j \sim A \cap V_j) < \varepsilon/2^j ,$$

so that

$$\mu(U_j \cap V_j \sim A) < \varepsilon/2^j ,$$

and hence (summing over j)

$$\mu\Big(\bigcup_{j=1}^{\infty} (U_j \cap V_j) \sim A\Big) < \varepsilon .$$

Since $\bigcup_{j=1}^{\infty} (U_j \cap V_j)$ is open and contains A, this completes the proof.

§2. HAUSDORFF MEASURE

If m is a non-negative real number, we define m-dimensional Hausdorff measure by

2.1 $$\mathcal{H}^m(A) = \lim_{\delta \downarrow 0} \mathcal{H}^m_\delta(A) , \quad A \subset X ,$$

where for each $\delta > 0$, $\mathcal{H}^m_\delta(A)$ is defined by

2.2 $$\mathcal{H}^m_\delta(A) = \inf \sum_{j=1}^{\infty} \omega_m \left(\frac{\operatorname{diam} C_j}{2}\right)^m$$

(ω_m = volume of unit ball in $\mathbb{R}^m$ in case m is a positive integer; ω_m any convenient constant > 0 otherwise), where the inf is taken over all countable collections $C_1, C_2, \ldots$ of subsets of X such that $\operatorname{diam} C_j < \delta$ and $A \subset \bigcup_{j=1}^{\infty} C_j$.

Notice that the limit in 2.1 always exists (although it may be $+\infty$) because $\mathcal{H}^m_\delta(A)$ is a decreasing function of δ; thus $\mathcal{H}^m(A) = \sup_{0<\delta} \mathcal{H}^m_\delta(A)$.

2.3 REMARKS

(1) Since $\operatorname{diam} C_j = \operatorname{diam} \bar{C}_j$ we can add the additional requirement in definition 2.2 that the C_j be *closed* without changing the value of $\mathcal{H}^m(A)$; indeed since for any $\varepsilon > 0$ we can find an open set $U_j \supset C_j$ with $\operatorname{diam} U_j < \operatorname{diam} C_j + \varepsilon/2^j$, we could also take the C_j to be *open* , *except in case* $m = 0$.

(2) Evidently $\mathcal{H}^m_\delta(A) < \infty$ $\forall m \geq 0$, $\delta > 0$ in case A is a totally bounded subset of X .

One easily checks from the definition of $\mathcal{H}^m_\delta$ that

$$\mathcal{H}^m_\delta(A \cup B) = \mathcal{H}^m_\delta(A) + \mathcal{H}^m_\delta(B) \quad \text{if} \quad d(A,B) > 2\delta ,$$

hence

$$\mathcal{H}^m(A \cup B) = \mathcal{H}^m(A) + \mathcal{H}^m(B) \quad \text{whenever} \quad d(A,B) > 0 ,$$

and therefore all Borel sets are $\mathcal{H}^m$-measurable by the Caratheodory criterion 1.2. It follows from this and Remark 2.3(1) that *each of the measures* $\mathcal{H}^m$ *is Borel-regular.*

<u>Note</u>: It is *not* true in general that the Borel sets are $\mathcal{H}^m_\delta$-measurable for $\delta > 0$; for instance if $n \geq 2$ then one easily checks that the half-space $\{x = (x^1,\ldots,x^n) \in \mathbb{R}^n : x_n > 0\}$ is not $\mathcal{H}^1_\delta$-measurable.

We will later show that for each integer $n \geq 1$ $\mathcal{H}^n$ agrees with the usual definition of n-dimensional volume measure on an n-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$, $k \geq 0$. As a first step we want to prove that $\mathcal{H}^n$ and $\mathcal{L}^n$ (n-dimensional Lebesgue measure) agree on $\mathbb{R}^n$. First recall one of the standard definitions of $\mathcal{L}^n$:

If K denotes the collection of all "n-dimensional cubes" I of the form $I = (a_1,a_1+\ell) \times (a_2,a_2+\ell) \times \ldots \times (a_n,a_n+\ell)$, where $a_i \in \mathbb{R}$ and $\ell > 0$, and if $|I|$ = volume of $I = \ell^n$, then

2.4 $$L^n(A) = \inf \sum_j |I_j|$$

where the inf is taken over all countable (or finite) collections $\{I_1,I_2,\ldots\} \subset K$ with $A \subset \bigcup_j I_j$. One easily checks that L^n is *uniquely characterized* among measures on $\mathbb{R}^n$ by the properties

$$L^n(I) = |I| \ \forall I \in K \ , \quad L^n(A) = \inf_{\substack{U \supset A \\ U \text{ open}}} L^n(U) \ \forall A \subset \mathbb{R}^n \ .$$

We can now show

(*) $$H^n_\delta(A) \le L^n(A) \quad \forall \delta > 0$$

as follows. Let $\varepsilon > 0$ and choose $I_1,I_2,\ldots \in K$ so that $A \subset \bigcup_k I_k$ and

$$\sum_k |I_k| \le L^n(A) + \varepsilon \ .$$

Now for each bounded open set $U \subset \mathbb{R}^n$ and each $\delta > 0$ we can select a pair-wise disjoint family of closed balls $B_1,B_2,\ldots$ with $\bigcup_{j=1}^\infty B_j \subset U$, diam $B_j < \delta$, and $L^n(U \sim \bigcup_{j=1}^\infty B_j) = 0$. (To see this first decompose U as a union $\bigcup_{j=1}^\infty C_j$ of closed cubes C_j of diameter $< \delta$ and with pairwise disjoint interiors, and for each $j \ge 1$ select a ball $B_j \subset$ interior C_j with diam $B_j > \frac{1}{2}$ edge-length of C_j . Then $L^n(B_j) > \theta_n L^n(C_j)$, $\theta_n = \omega_n/4^n$, and it follows $L^n(U \sim \bigcup_{j=1}^\infty B_j) < (1-\theta_n)L^n(U)$. Thus $L^n(U \sim \bigcup_{j=1}^N B_j) \le (1-\theta_n)L^n(U)$ for suitable $\theta_n \in (0,1)$. Since $U \sim \bigcup_{j=1}^N B_j$ is open, we can repeat the argument inductively to obtain the required collection of balls.) Then take $U = I_k$ and such a collection of balls $\{B_j\}$. Since $L^n(Z) = 0 \Rightarrow H^n_\delta(Z) = 0$ for each subset $Z \subset X$ (by definitions 2.2, 2.4) we then have (writing ρ_j = radius B_j)

$$\mathcal{H}^n_\delta(I_k) = \mathcal{H}^n_\delta(\bigcup_{j=1}^{\infty} B_j) \le \sum_{j=1}^{\infty} \omega_n \rho_j^n$$

$$= \sum_{j=1}^{\infty} \mathcal{L}^n(B_j) = \mathcal{L}^n(\bigcup_{j=1}^{\infty} B_j) = \mathcal{L}^n(I_k) = |I_k| ,$$

and hence

$$\mathcal{H}^n_\delta(A) \le \mathcal{H}^n_\delta(\bigcup_k I_k) \le \sum_k \mathcal{H}^n_\delta(I_k) \le \mathcal{L}^n(A) + \varepsilon .$$

Thus 2.5 is established.

To prove the reverse inequality

(**) $$\mathcal{L}^n(A) \le \mathcal{H}^n_\delta(A) \quad \forall \delta > 0 , \quad A \subset \mathbb{R}^n ,$$

we are going to need the inequality

2.5 $$\mathcal{L}^n(A) \le \omega_n \left(\frac{\text{diam } A}{2}\right)^n \quad \forall A \subset \mathbb{R}^n .$$

This is called the *isodiametric inequality* ; it asserts that among all sets $A \subset \mathbb{R}^n$ with a given diameter ρ , the ball with diameter ρ has the largest $\mathcal{L}^n$ measure. It is proved by *Steiner symmetrization* (see [HR] or [FH1] for the details).

Now suppose $\delta > 0$, $A \subset \mathbb{R}^n$, and let $C_1, C_2, \ldots$ be any countable collection with $A \subset \bigcup_j C_j$ and $\text{diam } C_j < \delta$. Then

$$\mathcal{L}^n(A) \le \mathcal{L}^n(\bigcup_j C_j) \le \sum_j \mathcal{L}^n(C_j)$$

$$\le \sum_j \omega_n \left(\frac{\text{diam } C_j}{2}\right)^n \quad \text{(by 2.5)} .$$

Taking the inf over all such collections $\{C_j\}$ we have (**).

Thus we have proved:

2.6 THEOREM

$L^n(A) = H^n(A) = H^n_\delta(A)$ *for every* $A \subset \mathbb{R}^n$ *and* $\delta > 0$.

§3. DENSITIES

Next we want to introduce the notion of n-dimensional density of a measure μ on X . For any measure μ on X , any subset $A \subset X$, and any point $x \in X$, we define the n-dimensional upper and lower densities $\Theta^{*n}(\mu,A,x)$, $\Theta^n_*(\mu,A,x)$ by

$$\Theta^{*n}(\mu,A,x) = \limsup_{\rho\downarrow 0} \frac{\mu(A\cap B_\rho(x))}{\omega_n \rho^n}$$

$$\Theta^n_*(\mu,A,x) = \liminf_{\rho\downarrow 0} \frac{\mu(A\cap B_\rho(x))}{\omega_n \rho^n}$$

(where $B_\rho(x)$ denotes the closed ball). In case $A = X$ we simply write $\Theta^{*n}(\mu,x)$ and $\Theta^n_*(\mu,x)$ to denote these quantities, so that $\Theta^{*n}(\mu,A,x) = \Theta^{*n}(\mu \llcorner A,x)$, $\Theta^n_*(\mu,A,x) = \Theta^n_*(\mu \llcorner A,x)$.

3.1 REMARK One readily checks that if all Borel sets are μ-measurable then $\mu(A\cap B_\rho(x)) \ge \limsup_{y\to x} \mu(A\cap B_\rho(y))$ for each fixed $\rho > 0$, so that $\mu(A\cap B_\rho(x))$ is a Borel-measurable function of x for each fixed $\rho > 0$. Hence $\Theta^{*n}(\mu,A,x)$ and $\Theta^n_*(\mu,A,x)$ are both Borel measurable (and hence μ-measurable) functions of $x \in X$. Notice that it is *not* necessary that A be μ-measurable.

If $\Theta^{*n}(\mu,A,x) = \Theta^n_*(\mu,A,x)$ then the common value will be denoted $\Theta^n(\mu,A,x)$.

Appropriate information about the upper density gives connections between μ and $\mathcal{H}^n$. Specifically we have

3.2 THEOREM *Let* μ *be a Borel-regular measure on* X *and* $t \geq 0$.

(1) *If* $A_1 \subset A_2 \subset X$ *and* $\Theta^{*n}(\mu, A_2, x) \geq t$ *for all* $x \in A_1$, *then* $t\,\mathcal{H}^n(A_1) \leq \mu(A_2)$.

(2) *If* $A \subset X$ *and* $\Theta^{*n}(\mu, A, x) \leq t$ *for all* $x \in A$, *then* $\mu(A) \leq 2^n t\,\mathcal{H}^n(A)$.

An important case of (1) is when $A_1 = A_2$. Notice that we do *not* assume A, A_1, A_2 are μ-measurable.

Of the two propositions above, (2) is the more elementary and we could prove it immediately. (1) requires a covering lemma, so we defer both proofs until we have discussed this.

In the following covering theorem and its proof, we use the notation that if B is a ball $B_\rho(x) \subset X$, then $\hat{B} = B_{5\rho}(x)$.

3.3 THEOREM *If* $\mathcal{B}$ *is any family of closed balls in* X *with* $R \equiv \sup\{\operatorname{diam} B : B \in \mathcal{B}\} < \infty$, *then there is a pairwise disjoint subcollection* $\mathcal{B}' \subset \mathcal{B}$ *such that*

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{B \in \mathcal{B}'} \hat{B} \ ;$$

in fact we can arrange the stronger property

$$(*) \qquad B \in \mathcal{B} \Rightarrow \exists\, S \in \mathcal{B}' \text{ with } S \cap B \neq \emptyset \text{ and } \hat{S} \supset B .$$

Proof For $j = 1,2,\ldots$ let $\mathcal{B}_j = \{B \in \mathcal{B} : R/2^j < \operatorname{diam} B \leq R/2^{j-1}\}$, so that $\mathcal{B} = \bigcup_{j=1}^{\infty} \mathcal{B}_j$. Proceed to define $\mathcal{B}_j' \subset \mathcal{B}_j$ as follows:

(i) Let $\mathcal{B}_1'$ be any maximal pairwise disjoint subcollection of $\mathcal{B}_1$.

(ii) Assuming $j \geq 2$ and that $\mathcal{B}_1', \ldots, \mathcal{B}_{j-1}'$ are defined, let $\mathcal{B}_j'$ be a maximal pairwise disjoint subcollection of $\{B \in \mathcal{B}_j : B \cap B' = \phi$ whenever $B' \in \bigcup_{i=1}^{j-1} \mathcal{B}_i'\}$.

Then evidently if $j \geq 1$ and $B \in \mathcal{B}_j$ we must have

$$B \cap B' \neq \phi \quad \text{for some} \quad B' \in \bigcup_{i=1}^{j} \mathcal{B}_i'$$

(otherwise we contradict maximality of $\mathcal{B}_j'$) , and for such a pair B, B' we have $\operatorname{diam} B \leq R/2^{j-1} = 2R/2^j \leq 2 \operatorname{diam} B'$, so that $B \subset \hat{B}'$.

Thus we may take $\mathcal{B}' = \bigcup_{i=1}^{\infty} \mathcal{B}_i'$.

In the following corollary we use the terminology that a subset $A \subset X$ is covered *finely* by a collection $\mathcal{B}$ of balls, meaning that for each $x \in A$ and each $\varepsilon > 0$, there is a ball $B \in \mathcal{B}$ with $x \in B$ and $\operatorname{diam} B < \varepsilon$.

3.4 COROLLARY *If* $\mathcal{B}$ *is as in* 3.3, *if* A *is a subset of* X *covered finely by* $\mathcal{B}$, *and if* $\mathcal{B}' \subset \mathcal{B}$ *is as in* 3.3, *then*

$$A \sim \bigcup_{j=1}^{N} B_j \subset \bigcup_{B \in \mathcal{B}' \sim \{B_1, \ldots, B_N\}} \hat{B}$$

for each finite subcollection $\{B_1, \ldots, B_N\} \subset \mathcal{B}'$.

Proof If $x \in A \sim \bigcup_{j=1}^{N} B_j$, since $\mathcal{B}$ covers A finely and since $X \sim \bigcup_{j=1}^{N} B_j$ is open, we can then find $B \in \mathcal{B}$ with $B \cap (\bigcup_{j=1}^{N} B_j) = \phi$ and $x \in B$, and (by (*)) find $S \in \mathcal{B}'$ with $S \cap B \neq \phi$ and $\hat{S} \supset B$. Evidently then $S \neq B_j \ \forall j = 1, \ldots, N$, and hence $x \in \bigcup_{S \in \mathcal{B}' \sim \{B_1, \ldots, B_N\}} \hat{S}$.

Proof of (1) of Theorem 3.2 We can assume $\mu(A_2) < \infty$ and $t > 0$, otherwise the result is trivial. We can also assume the strict inequality

$$\Theta^{*n}(\mu, A_2, x) > t \quad \text{for} \quad x \in A_1$$

(because to obtain the conclusion of (1) for t equal to a given $t_1 > 0$, it clearly suffices to prove it for each $t < t_1$) .

For $\delta > 0$, let $\mathcal{B}$ (depending on δ) be the collection $\{$closed balls $B_\rho(x) : x \in A_1 , 0<\rho<\delta/2 , \mu(A_2 \cap B_\rho(x)) \geq t\,\omega_n \rho^n\}$. Evidently $\mathcal{B}$ covers A_1 finely and hence there is a disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ so that 3.3(*) holds. Since $\mu(A_2 \cap B) > 0$ for each $B \in \mathcal{B}$ and since $\mu(A_2) < \infty$ it follows that $\mathcal{B}'$ is a countable collection $\{B_1, B_2, \ldots\}$ and hence 3.4 implies

$$A_1 \sim \bigcup_{j=1}^{N} B_j \subset \bigcup_{j=N+1}^{\infty} \hat{B}_j \qquad \forall N \geq 1 .$$

Thus $A_1 \subset (\bigcup_{j=1}^{N} B_j) \cup (\bigcup_{j=N+1}^{\infty} \hat{B}_j)$ and hence by the definition 2.2 of $\mathcal{H}^n_\delta$ we have

$$\mathcal{H}^n_{5\delta}(A_1) \leq \sum_{j=1}^{N} \omega_n \rho_j^n + 5^n \sum_{j=N+1}^{\infty} \omega_n \rho_j^n \left(\rho_j = \frac{\operatorname{diam} B_j}{2}\right) .$$

Since $B_j \in \mathcal{B}$, we have

$$\sum_{j=1}^{\infty} \omega_n \rho_j^n \leq t^{-1} \sum_{j=1}^{\infty} \mu(A_2 \cap B_j) = t^{-1} \sum_{j=1}^{\infty} (\mu \llcorner A_2)(B_j)$$

$$= t^{-1} (\mu \llcorner A_2)(\bigcup_{j=1}^{\infty} B_j) \leq t^{-1} \mu(A_2) < \infty ,$$

and hence letting $N \to \infty$ we deduce

$$\mathcal{H}^n_{5\delta}(A_1) \leq t^{-1} \mu(A_2) .$$

Letting $\delta \downarrow 0$, we then have the required result.

Proof of (2) of Theorem 3.2 We may assume that

$$\Theta^{*n}(\mu,A,x) < t \quad \text{for all} \quad x \in A$$

because to prove the conclusion of (2) for a given $t = t_1 > 0$, it is clearly enough to prove it for each $t > t_1$. Thus if

$$A_k = \{x \in A : \mu(A \cap B_\rho(x)) \le t\omega_n\rho^n \quad \forall 0 < \rho < 1/k\}$$

then $A = \bigcup_{n=1}^{\infty} A_k$ and $A_{k+1} \supset A_k$, $k = 1,2,\ldots$. The A_k are not necessarily μ-measurable, but we still have $\lim_{k\to\infty} \mu(A_k) = \mu(\bigcup_{k=1}^{\infty} A_k)$ by virtue of the regularity of μ. Thus we will be finished if we can prove

$$\mu(A_k) \le 2^n t H^n(A_k) \quad \forall k \ge 1 .$$

Let $\delta \in (0,1/2k)$ and let $C_1, C_2, \ldots$ be any countable cover for A_k with diam $C_j < \delta$ and $C_j \cap A_k \ne \phi$ $\forall j$. For each j we can find an $x_j \in A_k$ so that $B_{2\rho_j}(x_j) \supset C_j$, $\rho_j = \frac{\text{diam } C_j}{2}$. Then since $2\rho_j < 1/k$ we have by definition of A_k that

$$\mu(C_j) \le \mu(B_{2\rho_j}(x_j)) \le 2^n t\omega_n\rho_j^n .$$

Hence

$$\mu(A_k) \le 2^n t \sum_{j=1} \omega_n \left(\frac{\text{diam } C_j}{2}\right)^n .$$

Taking inf over all such covers $\{C_j\}$ we then have (by definition of H^n_δ) that $\mu(A_k) \le 2^n t\, H^n_\delta(A_k)$. Thus we have the required inequality by letting $\delta \downarrow 0$.

As a corollary to Theorem 3.2 (1) we can easily prove the following.

3.5 THEOREM *If* μ *is Borel regular, if* A *is a* μ*-measurable subset of* X *and if* $\mu(A) < \infty$, *then*

$$\Theta^{*n}(\mu,A,x) = 0 \quad \textit{for} \quad H^n\text{ - a.e. } x \in X \sim A .$$

REMARK Of course $\mu = H^n$ is an important case.

Proof For $t > 0$ let $C_t = \{x \in X \sim A : \Theta^{*n}(\mu,A,x) \geq t\}$ and if $H^n(C_t) > 0$ we can (by Theorem 1.3(2)) find a closed set $E \subset A$ such that

$$(1) \qquad \mu(A \sim E) < t\, H^n(C_t) .$$

Since $X \sim E$ is open and $C_t \subset X \sim A \subset X \sim E$ we have $\Theta^{*n}(\mu, A \cap (X \sim E), x) = \Theta^{*n}(\mu,A,x) \geq t$ for $x \in C_t$. Thus we can apply Theorem 3.2(1) with $\mu \llcorner A$, C_t, $X \sim E$ in place of μ, A_1, A_2 to give $t\, H^n(C_t) \leq \mu(A \sim E)$, thus contradicting (1). Thus we conclude that $H^n(C_t) = 0$ $\forall t > 0$. In particular $H^n(\bigcup_{k=1}^{\infty} C_{1/k}) = 0$. Thus $\Theta^{*n}(\mu,A,x) = 0$ for H^n - a.e. $x \in X \sim A$.

We conclude this section with two important bounds for densities with respect to Hausdorff measure.

3.6 THEOREM *Suppose* A *is any subset of* X.

(1) *If* $H^n(A) < \infty$, *then* $\Theta^{*n}(H^n,A,x) \leq 1$ *for* H^n - a.e. $x \in A$.

(2) *If* $H^n_\delta(A) < \infty$ *for each* $\delta > 0$ *(note this is automatic if* A *is a totally bounded subset of* X*), then* $\Theta^{*n}(H^n_\infty,A,x) \geq 2^{-n}$ *for* H^n - a.e. $x \in A$.

3.7 REMARK Since $H^n \geq H^n_\delta \geq H^n_\infty$ (by definitions 2.1, 2.2) this theorem implies

$$2^{-n} \leq \Theta^{*n}(H^n,A,x) \leq 1 \quad \text{for } H^n\text{ - a.e. } x \in A$$

if $H^n(A) < \infty$.

Proof of 3.6 To prove (1), let $\varepsilon, t > 0$, let $A_t = \{x \in A : \Theta^{*n}(\mathcal{H}^n, A, x) \geq t\}$ and (using Theorem 1.3(1) with $\mu = \mathcal{H}^n \llcorner A$) choose an open set $U \supset A_t$ such that

$$\mathcal{H}^n(U \cap A) < \mathcal{H}^n(A_t) + \varepsilon .$$

Since U is open and since $A_t \subset U$ we have $\Theta^*(\mathcal{H}^n, A\cap U, x) \geq t$ for each $x \in A_t$. Hence Theorem 3.2(1) (with $\mathcal{H}^n \llcorner A, A_t, A\cap U$ in place of μ, A_1, A_2) implies that

$$t\, \mathcal{H}^n(A_t) \leq \mathcal{H}^n(A \cap U) \leq \mathcal{H}^n(A_t) + \varepsilon .$$

We thus have $\mathcal{H}^n(A_t) = 0$ for each $t > 1$. Since $\{x : \Theta^{*n}(\mathcal{H}^n, A, x) > 1\} = \bigcup_{j=1}^{\infty} A_{t_j}$ for any strictly decreasing sequence $\{t_j\}$ with $\lim t_j = 1$, we thus have $\mathcal{H}^n\{x : \Theta^{*n}(\mathcal{H}^n, A, x) > 1\} = 0$, as required.

To prove (2), suppose for contradiction that $\Theta^{*n}(\mathcal{H}^n_\infty \llcorner A, x) < 2^{-n}$ for each x in a set $B_0 \subset A$ with $\mathcal{H}^n(B_0) > 0$. Then for each $x \in B_0$ (by definition) we can select $\delta_x \in (0,1)$ such that

$$\mathcal{H}^n_\infty(A \cap B_\rho(x)) \leq \frac{1-\delta_x}{2^n}\, \omega_n \rho^n , \quad 0 < \rho < \delta_x .$$

Therefore, since $B_0 = \bigcup_{j=1}^{\infty} \{x \in B_0 : \delta_x > 1/j\}$ and since $\mathcal{H}^n_\delta(A \cap B_\rho(x)) \equiv \mathcal{H}^n_\infty(A \cap B_\rho(x))$ for any $\rho < \delta/2$ (by definition 2.2), we can select $\delta > 0$ and $B \subset B_0$ with $\mathcal{H}^n(B) > 0$ and

$$\text{(1)} \qquad \mathcal{H}^n_\delta(A \cap B_\rho(x)) \leq \frac{1-\delta}{2^n}\, \omega_n \rho^n , \quad 0 < \rho < \delta/2 , \quad x \in B .$$

Now using 2.2 again, we can choose sets $C_1, C_2, \ldots$ with $B \subset \bigcup_{j=1}^{\infty} C_j$, $C_j \cap B \neq \phi \ \forall j$, and

$$\text{(2)} \qquad \sum_j \omega_n(\rho_j/2)^n < \frac{1}{1-\delta}\, \mathcal{H}^n_\delta(B) , \quad \rho_j = \text{diam } C_j .$$

Now take $x_j \in C_j \cap B$, so that $B \subset A \cap (\bigcup_{j=1}^{\infty} B_{\rho_j}(x_j))$, and we conclude from (1), (2) that $\mathcal{H}^n_\delta(B) = 0$, hence $\mathcal{H}^n(B) = 0$, contradicting our choice of B .

§4. RADON MEASURES

In this section X is assumed to be locally compact and separable. On such a space we say that μ is a *Radon measure* if μ is Borel regular and if μ is finite on compact subsets of X . Notice that (by 1.3, 1.4) such a measure μ automatically has the properties

$$\mu(A) = \inf_{\substack{U \supset A \\ U \text{ open}}} \mu(U) , \quad A \subset X \text{ arbitrary}$$

and

$$\mu(A) = \sup_{\substack{K \subset A \\ K \text{ compact}}} \mu(K) , \quad A \subset X \ \mu\text{-measurable.}$$

The finiteness of Radon measures μ on compact subsets enables us to integrate continuous functions with compact support. Indeed if H is a Hilbert space with inner product $(,)$ and if $\mathcal{K}(X,H)$ denotes the space of continuous functions $X \to H$ with compact support, then associated with each Radon measure μ and each μ-measurable H-valued function $\nu : X \to H$ satisfying $|\nu| = 1$, μ-a.e. , we have the linear functional $L : \mathcal{K}(X,H) \to \mathbb{R}$ defined by

$$L(f) = \int_X (f,\nu)\, d\mu \ .$$

The following *Riesz representation theorem* shows that *every* linear functional $L : \mathcal{K}(X,H) \to \mathbb{R}$ is obtained as above, provided

$$(*) \qquad \sup\{L(f) : f \in K(X,H) \ , \ |f| \le 1 \ , \ \operatorname{spt} f \subset K\} < \infty$$

for each compact $K \subset X$.

4.1 THEOREM *Let* L *be any linear functional on* $K(X,H)$ *satisfying* $(*)$ *above. Then there is a Radon measure* μ *on* X *and a* μ*-measurable function* $\nu : X \to H$ *such that* $|\nu(x)| = 1$ *for* μ-a.e. $x \in X$ *and*

$$L(f) = \int_X (f,\nu)\, d\mu \qquad \forall f \in K(X,H) \ .$$

4.2 REMARK Notice that (as one readily checks by using Lusin's theorem to exhaust μ-almost all of X by an increasing sequence of compact sets on which ν is continuous), we have

$$\sup\{L(f) : f \in K(X,H) \ , \ |f| \le 1 \ , \ \operatorname{spt} f \subset V\} = \mu(V)$$

for every open $V \subset X$, assuming μ , ν are as in the theorem. For this reason the measure μ is called the *total variation measure* associated with the functional L .

Proof of 4.1 First *define* $\mu(V)$ on open sets V according to the identity of 4.2 above, and then for an arbitrary subset $A \subset X$ let

$$(1) \qquad \mu(A) = \inf_{\substack{A \subset V \\ V \text{ open}}} \mu(V) \ .$$

(Of course these definitions are not contradictory when A itself is open.) To check that μ is a Radon measure we proceed as follows. First, if $V, V_1, V_2, \ldots$ are open sets in X with $V \subset \bigcup_{j=1}^{\infty} V_j$, and if ω is any element of $K(X,H)$ with $\sup_X|\omega| \le 1$ and support $\omega \subset V$, then, by using the definition of μ and a partition of unity of support ω subordinate to to the sets $\{V_j\}_{j=1,2,\ldots}$, we have

$$|L(\omega)| \le \sum_{j=1}^{\infty} \mu(V_j) \ .$$

Taking sup over all such ω we thus get $\mu(V) \le \sum_{j=1}^{\infty} \mu(V_j)$. Then by (1) we see that

$$\mu(A) \le \sum_{j=1}^{\infty} \mu(A_j)$$

whenever $A, A_1, A_2, \ldots$ are subsets of X with $A \subset \bigcup_{j=1}^{\infty} A_j$. Thus μ is a measure on X . It is also clear from the definition of μ that

$$\mu(V_1 \cup V_2) = \mu(V_1) + \mu(V_2)$$

whenever V_1, V_2 are open subsets of X with $d(V_1,V_2) > 0$. Then by (1) we see that μ satisfies the Caratheodory criterion, and hence all Borel sets are measurable by Theorem 1.2. Thus we can conclude that μ is a Borel regular measure and since it is evidently finite on compact sets (by (*)) it is then a Radon measure.

Next let $K(X) = K(X,\mathbb{R})$ and $K_+(X) = \{f \in K(X) : f \ge 0\}$.

Define

$$\lambda(f) = \sup_{\substack{|\omega| \le f \\ \omega \in K(X,H)}} |L(\omega)| \ , \quad f \in K_+(X) \ .$$

Then by definition of μ we have

$$\sup_{\substack{f \in K_+(X) \\ \text{support } f \subset U}} \lambda(f) = \mu(U) \quad \forall \text{ open } U \subset X \ .$$

We in fact claim

$$(2) \qquad \lambda(f) = \int f d\mu \ , \quad f \in K_+(X) \ .$$

To see this we first note that $\lambda(cf) = c\,\lambda(f)$, c constant ≥ 0 , $f \in K_+(X)$. Further we claim that $\lambda(f+g) = \lambda(f) + \lambda(g)$ $\forall f,g \in K_+(X)$. Indeed the inequality

$\lambda(f+g) \geq \lambda(f) + \lambda(g)$ is obvious, and we prove the reverse inequality as follows. Let $\omega \in K(X,H)$ with $|\omega| \leq f+g$, and define ω_1, ω_2 by

$$\omega_1 = \begin{cases} \frac{f}{f+g}\,\omega & \text{if } f+g > 0 \\ 0 & \text{if } f+g = 0 \end{cases} \qquad \omega_2 = \begin{cases} \frac{g}{f+g}\,\omega & \text{if } f+g > 0 \\ 0 & \text{if } f+g = 0 \end{cases} .$$

One easily checks that then ω_1, $\omega_2 \in K(X,H)$. Then since $\omega = \omega_1 + \omega_2$ and $|\omega_1| \leq f$, $|\omega_2| \leq g$, we have $|L(\omega)| \leq \lambda(f) + \lambda(g)$. Taking sup over all such ω we then have $\lambda(f+g) = \lambda(f) + \lambda(g)$. To complete the proof that (2) holds we let $\varepsilon > 0$ and choose $t_0 = 0 < t_1 < \ldots < t_N$, $t_N > \sup f$, such that $t_i - t_{i-1} < \varepsilon$ and $\mu(f^{-1}(t_j)) = 0$ $\forall j = 1,\ldots,N$. (This is of course possible, because $\{t \in \mathbb{R} : \mu(f^{-1}(t)) > 0\}$ is clearly countable.) Write $U_j = \{x \in X : t_{j-1} < f < t_j\}$, $j = 1,\ldots,N$.

Now, by definition of μ, for each $\varepsilon > 0$ we can choose $h_j \in K_+(X)$ with support $h_j \subset U_j$, $h_j \leq 1$,

(3) $$\lambda(h_j) \geq \mu(U_j) - \varepsilon/N$$

and

(4) $$\mu(U_j \sim \{x : h_j(x) = 1\}) < \varepsilon/N .$$

Evidently (4) together with the definitions of λ, μ implies

$$\lambda(f - f\sum_{j=1}^{N} h_j) \leq \sup|f|\,\mu\{x : \sum_{i=1}^{N} U_i \sim \{x : h_j(x) = 1\}\}$$

$$\leq \sup|f|\ \varepsilon ,$$

and it readily follows that

$$\sum_{j=1}^{N} t_{j-1}\mu(U_j) - 2\varepsilon \sup|f| \leq \lambda(f \sum_{j=1}^{N} h_j) \leq \lambda(f) \leq \lambda(f \sum_{j=1}^{N} h_j) + \varepsilon \sup|f|$$

$$\leq \sum_{j=1}^{N} t_j\mu(U_j) + \varepsilon \sup|f| .$$

Since

$$\sum_{j=1}^{N} t_{j-1}\mu(U_j) \le \int f d\mu \le \sum_{j=1}^{N} t_j\mu(U_j)$$

we then have $\left|\lambda(f) - \int f d\mu\right| < 2\varepsilon \sup|f|$, and hence (2).

To complete the proof of the theorem, let $e \in H$ with $|e| = 1$, and consider the linear functional λ_e on $K(X)$ defined by $\lambda_e(f) = T(fe)$. Evidently by (2) ,

$$|\lambda_e(f)| \le \int |f| d\mu \qquad \forall f \in K(X)$$

and hence λ_e extends uniquely to a linear functional on $L^1(\mu)$. By the Riesz Representation Theorem for $L^1(\mu)$ functions (see e.g. [RH] for details - the proof is based on the Radon-Nikodym theorem) we have a bounded μ-measurable (in fact Borel-measurable) function ν_e on X such that

$$L(fe) = \int f\, \nu_e\, d\mu \qquad \forall f \in K(X) .$$

Taking $e_1, \ldots, e_n$ to be an orthonormal basis for H , and defining $\nu = \sum_{j=1}^{n} \nu^j e_j$, $\nu^i \equiv \nu_{e_i}$, one then easily checks that $L(g) = \int (g,\nu) d\mu$ for each $g \in K(X,H)$, as required. Furthermore (Cf. Remark 4.2) for each open $U \subset X$ we have

$$(5) \qquad \sup\{L(g) : g \in K(X,H) ,\ |g| \le 1 ,\ \operatorname{spt} g \subset U\} = \int_U |\nu| d\mu .$$

On the other hand the left side of (5) is $\mu(U)$ by definition of μ . Hence (from the arbitraryness of U) we conclude $|\nu| = 1$ μ-a.e. This completes the proof of Theorem 4.1.

4.3 REMARK Note that in case $H = \mathbb{R}$, Theorem 4.2 asserts that the linear functional L can be represented

$$L(f) = \int_X f\,\nu\,d\mu \qquad \forall F \in K(X,\mathbb{R})\ ,$$

where $\nu(x) = \pm 1$ for μ-a.e. $x \in X$. In the special case when L is *non-negative*, i.e. $L(f) \geq 0$ if $f \geq 0$, then one easily checks that $\nu \equiv +1$, so that the theorem gives

$$L(f) = \int_X f\,d\mu$$

in this case. Thus *we can identify the Radon measures on* X *with the non-negative linear functionals on* $K(X,\mathbb{R})$. (Note (*) is automatic if L is non-negative.)

Now for $U \subset X$ with U open and $\bar{U}$ compact, let L_U^+ denote the set of bounded (real-valued) linear functionals on $K_U(X) = \{$continuous functions $f : X \to \mathbb{R}$ with $\operatorname{spt} f \subset U\}$ which are non-negative on $K_U^+(X) = \{f \in K_U(X) : f \geq 0\}$. The Banach-Alaoglu theorem (see e.g. [Roy]) tells us that $\{\lambda \in L_U^+ : \|\lambda\| \leq 1\}$ is weak* compact. That is, given a sequence $\{\lambda_k\} \subset L_U^+$ with $\sup_{k\geq 1}\|\lambda_k\| < \infty$, we can find a subsequence $\{\lambda_{k'}\}$ and $\lambda \in L_U^+$ such that $\lim \lambda_{k'}(f) = \lambda(f)$ for each fixed $f \in K_U^+(X)$. Using the above Riesz Representation Theorem (and in particular Remark 4.3) together with an exhaustion of X by an increasing sequence $\{U_i\}$ of open sets with $\bar{U}_i$ compact $\forall i$, this evidently implies the following assertion concerning sequences of Radon measures on X.

4.4 THEOREM *Suppose* $\{\mu_k\}$ *is a sequence of Radon measures on* X *with* $\sup_{k\geq 1} \mu_k(U) < \infty$ *for each open* $U \subset X$ *with* $\bar{U}$ *compact. Then there is a subsequence* $\{\mu_{k'}\}$ *which converges to a Radon measure* μ *on* X *in the sense that*

$$\lim \mu_{k'}(f) = \mu(f) \quad \textit{for each} \quad f \in K(X)\ ,$$

where $K(X) = \{f : f$ *is a real-valued continuous function with compact support on* $X\}$. *Here we used the notation*

$$\mu(f) = \int_X f\,d\mu\ .$$

Now let μ be any Radon measure on X. We say that X has the *symmetric Vitali property* relative to μ if for every collection $\mathcal{B}$ of balls which covers its *set of centres* $A \equiv \{x : B_\rho(x) \in \mathcal{B}$ for some $\rho > 0\}$ *finely* (i.e. for each $x \in A$ we have $\inf\{\rho : B_\rho(x) \in \mathcal{B}\} = 0$), there is a countable pairwise disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ covering μ-almost all of A, provided $\mu(A) < \infty$.

4.5 REMARKS

(1) It is easy to see (from Corollary 3.4) that the locally compact separable metric space X has this property with respect to μ, provided $\mu(B_{5\rho}(x)) \le c\,\mu(B_\rho(x))$ whenever $B_\rho(x) \subset X$, where c is a fixed constant independent of x and ρ.

(2) Most importantly, in the special case when $X = \mathbb{R}^n$, we have the *symmetric Vitali property* with respect to μ for any Radon measure μ.

To justify this last remark we need first to recall the following *Besicovitch covering lemma* (see [FH1] or [HR] for a proof).

4.6 LEMMA *Suppose* $\mathcal{B}$ *is a collection of closed balls in* $\mathbb{R}^n$, *let* A *be the set of centres, and suppose the set of all radii of balls in* $\mathcal{B}$ *is a bounded set. Then there are sub-collections* $\mathcal{B}_1, \ldots, \mathcal{B}_N \subset \mathcal{B}$ $(N = N(n))$ *such that each* $\mathcal{B}_j$ *is a pairwise disjoint (or empty) collection, and* $\bigcup_{j=1}^N \mathcal{B}_j$ *still covers* A : $A \subset \bigcup_{j=1}^N \left(\bigcup_{B \in \mathcal{B}_j} B\right)$.

We emphasize that N is a certain fixed constant depending only on n.

Using this lemma we can easily justify Remark 4.5(2) : For a given Radon measure μ in $\mathbb{R}^n$ and for a given collection of balls $\mathcal{B}$ covering its own set of centres A finely, we first choose (from the set $\{B \in \mathcal{B} : \text{radius } B \le 1\}$)

pairwise disjoint collections $\mathcal{B}_1, \ldots, \mathcal{B}_N \subset \mathcal{B}$ such that $\bigcup_{j=1}^{N} \mathcal{B}_j$ covers A. Then for at least one $j \in \{1,\ldots,N\}$ we get

$$\mu(A \sim \bigcup_{B \in \mathcal{B}_j} B) \le (1-1/N)\mu(A)$$

and hence taking a suitable finite subcollection $\{B_1, \ldots, B_Q\} \subset \mathcal{B}_j$,

$$\mu(A \sim \bigcup_{k=1}^{Q} B_k) \le (1-1/2N)\mu(A) .$$

Since $\mathcal{B}$ covers A finely, and since $\bigcup_{k=1}^{Q} B_k$ is closed, the collection $\tilde{\mathcal{B}} = \{B \in \mathcal{B} : B \cap (\bigcup_{k=1}^{Q} B_k) = \phi\}$ covers $A \sim \bigcup_{k=1}^{Q} B_k$ finely. Thus we can repeat the argument with $\tilde{\mathcal{B}}$ in place of $\mathcal{B}$ and $A \sim \bigcup_{k=1}^{Q} B_k$ in place of A in order to select new balls $B_{Q+1}, \ldots, B_P \in \tilde{\mathcal{B}}$ such that

$$\mu(A \sim \bigcup_{k=1}^{P} B_k) \le (1-\frac{1}{2N})\,\mu(A \sim \bigcup_{k=1}^{Q} B_k)$$

$$\le (1-\frac{1}{2N})^2\,\mu(A) .$$

Continuing (inductively) in this way, we conclude that if $\mu(A) < \infty$ there is a pairwise disjoint sequence $B_1, B_2, \ldots$ of balls in $\mathcal{B}$ such that

$$\mu(A \sim \bigcup_{k=1}^{\infty} B_k) = 0 .$$

Thus Remark 4.5(2) is established.

4.7 THEOREM *Suppose μ_1, μ_2 are Radon measures on X, where X has the symmetric Vitali property with respect to μ_1. Then*

$$D_{\mu_1}\mu_2(x) \equiv \lim_{\rho \downarrow 0} \frac{\mu_2(B_\rho(x))}{\mu_1(B_\rho(x))}$$

exists μ_1-almost everywhere and is μ_1-measurable. Furthermore for any Borel set $A \subset X$

(1) $$\mu_2(A) = \int_A (D_{\mu_1}\mu_2)\,d\mu_1 + \mu_2^*(A) ,$$

where

$$\mu_2^* = \mu_2 \llcorner Z ,$$

where Z *is a Borel set of* μ_1*-measure zero* (Z *independent of* A).

In case X *also has the symmetric Vitali property with respect to* μ_2 *then* $D_{\mu_1}\mu_2$ *also exists* μ_2*-almost everywhere and*

(2) $$\mu_2^* = \mu_2 \llcorner \{x : D_{\mu_1}\mu_2(x) = +\infty\} .$$

(i.e. we may take $Z = \{x : D_{\mu_1}\mu_2(x) = +\infty\}$ in this case.)

4.8 REMARKS

(1) Of course by Remark 4.5(2), we always have 4.7(2) if $X = \mathbb{R}^n$.

(2) μ_2^* is called the singular part of μ_2 with respect to μ_1 . One readily checks that $\mu_2^* = 0$ if and only if all sets of μ_1-measure zero also have μ_2-measure zero. In this case we say that μ_2 is *absolutely continuous* with respect to μ_1. 4.7(1) then simply says

(*) $$\mu_2(A) = \int_A (D_{\mu_1}\mu_2)\,d\mu_1 , \quad A \subset X , \quad A \text{ a Borel set.}$$

Proof We can of course assume $\mu_1(X) < \infty$, $\mu_2(X) < \infty$ since μ_1 , μ_2 are Radon measures and X is locally compact and separable.

First consider the case when all sets of μ_1-measure zero also have μ_2-measure zero. In this case we want to prove (*) , and we have that X also has the symmetric Vitali property relative to μ_2 .

Let $\tilde{X} = X \sim \{x : \mu_1(B_\sigma(x)) = 0 \text{ for some } \sigma > 0\}$. Evidently $\tilde{X}$ is closed and (by separability) $\mu_1(X \sim \tilde{X}) = 0$, $\mu_1 = \mu_1 \llcorner \tilde{X}$. Let $\underline{D}_{\mu_1}\mu_2$ and

$\bar{D}_{\mu_1}\mu_2$ be defined on $\tilde{X}$ by

$$\underline{D}_{\mu_1}\mu_2(x) = \liminf_{\rho\downarrow 0} \frac{\mu_2(B_\rho(x))}{\mu_1(B_\rho(x))}$$

$$\bar{D}_{\mu_1}\mu_2(x) = \limsup_{\rho\downarrow 0} \frac{\mu_2(B_\rho(x))}{\mu_1(B_\rho(x))}$$

and define $\bar{D}_{\mu_1}\mu_2$, $\underline{D}_{\mu_1}\mu_2 \equiv \infty$ on $X \sim \tilde{X}$. Notice that $\bar{D}_{\mu_1}\mu_2$ and $\underline{D}_{\mu_1}\mu_2$ are Borel measurable.

We first prove that if $\alpha \in (0,\infty)$ then for any Borel set $A \subset X$,

(1) $$A \subset \{x \in \tilde{X} : \underline{D}_{\mu_1}\mu_2(x) < \alpha\} \Rightarrow \mu_2(A) \le \alpha\, \mu_1(A)$$

(2) $$A \subset \{x \in \tilde{X} : \bar{D}_{\mu_1}\mu_2(x) > \alpha\} \Rightarrow \mu_2(A) \ge \alpha\, \mu_1(A) .$$

To prove (1) we simply note that if $A \subset \{x \in \tilde{X} : \underline{D}_{\mu_1}\mu_2(x) > \alpha\}$, then for any open $V \supset A$ the collection $\mathcal{B} = \{B_\rho(x) : x \in A , B_\rho(x) \subset V ,$ $\mu_2(B_\rho(x)) \le \alpha\, \mu_1(B_\rho(x))\}$ covers A finely, so there is a countable disjoint subcollection $\{B_1, B_2, \dots\} \subset \mathcal{B}$ which covers μ_1-almost all of A (and hence μ_2-almost all of A).

Then

$$\mu_2(A) \le \mu_2\Big(\bigcup_{j=1}^{\infty} B_j\Big) \le \sum_{j=1}^{\infty} \mu_2\,(Bj) \le \alpha \sum_{j=1}^{\infty} \mu_1(B_j)$$

$$= \alpha\, \mu_1\,\Big(\bigcup_{j=1}^{\infty} B_j\Big) \le \alpha\, \mu_1(V) .$$

Taking inf over all such V , by (1.31) we have (1) as required.

The proof of (2) is almost identical and is left to the reader.

Notice particularly that if we let $\alpha \to \infty$ in (1) and use

$\mu_1(X\sim\tilde{X}) = 0$, then we deduce

(3) $$\mu_1\{x \in X : \bar{D}_{\mu_1}\mu_2(x) = +\infty\} = 0 .$$

Now let $a < b$ and $A = \{x \in \tilde{X} : \underline{D}_{\mu_1}\mu_2(x) < a < b < \bar{D}_{\mu_1}\mu_2(x)\}$. Then by (1), (2) above we have

$$\mu_2(A) \le a\,\mu_1(A) \quad \text{and also} \quad b\,\mu_1(A) \le \mu_2(A) ,$$

which implies that $\mu_1(A) = \mu_2(A) = 0$. Thus, by (3) together with the fact that $\{x \in \tilde{X} : \underline{D}_{\mu_1}\mu_2(x) < \bar{D}_{\mu_1}\mu_2(x)\} = \bigcup_{a,b \text{ rational}, a<b} \{x \in \tilde{X} : \underline{D}_{\mu_1}\mu_2(x) < a < b < \bar{D}_{\mu_1}\mu_2(x)\}$, we have that $\underline{D}_{\mu_1}\mu_2(x) = \bar{D}_{\mu_1}\mu_2(x)$ $(= D_{\mu_1}\mu_2(x)) < \infty$ for μ_1 almost all $x \in X$.

Next, to establish (*) we proceed as follows. For any Borel set $A \subset X$ let

$$\nu(A) = \int_A (D_{\mu_1}\mu_2)\,d\mu_1$$

and for any subset $A \subset X$ let $\nu(A) = \inf_{\substack{B\supset A \\ B \text{ Borel}}} \nu(B)$.

Then ν is evidently a Radon measure and

$$t_1\,\mu_1(A_{t_1,t_2}) \le \mu(A_{t_1,t_2}) \le t_2\,\mu_1(A_{t_1,t_2})$$

for any $0 < t_1 \le t_2$, $A_{t_1,t_2} = \{x \in A : t_1 < D_{\mu_1}\mu_2(x) < t_2\}$, A any Borel set. By then by (1), (2) we have

$$\frac{t_1}{t_2}\,\mu_2(A_{t_1,t_2}) \le \nu(A_{t_1,t_2}) \le \frac{t_2}{t_1}\,\mu_2(A_{t_1,t_2})$$

and it readily follows that $\nu = \mu_2$. Thus (*) is established.

In the general case (when it may be that $\mu_2(A) > 0$ when $\mu_1(A) = 0$)

select a Borel set B from the collection $\mathcal{A} = \{A \subset X : A$ is Borel, $\mu_1(X\sim A) = 0\}$ such that $\mu_2(B) = \inf_{A\in\mathcal{A}} \mu_2(A)$. (Take $B = \bigcap_{i=1}^{\infty} A_i$, where $A_i \in \mathcal{A}$, $\lim \mu_2(A_i) = \inf_{A\in\mathcal{A}} \mu_2(A)$.) Now if $A \subset B$ with $\mu_1(A) = 0$ then we must have $\mu_2(A) = 0$ also, otherwise we contradict the minimality of $\mu_2(B)$. Then we can apply the previous argument to the measure $\tilde{\mu}_2 = \mu_2 \llcorner B$, thus giving

$$\mu_2(A\cap B) = \int_A (D_{\mu_1}\mu_2)\, d\mu_1 \quad \forall \text{ Borel set } A \subset X .$$

Thus 4.7(2) holds with $\mu_2^* = \mu_2 \llcorner (X\sim B)$.

Finally, in case X also has the symmetric Vitali property with respect to μ_2 , the first part of the argument above establishes that $D_{\mu_1}\mu_2$ exists μ_2-almost everywhere (as well as μ_1-almost everywhere) in $\tilde{X}$ and (1) shows that if $A \subset \{x \in \tilde{X} : D_{\mu_1}\mu_2(x) < \infty\}$ $(= \bigcup_{n=1}^{\infty} \{x\in\tilde{X} : D_{\mu_1}\mu_2(x) < n\})$ and if $\mu_1(A) = 0$, then also $\mu_2(A) = 0$. We can therefore apply the above argument to $\tilde{\mu}_2 = \mu_2 \llcorner \{x \in \tilde{X} : D_{\mu_1}\mu_2(x) < \infty\}$. Since we trivially have $D_{\mu_1}\mu_2(x) = \infty$ for μ_2-a.e. $x \in X \sim \tilde{X}$, we then deduce 4.7(1) with μ_2^* as in 4.7(2).