ON ISOLATED SINGULARITIES OF MINIMAL SURFACES

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We here want to give a brief discussion of some questions related to minimal surfaces with isolated singularities; the many questions related to minimal surfaces with more complicated singular sets are not considered.

We first make our terminology precise. For simplicity of exposition we discuss embedded submanifolds of Euclidean space - making the necessary comments about the more general Riemannian setting at appropriate points.

M will denote a smooth n-dimensional embedded submanifold of $\mathbb{R}^{n+k}$, $n \geq 2$, $k \geq 1$, where we always use the term "embedded" to mean locally properly embedded. Thus for each $y \in M$ there is an open ball $B_\rho(y)$ with centre y and radius $\rho > 0$, and a C^2 diffeomorphism ψ of $B_\rho(y)$ onto $B_\rho(0)$ such that $\psi\big(M \cap B_\rho(y)\big) = \mathbb{R}^n \cap B_\rho(0)$. Here and subsequently we identify $\mathbb{R}^n$ with the subspace of $\mathbb{R}^{n+k}$ consisting of all points $\big(x^1, \ldots, x^{n+k}\big)$ such that $x^j = 0$ $\forall j = n+1, \ldots, n+k$.

M is said to be *minimal* if the mean curvature of M is identically zero. As is well-known, this condition is equivalent to the *local area minimizing property*: for each $y \in M$ there is some open ball $B_\rho(y) \subset \mathbb{R}^{n+k}$ such that

$$H^n\left(M \cap B_\rho(y)\right) \leq H^n\left(\tilde{M} \cap B_\rho(y)\right) \qquad \left(H^n = \text{k-dimensional}\right.$$

$$\left.\text{Hausdorff measure}\right)$$

whenever $\tilde{M} = \varphi(M)$, where $\varphi : B_\rho(y) \to B_\rho(y)$ is a C^1 diffeomorphism such that $\{x \in B_\rho(y) : \varphi(x) \neq x\}$ is contained in a compact subset of $B_\rho(y)$. This explains the use of the term minimal.

The *regular set* reg M of M is defined to be the set of all points $y \in$ clos M $\left(\text{closure of } M \text{ taken in } \mathbb{R}^{n+k}\right)$ such that $B_\rho(y) \cap$ clos M is an embedded n-dimensional C^2 submanifold of $\mathbb{R}^{n+k}$ for some $\rho > 0$. The *singular set* sing M of M is defined by sing M = clos M $\sim$ reg M. By definition sing M is a *closed* subset of $\mathbb{R}^{n+k}$ and reg M $\supset$ M; if the inclusion reg M $\supset$ M is strict, then there are removable singularities. We always assume here that such singularities have been removed, so that

$$M = \text{reg } M , \qquad \text{sing } M = \text{clos } M \sim M .$$

A point $y \in \mathbb{R}^{n+k}$ is said to be an *isolated* singularity of M if $y \in$ sing M and sing M $\cap B_\rho(y) = \{y\}$ for some $\rho > 0$.

The simplest examples of embedded minimal M with isolated singularities (indeed the only known codimension 1 examples until the work [CHS] to be described in §2 below) are the embedded *minimal cones*; that is, minimal M representable in the form

$$(0.1) \qquad M = \{\lambda y : y \in \Sigma, 0 < \lambda < \infty\} ,$$

where Σ is a compact embedded (n-1)-dimensional submanifold of S^{n+k-1}.

There are many such minimal cones (see e.g. [HL]).
A simple example (corresponding to $n = 3$, $k = 1$) is

$$M = \left\{ \left(x^1, x^2, x^3, x^4\right) \in \mathbb{R}^4 \sim \{0\} : \left(x^1\right)^2 + \left(x^2\right)^2 = \left(x^3\right)^2 + \left(x^4\right)^2 \right\} .$$

In this case the corresponding submanifold $\Sigma \subset S^3$ (as in (0.1))
is just the two dimensional flat torus

$$\left(\frac{1}{\sqrt{2}} S^1\right) \times \left(\frac{1}{\sqrt{2}} S^1\right) \subset S^3 \subset \mathbb{R}^4 \equiv \mathbb{R}^2 \times \mathbb{R}^2 .$$

More generally for any $p \geq 1$

$$M = \left\{ (x, y) \in \mathbb{R}^p \times \mathbb{R}^p \sim \{0\} : |x|^2 = |y|^2 \right\}$$

is a $(2p-1)$-dimensional minimal cone, corresponding to $\Sigma \subset S^{2p-1}$
defined by $\Sigma = \left(\frac{1}{\sqrt{2}} S^{p-1}\right) \times \left(\frac{1}{\sqrt{2}} S^{p-1}\right) .$

The study of minimal surfaces generally is closely
related to quasilinear P.D.E. theory, by virtue of the fact that U
is an open subset of $\mathbb{R}^n$ and if $u = \left(u^1, \ldots, u^k\right) : U \to \mathbb{R}^k$ is a
$C^2(U)$ function with values in $\mathbb{R}^k$, then

$$(0.2) \qquad M = \text{graph } u \equiv \left\{ (x, u(x)) : x \in U \right\}$$

is minimal if and only if u satisfies the system of equations

$$(0.3) \qquad \sum_{i,j=1}^{n} g^{ij} D_{ij} u^\ell = 0 \text{ on } U , \qquad \ell = 1, \ldots, k .$$

Here $\left(g^{ij}\right) = \left(g_{ij}\right)^{-1}$, $g_{ij} = \delta_{ij} + D_i u \cdot D_j u$. The system (0.3)
is called the *minimal surface system* (minimal surface *equation* in

case k = 1), and with respect to suitably chosen coordinate axes
any minimal $M \subset \mathbb{R}^{n+k}$ can be represented locally as graph u for
some u satisfying (0.3). All this is readily checked by directly
computing the mean curvature vector of M in case M has the
form (0.2). (See, for example, [O].)

We should make a point here: it is *not* in general
true that M can be represented in the form (0.2) in the neighbourhood
of an isolated singular point. Indeed in the codimension 1 case
(when k = 1), *no* M of the form (0.2) can have isolated singular
points. For this and related results see [FR], [B], [MM], [SL2, 3].
On the other hand in case n = 4 , k = 3 there is an example due
to Lawson and Osserman [LO] of an M having the form (0.2) and
having an isolated singularity. The example is also of the form (0.1)
for some compact smooth $\Sigma \subset S^3$.

The outline of the present article is as follows:
in §1 we discuss an old question (still unsettled) as to whether an
embedded minimal surface in $\mathbb{R}^3$ can have an isolated singularity.
§2 gives a brief summary of recent work of Caffarelli, Hardt, Simon
[CHS] on examples of minimal submanifolds with isolated singularities,
obtained by perturbation of minimal cones (as in (0.1)). In §3 we
briefly discuss the important question of whether or not a minimal
surface is asymptotic to a minimal cone near an isolated singular
point.

For reference in these subsequent sections we here
make some remarks concerning the *area* growth of minimal surfaces
near an isolated singularity. Thus we let $M \subset \mathbb{R}^{n+k}$ be as above and

we suppose that $\text{sing } M \cap B_{\rho_0} = \{0\}$ for some $\rho_0 > 0$. Here and subsequently we abbreviate $B_\rho = B_\rho(0)$. Since we assume M is locally properly embedded we have

$$(0.4) \qquad H^n\big(M \cap (B_\rho \sim B_\sigma)\big) < \infty , \qquad 0 < \sigma < \rho < \rho_0 .$$

Now by plugging in a function of the form $\psi(|x|)x$ in the first variation formula (see [AW] or [SL1] or [MS] for a discussion) we get the identity

$$(0.5) \qquad \int_M \left[n\psi(|x|) + |x|\psi'(|x|)\big|\nabla|x|\big|^2 \right] = 0 ,$$

provided support ψ is a compact subset of $(0, \rho_0)$. In (0.5), ∇ denotes gradient on M; that is $\nabla f(x)$ is the orthogonal projection of the ordinary $\mathbb{R}^{n+k}$ gradient of a function f onto the tangent space $T_x M$.

We now choose $\rho \in (0, \rho_0)$, $\sigma \in (0, \rho)$ and replace ψ by the function $\psi_{\sigma,\rho}$ defined by

$$\psi_{\sigma,\rho}(t) = \varphi_\sigma(t)\psi(t/\rho) ,$$

where $\varphi'_\sigma(t) \geq 0$ $\forall t$, $\varphi_\sigma(t) \equiv 0$ for $t < \sigma/2$, $\varphi(t) \equiv 1$ for $t > \sigma$, $\psi'(t) \leq 0$ $\forall t$, and $\psi(t) = 0$ for $t \geq 1$. Then (0.5) implies, for $0 < \sigma < \rho < \rho_0$, that

$$(0.6) \qquad n \int_M \varphi_\sigma(|x|)\psi(|x|/\rho) - \rho\frac{d}{d\rho} \int_M \varphi_\sigma(|x|)\psi(|x|/\rho)\big|\nabla|x|\big|^2 \leq 0 ,$$

where we have used $-\rho \frac{\partial}{\partial \rho}[\psi(t/\rho)] = t\frac{\partial}{\partial t}[\psi(t/\rho)]$ (the common value

being $(t/\rho)\psi'(t/\rho)$). Taking $\varepsilon \in (0, 1)$, $\psi \equiv 1$ on $[0, 1-\varepsilon]$

and multiplying through by ρ^{-n-1} and rearranging the resulting

expression, making use of the fact that $\rho^{-n}|\psi'(|x|/\rho)| \geq \frac{(1-\varepsilon)^n}{|x|^n}|\psi'(|x|/\rho)|$

(because $\psi(t/\rho) \equiv 1$ for $t \leq (1-\varepsilon)\rho$), we then deduce

$$-\frac{d}{d\rho}\left[\int_M \varphi_\sigma(|x|)\frac{1-|\nabla|x||^2}{(1-\varepsilon)^{-n}|x|^n}\psi(|x|/\rho)\right] + \frac{d}{d\rho}\left[\rho^{-n}\int_M \varphi_\sigma(|x|)\psi(|x|/\rho)\right] \geq 0 ,$$

so that in particular

$$(0.7) \quad \int_M \varphi_\sigma(|x|)\frac{1-|\nabla|x||^2}{(1-\varepsilon)^{-n}|x|^n}\left\{\psi(|x|/\rho)-\psi(|x|/\tau)\right\} + \tau^{-n}\int_M \varphi_\sigma(|x|)\psi(|x|/\tau)$$

$$\leq \rho^{-n}\int_M \varphi_\sigma(|x|)\psi(|x|/\rho)$$

$$\leq -n^{-1}\rho^{-n}\int_M \varphi_\sigma(|x|)\psi'(|x|/\rho)$$

where, in the last line, we used (0.6) again. This is valid for

$\sigma < \tau < \rho < \rho_0$. Holding ρ, τ fixed, and letting $\sigma \downarrow 0$ we deduce

by virtue of (0.4) that

$$(0.8) \qquad\qquad H^n(M \cap B_\tau) < \infty \ \forall \tau < \rho_0 .$$

Then we can let $\varphi_\sigma \uparrow 1$ and $\psi \uparrow \chi$, where χ is the characteristic

function of the interval $[0, 1)$. This gives

$$(0.9) \qquad \int_{M \cap (B_\rho \sim B_\tau)} |x|^{-n}(1-|\nabla|x||^2) + \tau^{-n}H^n(M \cap B_\tau)$$

$$\leq \rho^{-n}H^n(M \cap B_\rho) < \infty , \quad 0 < \tau < \rho < \rho_0 .$$

This is the well-known *monotonicity formula* for minimal surfaces,
which we have thus shown to be valid if M has an isolated
singularity at 0 . (Actually it is valid in the presence of
much more serious singularities - in fact for arbitrary stationary
varifolds - but in this case one needs *a-priori* to assume finiteness
of the area, which we did *not* need to assume here.)

A formula like (0.9) continues to hold in the more
general case when M is a submanifold of a Riemannian manifold N ;
in this case the B_ρ are Geodesic balls in N , ρ must be
sufficiently small, and there is an additional factor of the form
$(1+c\rho)$ on the right hand side (c a constant depending on N).

§1.   2-dimensional surfaces in $\mathbb{R}^3$

Here we suppose n = 2 , k = 1 ; thus M is an
embedded 2-dimensional minimal surface in $\mathbb{R}^3$. The question is
whether or not such surfaces can have an isolated singularity at 0 .

We shall here sketch the simple proof that such an
isolated singularity *cannot* exist if we make the additional assumption
that M is stable in $B_{\rho_0} \sim \{0\}$ for some $\rho_0 > 0$. Thus we shall
assume that sing M $\cap \left(B_{\rho_0} \sim \{0\}\right) = \emptyset$ and

(1.1)
$$\int_{M \cap B_{\rho_0}} |A|^2 \varphi^2 \leq \int_{M \cap B_{\rho_0}} |\nabla\varphi|^2$$

whenever support $\varphi \subset B_{\rho_0} \sim \{0\}$ is compact, where A denotes the

second fundamental form of M . ((1.1) is just the stability

inequality; see for example [SL1] for an elementary discussion.)

We want to prove $0 \in \text{reg } M$. Note first that

(1.1), taken together with the area bounds (0.9), implies

immediately that

(1.2) $\qquad \int_{M \cap B_\rho} |A|^2 < \infty \ \forall \rho \in \left(0, \rho_0\right)$.

(To check this, just take φ in (1.1) with $\varphi \equiv 1$ on $B_\rho \sim B_\sigma$,

$\varphi = 0$ on ∂B_{ρ_0} , $\varphi \equiv 0$ in $B_{\sigma/2}$, and $|\nabla\varphi| \leq c/\sigma$, then let

$\sigma \downarrow 0$ and use (0.9) to bound the right hand side.) Also by the

curvature estimates ([SSY] of [SS1]) for stable minimal surfaces we

have that

(1.3) $\qquad |A(x)| \leq c/|x|$, $x \in M \cap B_{\rho_0/2}$.

Now let $\rho_k \downarrow 0$ and let $M_k = \left\{\rho_k^{-1}x : x \in M\right\}$. By

(1.3) and the Arzela-Ascoli lemma we can select a subsequence

$\{k'\} \subset \{k\}$ such that $\left\{M_{k'}\right\}$ converges (in a C^1 sense) to a

minimal surface C with $\text{sing } C = \{0\}$. (Notice that we do this

by locally representing the $M_{k'}$ as graphs of functions satisfying

the minimal surface equation, which is possible in a uniform way by

(1.3), because (1.3) tells us that the second fundamental form $A_{k'}$

of $M_{k'}$ is uniformly bounded in any annular region $B_\rho \sim B_\sigma$ as

$k' \to \infty$.) By virtue of the monotonicity formula (0.9), we easily

check that $\int_{C \cap B_\rho} \left(1 - |\nabla|x||^2\right) = 0$ (∇ = gradient operator on C , so

that $\left| \nabla |x| \right| \equiv 1$ on C). Then C must be a cone with vertex at the origin, and $C \cap S^2$ is a 1-dimensional embedded minimal surface (i.e. a geodesic) in S^2, and hence must be a great circle in S^2. Thus we finally deduce that C is a plane through the origin. Without loss of generality, let us suppose that the unit normal of C is $e_3 = (0, 0, 1)$.

Since $M_{k'}$ approaches C locally (away from 0) in a C^1 fashion, we see that

$$\sup_{M \cap \left(B_{\rho_{k'}} \sim B_{\rho_{k'/2}} \right)} \left(1 - |\nu \cdot e_3| \right) \to 0 \quad \text{as} \quad k' \to \infty \,,$$

where ν = unit normal of M. In particular if we take any $r > k$ and if we let $\Sigma_{k,r}$ be any component of $M \cap \left(B_{\rho_{k'}} \sim B_{\rho_{r'}} \right)$, and if we write $L_{k,r} = \partial \Sigma_{k,r}$, then we see that the Gauss map $\nu : M \to S^2$ takes $L_{k,r}$ to an ε-neighbourhood of e_3, where $\varepsilon \to 0$ as $k, r \to \infty$.

On the other hand the Jacobian (area magnification factor) of the Gauss map is minus the Gauss curvature of M, which is $\frac{1}{2}|A|^2$. Hence the area of the Gauss image of $\Sigma_{k,r}$ is $\frac{1}{2} \int_{\Sigma_{k,r}} |A|^2$, which converges to zero as $k \to \infty$ by (1.2), and furthermore the Gauss map is either a constant map on an *open* map (because the Gauss curvature vanishes at only isolated points of $\Sigma_{k,r}$ unless $\Sigma_{k,r}$ is contained in a plane). We therefore deduce that the Gauss image of $\Sigma_{k,r}$ must be contained in the same ε-neighbourhood of e_3 as $L_{k,r}$, where $\varepsilon \to 0$ as $k \to \infty$.

Letting $r \to \infty$, holding k fixed (but large), we can thus deduce that the oscillation of ν on any component of $B \cap M$ is small, $B = B_{\rho_{k'}}$. Writing $B' = B_{\rho_{k'}/2}$, it follows that for each component M^* of $B \cap M$ with $0 \in \operatorname{clos} M^*$ we have a minimal graph G with

$$0 \in (\operatorname{clos} G) \cap B' = (\operatorname{clos} M^*) \cap B' \ ,$$

provided k' is sufficiently large. (In checking this one needs to use the fact that the oscillation of ν on M^* is small, together with the fact that M^* is connected, embedded, and $\partial M^* \subset \partial B$.) But by the discussion of the introduction, we know that isolated singularities of codimension 1 minimal graphs are removable, hence we have

$$\operatorname{clos} M^* \cap B' = G \ ,$$

where $G = \operatorname{graph} u$, with $u(0) = 0$ and with u satisfying the minimal surface equation in some neighbourhood of the origin in $\mathbb{R}^2$.

This essentially completes the proof, except that we have to check that there cannot be more than one component M^* of $M \cap B$ whose closure contains the origin. However if there were another, then we could apply the above argument to it in order to deduce that there are *two* minimal graphs G_1, G_2 with $G_1 \cap G_2 = \{0\}$. Since the difference of two solutions of the minimal surface equation satisfies the maximum principle, this is impossible.

Notice that the above result has been extended to the case $2 < n \leq 6$; thus if $2 < n \leq 6$ ($k = 1$ and M stable in

$B_{\rho_0} \sim \{0\}$ still being assumed), then M cannot have an isolated

singularity at 0 . Indeed if $H^n\left(M \cap B_{\rho_0}\right) < \infty$ (which is automatic

from (0.9) in case M has an isolated singularity at 0 and ρ_0 is

sufficiently small) and if M is stable in $B_{\rho_0} \sim$ sing M , then

(1.4) $\qquad H^{n-2}\left(\text{sing } M \cap B_{\rho_0}\right) = 0 \Rightarrow \text{sing } M \cap B_{\rho_0} = \emptyset$.

(In particular this guarantees isolated singularities are removable

for $2 < n \leq 6$.) (1.4) is proved in the recent work of Schoen and

Simon [SS1]. The results of [SS1] are actually only stated for the

case when M is stable in B_{ρ_0} , but the reader will see that the

proofs of [SS1] actually apply to the case when M is only stable

in $B_{\rho_0} \sim$ sing M (in the sense that (1.1) holds for any

$\varphi \in C_0^1\left(B_{\rho_0}\right)$).

      All this generalizes to the case when M is a

codimension 1 embedded submanifold of a general Riemannian manifold,

and to the case when M has bounded (rather than zero) mean

curvature. In fact [SS1] is already presented in this setting, and

the main change needed in the above 2-dimensional argument relates to

the fact that the Gauss map is no longer necessarily open. One then

needs to use the iterative procedure described in §2 of [SS2] to prove

that the oscillation of ν on M^* is small as before.

§2.   Examples of minimal surfaces with isolated singularities

The material in this section is an outline of some parts of recent joint work [CHS].

Let Σ be any smooth $(n-1)$-dimensional submanifold of S^{n+k-1} , $1 \leq k$, $n \geq 2$, and let C be the cone over Σ . Thus

$$C = \{\lambda \omega : \lambda > 0, \; \omega \in \Sigma\}$$

and C has an isolated singularity at 0 unless it is a n-dimensional plane through the origin. Write also

$$C_r = C \cap B_r \; ,$$

so that

$$\partial C_1 = \Sigma \; .$$

For the moment we are not assuming C is minimal.

We consider first the linear Dirichlet problem

(2.1)
$$\begin{cases} L_C u = f \quad \text{on} \quad C_1 \\[2mm] u = \psi \quad \text{on} \; \Sigma \; , \end{cases}$$

where f, ψ are given functions on C_1 and Σ respectively, and where L_C is a linear second order operator of the form

$$L_C u = \Delta_C u + Q u \; ,$$

where Δ_C denotes the Laplace-Beltrami operator for C , and Q has

the form

$$Q(x) = r^{-2}q(\omega) \ , \quad x \in C \ , \quad r = |x| \ , \quad \omega = x/|x| \in \Sigma \ .$$

Since C is a cone, we have

(2.2) $\qquad L_C u = \dfrac{1}{r^{n-1}} \dfrac{\partial}{\partial r}\Big(r^{n-1} \dfrac{\partial u}{\partial r}\Big) + r^{-2}\big(\Delta_\Sigma u + q(\omega)u\big)$

where on the right $u = u(r, \omega)$, $0 < r < 1$, $\omega \in \Sigma$ and Δ_Σ
(the Laplace-Beltrami operator for Σ) acts on $u(r, \omega)$ as a
function of $\omega \in \Sigma$ with r fixed.

Our initial aim in this section is to discuss the
following simple question concerning solutions of (2.1):
Suppose $p > 0$,

(2.3) $\qquad \|f\|_r \le c \ r^{p-2} \ , \qquad \forall 0 < r < 1 \ ,$

where $\|f\|_r = \big(\int_\Sigma f^2(r\omega)d\omega\big)^{\frac{1}{2}}$; then for which boundary data ψ in
(2.1) can we find a solution u of (2.1) such that u decays near
0 in the sense

(2.4) $\qquad \|u\|_r \le c' \ r^p \ , \qquad \forall 0 < r < 1 \ .$

(Notice that the requirement $\|f\|_r \le c \ r^{p-2}$ is evidently necessary
for there to exist *any* such boundary data ψ in view of the form
of the operator L_C .)

We write $L_\Sigma = \Delta_\Sigma + q$ and we let

$$\mu_1 \le \mu_2 \le \ldots \qquad \big(\mu_k \to \infty \ \text{as} \ k \to \infty\big)$$

denote the eigenvalues of L_Σ, and let

$$\varphi_1, \varphi_2, \ldots$$

be a corresponding orthonormal basis for $L^2(\Sigma)$. (Notice that these φ_j are automatically of class $C^{2,\alpha}$ on Σ by virtue of the assumption that q is $C^{0,\alpha}$.) Thus any $u \in L^2(C_1)$ can be written

$$u(x) \equiv u(r\omega) = \sum_{j=1}^{\infty} a_j(r)\varphi_j(\omega)$$

where $\sum_{j=1}^{\infty} a_j^2(r) < \infty$ a.e. $r \in [0, 1]$, and (by (2.2)) such a u is a solution of the equation $L_c u = f$ (in the generalized sense) if and only if

$$(2.5) \qquad r^2 a_j'' + (n-1)r a_j' - \mu_j a_j(r) = r^2 f_j(r) , \qquad j = 1, 2, \ldots$$

(in the generalized sense), where

$$f_j(r) = \int_\Sigma f(r\omega)\varphi_j(\omega)\,d\omega \equiv \left(f(r\omega), \varphi_j(\omega) \right)_{L^2(\Sigma)}$$

where $0 < r < 1$.

The homogeneous equation (i.e. (2.5) with $f_j \equiv 0$) has solutions $c r^\gamma$, where γ is any root of

$$(2.5)' \qquad\qquad \gamma^2 + (n-2)\gamma - \mu_j = 0 ,$$

so

$$\gamma = -\frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu_j} \quad .$$

For each j we let γ_j be the root corresponding to the plus sign.

Then since $w_j = r^{-\gamma_j} a_j$ satisfies the equation
$$\left(r^{n-1+2\gamma_j} w_j'\right)' = r^{n-1+\gamma_j} f_j$$ we see that there are solutions

$$(2.6) \quad a_j(r) = \mathrm{Re}\left\{\alpha_j r^{\gamma_j} + r^{\gamma_j} \int_{\beta_j}^{r} s^{-(n-1)-2\gamma_j} \int_0^s \tau^{n-1+\gamma_j} f_j(\tau)d\tau ds\right\} \quad ,$$

where α_j, β_j are constants with $\beta_j > 0$. We now make the additional assumption on p that there is an integer $j_0 \geq 1$ such that $p > 0$ (as before) and

$$(2.7) \qquad \mathrm{Re}\left(\gamma_{j_0}\right) < p < \mathrm{Re}\left(\gamma_{j_0+1}\right) \quad .$$

The expressions for a_j in (2.6) make sense, and give a solution of (2.5), provided that we take

$$\begin{cases} \alpha_j \in \mathbb{R} \text{ arbitrary, and } \beta_j = 1 \text{ for } j \geq j_0 + 1 \\ \alpha_j = 0 \text{ , and } \beta_j = 0 \text{ for } j \leq j_0 \end{cases} .$$

Indeed one easily checks that then

$$u = \sum_{j=1}^{\infty} a_j(r)\varphi_j(\omega)$$

satisfies (2.1) with

$$\psi = \sum_{j \geq j_0 + 1} \alpha_j \varphi_j + \sum_{j=1}^{j_0} Re \left(\int_0^1 s^{-(n-1)-2\gamma_j} \int_0^s \tau^{n-1+\gamma_j} f_j(\tau) d\tau ds \right)$$

and that

$$(2.8) \quad \|u\|_r \leq c \left[\sum_{j \geq j_0 + 1} \alpha_j^2 + \sup_{0 < \tau < 1} \tau^{2-p} \|f\|_\tau \right] r^p , \quad 0 < r < 1$$

as required. Furthermore one readily checks that u is the unique solution of (2.1) satisfying $\sup_{0 < r < 1} r^{-p} \|u\|_r < \infty$, together with the boundary data

$$(2.9) \quad \left(\ell im_{r \uparrow 1} u(r\omega), \varphi_j(\omega) \right)_{L^2(\Sigma)} = \alpha_j , \quad j \geq j_0 + 1 .$$

(We want to emphasize particularly that we have no control over the numbers $\left(\ell im_{r \uparrow 1} u(r\omega), \varphi_j(\omega) \right)_{L^2(\Sigma)}$, $j \leq j_0$; these are uniquely determined by f .)

Using the notation $\Pi_{j_0} : L^2(\Sigma) \to L^2(\Sigma)$ to denote the operator $w \mapsto \sum_{j \geq j_0 + 1} \left(w, \varphi_j \right)_{L^2(\Sigma)} \varphi_j$, we have thus proved that for any given $\psi \in L^2(\Sigma)$ there is a unique solution u of

$$(2.10) \quad \begin{cases} L_C u = f \quad \text{on} \quad C_1 \\ \Pi_{j_0} (u - \psi) = 0 \quad \text{on} \quad \partial C_1 \end{cases}$$

with $\sup_{0 < r < 1} r^{-p} \|u\|_r < \infty$; this solution satisfies

$(2.10)'$ $\quad \|u\|_r \leq c \left[\left\| \Pi_{j_0} \psi \right\|_{L^2(\Sigma)} + \sup_{0<\tau<1} \tau^{2-p} \|f\|_\tau \right] r^p$,

$$\left\| \left(1-\Pi_{j_0}\right) u \big|_\Sigma \right\|_{L^2(\Sigma)} \leq c \sup_{0<\tau<1} \tau^{2-p} \|f\|_\tau \ ,$$

where $u|_\Sigma(\omega) = \lim_{r\uparrow 1} u(r\omega)$. Of course we are still assuming (2.3) and (2.7).

To describe applications of this to non-linear equations (and to minimal surfaces) consider the Banach spaces

$$\mathcal{B}_p = \left\{ u \in C^2(C_1) \ : \ \|u\|_{\mathcal{B}_p} \right.$$

$$\left. \equiv \sup_{0<r<1} r^{-p} \left(\sum_{j=0}^{2} r^j |\nabla^j u|_{0;r} + r^{2+\alpha} [\nabla^2 u]_{\alpha;r} \right) < \infty \right\} \ ,$$

where we use the notation that $\nabla^2 u$ denotes the covariant Hessian of u ,

$$|w|_{0;r} = \sup_{\partial C_r} |w| \ , \quad [w]_{\alpha;r} = \sup_{\substack{x,y \in C_r \sim C_{r/2} \\ x \neq y}} \frac{|w(x)-w(y)|}{|x-y|^\alpha} \ ,$$

and where α is a fixed constant in the interval $(0, 1)$.

Now consider the quasilinear problem of finding $u \in \mathcal{B}_p$ such that

(2.11)
$$\begin{cases} Lu + N(u) = f & \text{on } C_1 \\ \Pi_{j_0}(u-\psi) = 0 & \text{on } \Sigma \ , \end{cases}$$

where f , ψ , j_0 are as above but where we now make the additional assumption

(2.12) $$\mathrm{Re}\ \gamma_{j_0} \geq 1\ ,$$

so that (2.7) implies $p > 1$. Concerning the operator N we assume

(2.13) $\quad Nu = N^{(1)}(x,\ u/r,\ \nabla u) \cdot \nabla^2 u + r^{-1}N^{(2)}(x,\ u/r,\ \nabla u)\ .$

Here, for $(x,\ z,\ p) \in C_1 \times \mathbb{R} \times \mathbb{R}^{n+k}$, $N^{(1)}(x,\ z,\ p)$ is a symmetric

bilinear form on $\mathbb{R}^{n+k}$ and $N^{(1)}(x,\ u/r,\ \nabla u) \cdot \nabla^2 u$ should be taken

to mean trace $S_1 \circ S_2$, where S_1, S_2 are the symmetric bilinear

transformations on $T_x C \times T_x C$ associated with

$N^{(1)}(x,\ u/r,\ \nabla u)\big|_{T_x C \times T_x C}$ and $\nabla^2 u(x)$. We also assume that

$N^{(1)}, N^{(2)}$ have a C^2 dependence on $(x,\ z,\ p)$ $(x \neq 0)$ and satisfy

the structural conditions

(2.14)
$$
\left\{
\begin{aligned}
&r\left(\big|N_x^{(1)}(x,\ z,\ p)\big| + \big|N_x^{(2)}(x,\ z,\ p)\big|\right) + \big|N_z^{(1)}(x,\ z,\ p)\big| \\[2mm]
&\quad + \big|N_p^{(1)}(x,\ z,\ p)\big| + r\sum_{j=1}^{2}(|z|+|p|)^{1-j}\left(\big|N_{xz}^{(j)}(x,z,p)\big|+\big|N_{xp}^{(j)}(x,z,p)\big|\right) \\[2mm]
&\quad + \sum_{j=1}^{2}\left(\big|N_{zz}^{(j)}(x,\ z,\ p)\big| + \big|N_{zp}^{(j)}(x,\ z,\ p)\big| + \big|N_{pp}^{(j)}(x,\ z,\ p)\big|\right) \leq \mu \\[4mm]
&\big|N^{(1)}(x,\ z,\ p)\big| + \big|N_z^{(2)}(x,\ z,\ p)\big| + \big|N_p^{(2)}(x,\ z,\ p)\big| \leq \mu(|z| + |p|) \\[4mm]
&\big|N^{(2)}(x,\ z,\ p)\big| \leq \mu(|z| + |p|)^2\ .
\end{aligned}
\right.
$$

(Here subscripts denote partial derivatives with respect to the indicated variables.)

Subject to these assumptions, the result described in (2.10) above, together with a standard application of the implicit function theorem (or the contraction mapping principle), enables us to assert that if (2.3), (2.7), (2.11), (2.12), (2.14) hold, then there is a constant $\beta_0 > 0$ such that the problem

(2.15)
$$\begin{cases} Lu + N(u) = \beta f & \text{on} \quad C_1 \\ \Pi_{j_0}(u-\beta\psi) = 0 & \text{on} \quad \Sigma \end{cases}$$

is solvable for any $\beta \leq \beta_0$, with $u \in \mathcal{B}_p$ satisfying

(2.15)'
$$\|u - u_1\|_{\mathcal{B}_p} \leq c\beta^2 ,$$

where u_1 is the solution of the linear problem (2.10) with βf, $\beta\psi$ in place of f, ψ.

If C_1 is *minimal* of codimension 1 (i.e. k = 1), and if M is the graph of a function $u \in \mathcal{B}_p$ over C_1 :

$$M = \{x + \nu(x)u(x) : x \in C_1\} \quad (\nu = \text{unit normal of } C_1) ,$$

then M is a minimal hypersurface if and only if u satisfies $M_C u = 0$ on C_1, where M_C is the minimal surface operator (i.e. the mean curvature operator). As is well known, M_C has the form

$$M_C(u) = L_C u + N(u) ,$$

with N as in (2.13), (2.14) and with L_C as in (2.2) with

(2.16) $$q = |A|^2 \quad \text{on} \quad \Sigma \, ,$$

A = second fundamental form of C . Therefore the above general existence result gives a large new class of examples of minimal surfaces with isolated singularities. Specifically:

THEOREM 1. *Suppose* C *is minimal,* $k = 1$, q *is as in* (2.16), *and suppose* j_0 *is chosen so that* (2.7), (2.12) *hold for some* $p > 1$. *Then for any given* $\psi \in C^{2,\alpha}(\Sigma)$ *with* $\left(1 - \Pi_{j_0}\right)\psi = 0$, *there is a 1-parameter family* $\{u_\beta\}_{0 < \beta \leq \beta_0} \subset B_p$ *of solutions of* $M_C u = 0$ *such that*

$$\overset{\bullet}{u}_0\Big|_\Sigma = \psi \, .$$

Here $\overset{\bullet}{u}_0(x) = \lim\limits_{\beta \downarrow 0} \beta^{-1} u_\beta(x)$, $x \in C_1 \cup \Sigma$. In fact we have

$$\overset{\bullet}{u}_0 = \text{Re} \sum_{j \geq j_0 + 1} r^{\gamma_j} \left(\psi, \varphi_j\right)_{L^2(\Sigma)} \varphi_j \quad \text{on} \quad C_1 \, .$$

All the above discussion extends straightforwardly to the case when our functions are sections of the normal bundle $TC^\perp$ of C , provided we take

(2.17) $$L_C = \Delta_C^\perp + r^{-2} q(\omega) \, ,$$

where $\Delta_C^\perp$ is the normal Laplacian (see [SJ]) and $q(\omega) : T_\omega \Sigma^\perp \to T_\omega \Sigma^\perp$

is linear with smooth dependence on ω . If C is minimal, the minimal surface operator M_C in this arbitrary codimension case is an operator taking smooth sections of $TC^{\perp}$ to smooth sections of $TC^{\perp}$ with the form

$$M_C = L_C + N ,$$

with L_C as in (2.17) with

(2.18) $$q(\omega) = A_{\omega}$$

where $A_{\omega} : \left(T_{\omega}\Sigma\right)^{\perp} \to \left(T_{\omega}\Sigma\right)^{\perp}$ is defined by

$$v_2 \cdot A_{\omega}\left(v_1\right) = \text{trace} \left(v_1 \cdot A_{\omega}\right) \circ \left(v_2 \cdot A_{\omega}\right) ,$$

where A is the second fundamental form of Σ (thus $v_1 \cdot A_{\omega}$ is a bilinear form on $T_{\omega}\Sigma \times T_{\omega}\Sigma$), and where N satisfies similar structural conditions to the N in (2.13), (2.14). (As before, $M_C u = 0$ on C_1 if and only if graph $u \equiv \left\{x + u(x) : x \in C_1\right\}$ is an n-dimensional minimal submanifold of $\mathbb{R}^{n+k}$.)

We thus have in particular the following theorem.

THEOREM 2. *Suppose* C *is minimal,* $k \geq 1$, q *is as in* (2.18) *and suppose* j_0 *is chosen so that* (2.7), (2.12) *hold for some* $p > 1$. *Then for any given* $\psi \in C^{2,\alpha}\left(\Sigma, T\Sigma^{\perp}\right)$ *there is a 1-parameter family* $\left\{u_{\beta}\right\}_{0 < \beta \leq \beta_0} \subset \overset{\perp}{B}_p$ *of solutions of* $M_C u = 0$ *such that*

$$\overset{\bullet}{u}_0\Big|_{\Sigma} = \psi .$$

Here the notation is as follows: $\overset{\bullet}{u}_0 = \underset{\beta \downarrow 0}{\lim} \beta^{-1} u_\beta$ as before, $\mathcal{B}_p^\perp$

denotes the set of $C^{2,\alpha}(C_1, \mathbb{R}^{n+k})$ functions $w = (w^1, \ldots, w^{n+k})$

such that each component w^j is in $\mathcal{B}_p$ and such that

$w(x) \in T_x C^\perp$ $\forall x \in C_1$, and $C^{2,\alpha}(\Sigma, T\Sigma^\perp)$ denotes the set of

$C^{2,\alpha}(\Sigma, \mathbb{R}^{n+k})$ functions $w = (w^1, \ldots, w^{n+k})$ such that each

component w^j is in $C^{2,\alpha}(\Sigma)$ and $w(x) \in T_x \Sigma^\perp$ $\forall x \in \Sigma$.

The question now arises whether or not these perturbed

solutions are stable in case the underlying minimal cone is stable.

To facilitate further discussion we first derive a criterion

(involving eigenvalues of L_Σ) for the cone C to be stable.

Since there are no 2-dimensional minimal cones other than the 2 planes,

we assume $n \geq 3$ for the remainder of this section.

To begin , let us suppose C is an arbitrary cone

(not necessarily minimal) and let $L_C = \Delta_C + Q$ as before.

Let $\zeta \in C_0^2(C_1)$ be arbitrary, and write

$\zeta(r\omega) = \sum\limits_{j=1}^\infty a_j(r) \, \varphi_j(\omega)$, where $a_j \in C_0^1((0, 1))$. Then by (2.2)

$$\int_{C_1} \zeta L_C \zeta = \int_0^1 \int_\Sigma \sum_{j=1}^\infty \left[\frac{\partial}{\partial r}(r^{n-1} a_j') \varphi_j(\omega) + r^{n-3} a_j (L_\Sigma \varphi_j) \right] \left(\sum_{k=1}^\infty a_k \varphi_k \right) d\omega \, dr \; ,$$

so that since $L_\Sigma \varphi_j = -\mu_j \varphi_j$ and since φ_j are orthonormal in

$L^2(\Sigma)$, we get

(2.19) $\int_{C_1} \zeta L_C \zeta = -\int_0^1 \left[r^{n-1} \sum\limits_{j=1}^\infty (a_j')^2 + r^{n-3} \sum\limits_{j=1}^\infty \mu_j a_j^2 \right] dr$.

Now using integration by parts we have

(2.20) $-\int_0^1 r^{n-3} \sum_{j=1}^{\infty} \mu_j a_j^2 \, dr \le \int_0^1 \sum_{j=1}^{\infty} \mu_j^- a_j^2 r^{n-3} \, dr$,

$$\mu_j^- = \max\{-\mu_j, 0\} \ ,$$

and

(2.21) $\int_0^1 \sum_{j=1}^{\infty} \mu_j^- a_j^2 r^{n-3} dr = -\frac{2}{n-2} \int_0^1 r^{n-2} \sum_{j=1}^{\infty} \mu_j^- a_j a_j' \, dr$

$$\le \frac{2}{n-2} \left(\int_0^1 r^{n-1} \sum_{j=1}^{\infty} \mu_j^- a_j'^2 dr\right)^{\frac{1}{2}} \left(\int_0^1 r^{n-3} \sum_{j=1}^{\infty} \mu_j^- a_j^2 dr\right)^{\frac{1}{2}} \ ,$$

so that

$$\int_0^1 r^{n-3} \sum_{j=1}^{\infty} \mu_j^- a_j^2 \, dr \le \frac{4}{(n-2)^2} \int_0^1 r^{n-1} \sum_{j=1}^{\infty} \mu_j^- \left(a_j'\right)^2 \, dr$$

$$\le \frac{4 \, \mu_1^-}{(n-2)^2} \int_0^1 r^{n-1} \sum_{j=1}^{\infty} \left(a_j'\right)^2 \, dr \ ,$$

using this in (2.20) we see that $\int_{C_1} \zeta L_C \zeta \le 0 \quad \forall \zeta \in C_0^2(C_1)$ provided

(2.22) $\dfrac{4}{(n-2)^2} \, \mu_1 \ge -1$.

Since $\int_{C_1} \zeta L_C \zeta = -\int_{C_1} \left(|\nabla \zeta|^2 - Q\zeta^2\right)$, we have thus shown

(2.23) $\int_{C_1} Q\zeta^2 \le \int_{C_1} |\nabla \zeta|^2 \quad \forall \zeta \in C_0^1(C_1)$

provided that (2.22) holds.

The condition (2.22) is equivalent to the requirement that all the roots γ in (2.5)' are real. We then easily see (since a complex root γ_1 gives oscillatory solutions

$$\zeta_1 = \mathrm{Re}\left(r^{\gamma_1} \varphi_1\right) \quad \text{for} \quad L_C \, \zeta_1 = 0)$$

$$\inf_{\zeta \in C_0^2(C_1)} \int_{C_1} |\nabla \zeta|^2 - Q\zeta^2 < 0$$

if (2.22) fails. Thus we conclude that (2.23) holds if and only if (2.22) holds.

In case C is minimal of codimension 1 and $Q = r^{-2}|A|^2(\omega)$ (A = second fundamental form of C, restricted to Σ) (2.23) is then exactly the stability inequality for C (cf. (1.1)). This extends routinely to the case of arbitrary codimension (as for Theorem 2). In this case the stability inequality requires

$$(2.24) \quad \int_{C_1} A(\zeta) \cdot \zeta \leq \int_{C_1} |\nabla^{\perp} \zeta|^2 , \quad \forall \zeta \in C_0^1(C_1, TC^{\perp}) ,$$

and by the appropriate modifications of the argument above, we see that this is true if and only if (2.22) holds, where now μ_1 is the minimum eigenvalue of the operator

$$L_{\Sigma} \zeta \equiv \Delta_{\Sigma}^{\perp} \zeta + A(\zeta) ,$$

where $\Delta_{\Sigma}^{\perp}$ is the normal Laplacian for Σ (see [SJ]) and where A is as in (2.18).

The condition (2.22) for stability (at least in the

codimensional 1 case) was first derived by J. Simons [SJ; Lemma 6.1.6]. R. Schoen has pointed out that it is equivalent to the requirement that Σ be conformally equivalent to a positive scalar curvature manifold.

It is instructive to check the condition (2.22) for the codimension 1 examples $\Sigma = \left(\frac{1}{\sqrt{2}} S^p\right) \times \left(\frac{1}{\sqrt{2}} S^p\right)$. One readily checks in this case that $|A|^2(\omega) \equiv 2p$ on Σ , so that μ_1 is the minimum eigenvalue of $\Delta_\Sigma + 2p$, which is trivially $-2p$. (The minimum eigenvalue of Δ_Σ is zero.) Since $n = 2p + 1$ in this case, the criterion (2.22) tells us that the cone C over Σ is stable if and only if

$$\frac{(-n+1)}{(n-2)^2} \geq -\tfrac{1}{4}$$

i.e.

$$n \geq 7 .$$

(Cf. [SJ].) Similar considerations show that if Σ is a codimension 1 minimal submanifold of S^n , with second fundamental form having constant length κ , then the cone over Σ is stable if and only if

$$\kappa \leq \frac{n-2}{2} .$$

We note that the argument above actually establishes

$$\int_{C_1} \left(|\nabla \zeta|^2 - Q\zeta^2\right) \geq \left[1 - \frac{4\mu_1^-}{(n-2)^2}\right] \int_{C_1} (\partial\zeta/\partial r)^2 ,$$

so that if we have *strict* inequality in (2.22), then

$$(2.25) \qquad \int_{C_1} \left(|\nabla \zeta|^2 - Q\zeta^2 \right) \geq \theta \int_{C_1} \left(\partial \zeta / \partial r \right)^2 \quad \forall \zeta \in C_0^1(C_1)$$

for some constant $\theta \in (0, 1)$. Using the identity

$$\int_\Sigma \int_0^1 \frac{\partial}{\partial r} \left(r^{n-2} \zeta^2 (r\omega) \right) dr \, d\omega = 0 , \text{ so that}$$

$$\left(\frac{n-2}{2} \right)^2 \int_{C_1} \zeta^2 / r^2 \leq \int_{C_1} \left(\partial \zeta / \partial r \right)^2 , \text{ we then have}$$

$$(2.26) \qquad \int_{C_1} \left(|\nabla \zeta|^2 - Q\zeta^2 \right) \geq \theta' \int_{C_1} \zeta^2 / r^2 \quad \forall \zeta \in C_0^1(C_1) ,$$

where $\theta' > 0$ is positive.

Notice that (by virtue of the standard representation of minimum eigenvalue in terms of Rayleigh quotient) (2.23) is equivalent to the assertion that the eigenvalue problem

$$(2.27) \qquad \begin{cases} L_C u + \dfrac{\lambda}{r^2} u = 0 \quad \text{on} \quad C_{\varepsilon,1} \equiv \{x \in C : \varepsilon < |x| < 1\} \\[2mm] \qquad u = 0 \quad \text{on} \quad \partial C_{\varepsilon,1} \end{cases}$$

has minimum eigenvalue $\lambda_1^\varepsilon \geq 0 \quad \forall \varepsilon > 0$, while (2.26) is equivalent to $\lambda_1^\varepsilon \geq \theta' > 0$ (θ' independent of ε).

That is, (2.22) $\Leftrightarrow \lambda_1^\varepsilon \geq 0 \quad \forall \varepsilon > 0 \Leftrightarrow$ (2.23); and strict inequality in (2.22) $\Leftrightarrow \lambda_1^\varepsilon \geq \theta > 0$ for some θ independent of ε

$$\Leftrightarrow (2.26) \text{ for some } \theta' > 0 .$$

In view of this, and the fact that the u_β obtained

in Theorems 1, 2 lie in $\mathcal{B}_p$ ($\mathcal{B}_p^\perp$ respectively) we now easily deduce the following theorem.

THEOREM. *If C is strictly stable in the sense that strict inequality holds in* (2.22), *then there is* $\beta_0 > 0$ *such that all the minimal graphs of Theorems 1,2 are stable for* $\beta \leq \beta_0$.

(If $\mu_1 = \dfrac{(n-2)^2}{4}$, then the minimal graphs of Theorems 1, 2 are stable in the weak sense that the stability inequality holds for $\zeta \in \mathcal{B}_q$ (respectively $\mathcal{B}_q^\perp$) with $\zeta = 0$ and Σ and $\|\zeta\|_{\mathcal{B}_q} \leq K$, provided $q > -\left(\dfrac{n-2}{2}\right)$ and provided β is sufficiently small, depending on K).

§3.   The question of classifying isolated singularities – uniqueness of tangent cones.

To facilitate the discussion of this section let us assume that M has an isolated singularity in the following very strict sense:

DEFINITION.   M is said to have an isolated singularity at 0 in the *strong sense* if sing $M \cap B_\rho = \{0\}$ for some $\rho > 0$ and if the second fundamental form A of M satisfies

(3.1) $$|A(x)| \leq c/|x| , \quad 0 < |x| < \rho .$$

One would of course ultimately like to understand the general case when it is simply assumed $M \cap B_\rho = \{0\}$, but for our present purposes

the above assumption is convenient. Notice that (by the compactness and regularity theory developed in [SS1]) (3.1) will automatically hold for a seven dimensional stable embedded minimal hypersurface with an isolated singularity at 0 .

In view of the examples of the previous section, one might be led to conjecture that M is always asymptotic to a minimal cone C near an isolated singular point at 0 , at least in the sense

$$(3.2) \qquad \lim_{\rho \downarrow 0} \left(\sup_{\substack{x \in M \cap B_\rho \\ y \in C \cap B_\rho}} |x-y| \Big/ \rho \right) = 0$$

for some minimal cone C . This would in fact be extremely illuminating, because it is elementary to check that (3.1), (3.2) imply that sing C = {0} and that the spherical nearest point projection ψ (taking a point $y \in M \cap \partial B_\sigma$ to the point $z \in C \cap \partial B_\sigma$ with least distance, measured in ∂B_σ , from y) is a smooth covering projection. Thus we get a precise description of the exact nature of the isolated singularity at 0 in terms of the minimal cone C and an integer multiplicity (which is constant on the components of C).

Unfortunately, (3.2) appears very difficult to prove, even if we assume the strong condition (3.1). It is true that there always exist tangent cones in the sense that if $\{\rho_k\}$ is a sequence of positive numbers converging to zero, then there is a subsequence $\{\rho_{k'}\}$ and a cone C

[SS2] R. Schoen, L. Simon, Regularity of simply connected
         surfaces with quasiconformal Gauss map
         (To appear in Ann. Math. Studies volume on Minimal
         Surfaces).

[SSY] R. Schoen, L. Simon, S.-T. Yau, Curvature estimates
         for minimal hypersurfaces, *Acta Math.* 134 (1975),
         275-288.

[SL1] L. Simon, Survey lectures on minimal surfaces
         (To appear in Ann. Math. Studies volume on Minimal
         Surfaces).

[SL2] L. Simon, On a theorem of De Giorgi and Stampacchia,
         Math. Z. 155 (1977), 199-204.

[SL3] L. Simon, On some extensions of Bernstein's
         Theorem, *Math. Z.* 154 (1977), 265-273.

[SJ]   J. Simons, Minimal varieties in Riemannian manifolds,
         Ann. of Math. 88 (1968), 62-105.

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