17. ALGORITHMS FOR FINITE RANK METHODS

As pointed out in Section 11, one approach to solving an eigenvalue problem for $T \in BL(X)$ is to consider a nearby operator $T_0 \in BL(X)$ which is simpler than T, and first solve an eigenvalue problem for T_0. For example, if T is a large full matrix then T_0 can be a much smaller matrix, or a matrix with some special structure like tridiagonality or sparsity. We can attempt to refine the eigenelements λ_0 and φ_0 of T_0 for obtaining approximations of corresponding eigenelements λ and φ of T. Several iterative procedures of this kind are given in Section 11 when λ_0 is a simple eigenvalue of T_0. In practice, one often chooses T_0 to be a bounded operator of finite rank. In the present section, we describe the step by step construction of the refinement schemes of Section 11 when $T_0 \in BL(X)$ is of finite rank. The relevant algorithms can be implemented on a computer. (Many of the results of this section appear in the thesis [DE].)

Let us first study the spectrum of a bounded operator T_0 of finite rank. Since T_0 is a compact operator, one can appeal to the well known results for the spectra of compact operators. However, we prefer to give an independent treatment.

If the dimension of X is greater than the rank of T_0 (in particular, if X is infinite dimensional), then $T_0x = 0$ for some nonzero $x \in X$, i.e., 0 is an eigenvalue of T_0. Next, let $\lambda_0 \neq 0$. If λ_0 is not an eigenvalue of T_0, i.e., if $T_0 - \lambda_0 I$ is one to one, then we show that $T_0 - \lambda_0 I$ is also onto, so that λ_0 is not a spectral value of T_0. Let $\tilde{T} = (T_0 - \lambda_0 I)|_{R(T_0)}$. Since $R(T_0)$ is finite dimensional and $\tilde{T}$ is one to one, we see that $\tilde{T}$ maps $R(T_0)$

onto $R(T_0)$. Let $y \in X$. Since $T_0 y \in R(T_0)$, consider $\tilde{x} \in R(T_0)$ such that $(T_0-\lambda_0 I)\tilde{x} = \tilde{T}\tilde{x} = T_0 y$. If we let $x = (\tilde{x}-y)/\lambda_0$, we see that $(T_0-\lambda_0 I)x = (T_0 y - T_0 y + \lambda_0 y)/\lambda_0 = y$. Hence $T_0 - \lambda_0 I$ is onto. Thus, every nonzero spectral value of T_0 is an eigenvalue of T_0 .

In order to find the nonzero eigenvalues of T_0 , we first set up some notations.

Consider $x_1,\ldots,x_n$ in X and $x_1^*,\ldots,x_n^*$ in X^* . Then the map

$$(17.1) \qquad T_0 x = \sum_{i=1}^{n} \langle x,x_i^*\rangle x_i \ , \quad x \in X \ .$$

is a bounded operator on X , and since $R(T_0) \subset \mathrm{span}\{x_1,\ldots,x_n\}$, it is of finite rank. Conversely, (3.8) shows that every bounded operator of finite rank is of this form. We shall assume throughout this section that T_0 is given by (17.1). Then it is easy to see that its adjoint T_0^* is given by

$$(17.2) \qquad T_0^* x^* = \sum_{i=1}^{n} \langle x^*,x_i\rangle x_i^* \ , \quad x^* \in X^* \ .$$

Consider the matrix

$$A = [a_{i,j}] \ , \quad a_{i,j} = \langle x_j,x_i^*\rangle \ , \quad i,j = 1,\ldots,n \ .$$

Let $\mathbb{C}^n$ denote the set of all column vectors $\underset{\sim}{c} = [c(1),\ldots,c(n)]^t$, $c(i) \in \mathbb{C}$ for $i = 1,\ldots,n$. As pointed out in Section 1, we denote the operator induced by the matrix A on $\mathbb{C}^n$ also by A .
Consider the linear map $F : X \to \mathbb{C}^n$ given by

$$(17.3) \qquad Fx = [\langle x,x_1^*\rangle,\ldots,\langle x,x_n^*\rangle]^t \ , \quad x \in X \ ,$$

and the linear map $G : \mathbb{C}^n \to X$ given by

$$(17.4) \qquad G\underset{\sim}{c} = c(1)x_1 + \ldots + c(n)x_n \ , \quad \underset{\sim}{c} \in \mathbb{C}^n \ .$$

Then

$$GF = T_0 \quad \text{and} \quad FG = A \ . \tag{17.5}$$

Of these, the first equality is immediate, and the second follows since for every $\underset{\sim}{c} \in \mathbb{C}^n$,

$$\begin{aligned}
FG\underset{\sim}{c} &= c(1)Fx_1 + \dots + c(n)Fx_n \\
&= \sum_{i=1}^{n} c(i)[\langle x_i, x_1^* \rangle, \dots, \langle x_i, x_n^* \rangle]^t \\
&= \left[\sum_{i=1}^{n} c(i)\langle x_i, x_1^* \rangle, \dots, \sum_{i=1}^{n} c(i)\langle x_i, x_n^* \rangle\right]^t \\
&= \left[(A\underset{\sim}{c})(1), \dots, (A\underset{\sim}{c})(n)\right]^t \\
&= A\underset{\sim}{c} \ .
\end{aligned}$$

PROPOSITION 17.1 Let $0 \neq \lambda_0 \in \mathbb{C}$. Then λ_0 is an eigenvalue, a semisimple eigenvalue or a simple eigenvalue of T_0 if and only if it is an eigenvalue, a semisimple eigenvalue or a simple eigenvalue of A, respectively. If $\underset{\sim}{u}$ is an eigenvector of A corresponding to λ_0, then $G\underset{\sim}{u}$ is an eigenvector of T_0 corresponding to λ_0; if x is an eigenvector of T_0 corresponding to λ_0, then Fx is an eigenvector of A corresponding to λ_0.

Proof For $k = 1,2$, let

$$V_k = Z((T_0-\lambda_0 I)^k) \quad \text{and} \quad W_k = Z((A-\lambda_0 I)^k) \ .$$

We show that

$$F(V_k) = W_k \ , \quad G(W_k) = V_k \ ,$$

and the maps $F|_{V_k}$, $G|_{W_k}$ are one to one.

It follows by (17.5) that for $x \in V_k$,

$$(A-\lambda_0 I)^k Fx = (FG-\lambda_0 I)^k Fx = F(GF-\lambda_0 I)^k x = F(T_0-\lambda_0 I)^k x = \underset{\sim}{0} \ ,$$

so that $Fx \in W_k$. Similarly, for $\underset{\sim}{c} \in W_k$,

$$(T_0-\lambda_0 I)^k G\underset{\sim}{c} = (GF-\lambda_0 I)^k G\underset{\sim}{c} = G(FG-\lambda_0 I)^k \underset{\sim}{c} = G(A-\lambda_0 I)^k \underset{\sim}{c} = 0 ,$$

so that $G\underset{\sim}{c} \in V_k$. Thus, $F(V_k) \subset W_k$ and $G(W_k) \subset V_k$.

Next, let $x \in V_k$ and $Fx = \underset{\sim}{0}$. Then

$$T_0 x = GFx = G(\underset{\sim}{0}) = 0 ,$$

so that $T_0^2 x = 0$ also. But since $x \in V_k$, we have $(T_0-\lambda_0 I)^k x = 0$. This implies $\lambda_0^k x = 0$, i.e., $x = 0$, as $\lambda_0 \neq 0$. Thus, $F|_{V_k}$ is one to one. In an exactly similar manner, it follows that $G|_{W_k}$ is one to one. Now, since $W_k \subset \mathbb{C}^n$ is finite dimensional, we see that W_k and V_k have the same finite dimension and $F(V_k) = W_k$, $G(W_k) = V_k$ for $k = 1,2$.

We immediately note that λ_0 is an eigenvalue of T_0 if and only if $V_1 \neq \{0\}$ if and only if $W_1 \neq \{0\}$ if and only if λ_0 is an eigenvalue of A . Since the n-dimensional operator A has at most n distinct eigenvalues, the same is true about the nonzero eigenvalues of T_0 . As every nonzero spectral value of T_0 is an eigenvalue, we see that $\sigma(T_0)$ consists of at most $n + 1$ (isolated) points.

If $0 \neq \lambda_0 \in \sigma(T_0)$, then by considering a simple closed rectifiable curve Γ_0 which isolates λ_0 from 0 as well as the rest of $\sigma(T_0)$, we see that the range of the spectral projection P_0 associated with T_0 and λ_0 is contained in the range of T_0 (cf. Problem 6.7 and (7.19)). Since T_0 is of finite rank, it follows that λ_0 is an eigenvalue of T_0 of finite algebraic multiplicity. Since A is itself a finite dimensional operator, it is obvious that every eigenvalue of A has finite algebraic multiplicity.

Now, by Theorem 7.5(b) and Proposition 7.3, λ_0 is a semisimple eigenvalue of T_0 if and only if $V_2 = V_1 \neq \{0\}$ if and only if $W_2 = W_1 \neq \{0\}$ if and only if λ_0 is a semisimple eigenvalue of A. Also, λ_0 is a simple eigenvalue of T_0 if and only if $V_2 = V_1$ and $\dim V_1 = 1$, which is the case if and only if $W_2 = W_1$ and $\dim W_1 = 1$, i.e., λ_0 is a simple eigenvalue of A. The statements regarding the eigenvectors of T_0 and A corresponding to λ_0 are also clear. //

COROLLARY 17.2 Let $0 \neq \lambda_0$ be a simple eigenvalue of A and let $\underset{\sim}{u}$ be a corresponding eigenvector. Let $\underset{\sim}{v}$ be the eigenvector of A^H corresponding to $\bar{\lambda}_0$ such that $\underset{\sim}{v}^H\underset{\sim}{u} = 1/\lambda_0$. Then λ_0 is a simple eigenvalue of T_0. Also,

$$(17.7)\quad \varphi_0 = u(1)x_1 + \ldots + u(n)x_n \quad\text{and}\quad \varphi_0^* = v(1)x_1^* + \ldots + v(n)x_n^*$$

are eigenvectors of T_0 and T_0^* corresponding to λ_0 and $\bar{\lambda}_0$, respectively, and they satisfy $\langle\varphi_0,\varphi_0^*\rangle = 1$. In fact,

$$(17.8)\qquad \langle\varphi_0,x_j^*\rangle = \lambda_0 u(j)\ ,\ j = 1,\ldots,n\ .$$

The spectral projection P_0 associated with T_0 and λ_0 is given by

$$(17.9)\qquad P_0x = \sum_{i,j=1}^{n} u(i)\overline{v(j)}\langle x,x_j^*\rangle x_i\ .$$

Proof The spectral subspace associated with A and λ_0 is one dimensional; if $\underset{\sim}{u}$ is a nonzero vector in this space, it follows by Theorem 8.3 that there is $\underset{\sim}{w}$ in the one dimensional spectral subspace associated with $A^* = A^H$ and $\bar{\lambda}_0$ such that $\langle\underset{\sim}{u},\underset{\sim}{w}\rangle = \underset{\sim}{w}^H\underset{\sim}{u} = 1$. We can then let $\underset{\sim}{v} = \underset{\sim}{w}/\bar{\lambda}_0$.

By Proposition 17.1, λ_0 is a simple eigenvalue of T_0 and $\varphi_0 = G\underset{\sim}{u} = u(1)x_1 + \ldots + u(n)x_n$ is a corresponding eigenvector of T_0. Next, (17.2) shows that

$$T_0^* x^* = \sum_{i=1}^{n} \langle x^*, x_i^{**} \rangle x_i^* , \quad x^* \in X^* ,$$

where $\langle x_i^{**}, x^* \rangle \equiv \langle x_i, x^* \rangle$ for $i = 1,\ldots,n$ and $x^* \in X^*$. Also,

$$\langle x_j^*, x_i^{**} \rangle = \overline{\langle x_i^{**}, x_j^* \rangle} = \overline{\langle x_i, x_j^* \rangle} = \overline{a_{j,i}} .$$

Thus, we can apply Proposition 17.1 to T_0^*, its simple eigenvalue $\bar{\lambda}_0$ and the operator A^*. Hence $\varphi_0^* = v(1)x_1^* + \ldots + v(n)x_n^*$ is an eigenvector of T_0^* corresponding to $\bar{\lambda}_0$. Now, for $j = 1,\ldots,n$,

$$\langle \varphi_0, x_j^* \rangle = \langle \sum_{i=1}^{n} u(i)x_i, x_j^* \rangle = \sum_{i=1}^{n} \langle x_i, x_j^* \rangle u(i) = (A\underset{\sim}{u})(j) = \lambda_0 u(j) ,$$

since $A\underset{\sim}{u} = \lambda_0 \underset{\sim}{u}$. This proves (17.8). Hence

$$\langle \varphi_0, \varphi_0^* \rangle = \sum_{j=1}^{n} \langle \varphi_0, x_j^* \rangle \overline{v(j)} = \lambda_0 \sum_{i=1}^{n} u(j)\overline{v(j)} = 1 .$$

Finally, (17.9) follows from

$$P_0 x = \langle x, \varphi_0^* \rangle \varphi_0 = \sum_{i=1}^{n} u(i)\langle x, \varphi_0^* \rangle x_i = \sum_{i,j=1}^{n} u(i)\overline{v(j)}\langle x, x_j^* \rangle x_i . \qquad //$$

REMARK 17.3 The converse of the above result also holds, i.e., if $0 \neq \lambda_0$ is a simple eigenvalue of T_0, and φ_0 (resp., φ_0^*) is an eigenvector of T_0 (resp, T_0^*) corresponding to λ_0 (resp., $\bar{\lambda}_0$) such that $\langle \varphi_0, \varphi_0^* \rangle = 1$, then λ_0 is a simple eigenvalue of $\underset{\sim}{A}$, and

$$\underset{\sim}{u}' = [\langle \varphi_0, x_1^* \rangle, \ldots, \langle \varphi_0, x_n^* \rangle]^t \quad \text{and} \quad \underset{\sim}{v}' = [\langle \varphi_0^*, x_1 \rangle, \ldots, \langle \varphi_0^*, x_n \rangle]^t$$

are eigenvectors of A and A^* corresponding to λ_0 and $\bar{\lambda}_0$, respectively, such that

$$(\underset{\sim}{v}')^H \underset{\sim}{u}' = \sum_{i=1}^{n} \langle \varphi_0, x_i^* \rangle \overline{\langle \varphi_0^*, x_i \rangle} = \langle \sum_{i=1}^{n} \langle \varphi_0, x_i^* \rangle x_i, \varphi_0^* \rangle$$

$$= \langle T_0\varphi_0, \varphi_0^* \rangle = \langle \lambda_0\varphi_0, \varphi_0^* \rangle = \lambda_0 \ .$$

Let us recall the following iteration schemes for approximating eigenelements λ and φ of an operator $T \in BL(X)$ which we have discussed in Section 11.

Let λ_0 be a simple eigenvalue of $T_0 \in BL(X)$ with a corresponding eigenvector φ_0 . Let φ_0^* be the eigenvector of T_0^* corresponding to $\bar{\lambda}_0$ such that $\langle \varphi_0, \varphi_0^* \rangle = 1$. Let P_0 and S_0 denote the associated spectral projection and the reduced resolvent.

(i) The Rayleigh-Schrödinger iteration scheme (11.18) :

$$\varphi_j = \varphi_{j-1} + S_0\left[-(T-\lambda_1 I)\varphi_{j-1} + \sum_{k=2}^{j} (\lambda_k - \lambda_{k-1})\varphi_{j-k}\right]$$

(ii) The fixed point iteration scheme (11.19):

$$\varphi_j = \varphi_{j-1} + S_0\left[-T\varphi_{j-1} + \lambda_j\varphi_{j-1}\right]$$

(iii) The modified fixed point iteration scheme (11.31):

$$\varphi_j = \frac{T\varphi_{j-1}}{\lambda_j} + \frac{S_0}{\lambda_j}\left[-T^2\varphi_{j-1} + \frac{\langle T^2\varphi_{j-1}, \varphi_0^* \rangle}{\lambda_j} T\varphi_{j-1}\right]$$

(iv) The Ahués iteration scheme (11.35):

$$\varphi_j = \frac{T\varphi_{j-1}}{\lambda_j} + \frac{S_0}{\lambda_j}\left[-T^2\varphi_{j-1} + \lambda_j T\varphi_{j-1}\right] .$$

Recall that in all these cases,

$$\lambda_j = \langle T\varphi_{j-1}, \varphi_0^* \rangle .$$

We remark that in the case of all the above iteration schemes,

$$\langle \varphi_j, \varphi_0^* \rangle = \langle \varphi_0, \varphi_0^* \rangle = 1$$

for $j = 1,2,\ldots,$ as can be proved by induction on j. Hence if we let y_{j-1} equal $-(T-\lambda_1 I)\varphi_{j-1} + \sum_{k=2}^{j} (\lambda_k - \lambda_{k-1})\varphi_{j-1}$, or $-T\varphi_{j-1} + \lambda_j \varphi_{j-1}$, or $-T^2\varphi_{j-1} + \langle T^2\varphi_{j-1}, \varphi_0^* \rangle T\varphi_{i-1}/\lambda_j$, then $\langle y_{j-1}, \varphi_0^* \rangle = 0$, i.e., $P_0 y_{j-1} = 0$. Thus, to implement the first three schemes, we need to calculate

(i) $\langle Tx, \varphi_0^* \rangle$ for various $x \in X$ and

(ii) $S_0 y$ for various $y \in X$ which satisfy $P_0 y = 0$.

We shall comment on the implementation of the fourth scheme in Remark (17.7).

We have already seen in Corollary 17.2 how to find λ_0, φ_0 and φ_0^* in case T_0 is of finite rank. In the next result we give a procedure for finding $S_0 y$, where $y \in X$ and $P_0 y = 0$.

PROPOSITION 17.4 Let $0 \neq \lambda_0$, $\underset{\sim}{u}$, φ_0, $\underset{\sim}{v}$ and φ_0^* be as in Corollary 17.2. For $y \in X$ with $P_0 y = 0$, the unique solution x of the system

$$(T_0 - \lambda_0 I)x = y, \quad P_0 x = 0 \tag{17.9}$$

is given by

$$x = S_0 y = \frac{1}{\lambda_0}\left[-y + \sum_{i=1}^{n} \alpha(i) x_i\right] \tag{17.10}$$

where $\underset{\sim}{\alpha} = [\alpha(1),\ldots,\alpha(n)]^t$ is the unique solution of

$$(A - \lambda_0 I)\underset{\sim}{\alpha} = [\langle y, x_1^* \rangle, \ldots, \langle y, x_n^* \rangle]^t, \quad \underset{\sim}{v}^H \underset{\sim}{\alpha} = 0. \tag{17.11}$$

Also, we have

$$\langle x, x_i^* \rangle = \alpha(i) \ , \quad i = 1,\ldots,n \ . \tag{17.12}$$

Proof Since $S_0|_{(I-P_0)X}$ is the inverse of $(T_0-\lambda_0 I)|_{(I-P_0)X}$, it is clear that $x = S_0 y$ is the unique solution of (17.9). Now, we have

$$x = \frac{1}{\lambda_0}(-y + T_0 x) = \frac{1}{\lambda_0}(-y + GFx) \ .$$

Thus, to obtain x, it is enough to find Fx. Now,

$$Fx = [\langle x, x_1^* \rangle, \ldots, \langle x, x_n^* \rangle]^t \ ,$$

$$\underset{\sim}{v}^H Fx = \sum_{i=1}^{n} \langle x, x_i^* \rangle \overline{v(i)} = \langle x, \sum_{i=1}^{n} v(i) x_i^* \rangle = \langle x, \varphi_0^* \rangle = 0 \ ,$$

since $P_0 x = 0$. Similarly, $Fy = [\langle y, x_1^* \rangle, \ldots, \langle y, x_n^* \rangle]^t$ and $\underset{\sim}{v}^H Fy = 0$, since $P_0 y = 0$. Also,

$$(A-\lambda_0 I)Fx = (FG-\lambda_0 I)Fx = F(GF-\lambda_0 I)x = F(T_0-\lambda_0 I)x = Fy \ .$$

Since λ_0 is a simple eigenvalue of A and $\underset{\sim}{v}$ is an eigenvector of A^* corresponding to $\bar{\lambda}_0$, we see that whenever $\underset{\sim}{v}^H \underset{\sim}{\beta} = 0$, the system

$$(A-\lambda_0 I)\underset{\sim}{\alpha} = \underset{\sim}{\beta} \ , \quad \underset{\sim}{v}^H \underset{\sim}{\alpha} = 0$$

has a unique solution in $\mathbb{C}^n$. We have just now seen that if $\underset{\sim}{\beta} = Fy$, then $\underset{\sim}{\alpha} = Fx$ is such a solution. In particular, for $i = 1,\ldots,n$, we have $\alpha(i) = Fx(i) = \langle x, x_i^* \rangle$. This establishes (17.12) and completes the proof. //

We are now in a position to find the iterates λ_j and φ_j for a variety of iteration schemes. The problem of solving the operator equations $(T_0-\lambda_0 I)x = y$, $P_0 x = 0$ (where $P_0 y = 0$) is reduced to solving matrix equations in the following manner. A similar problem is considered in [WH] when $\lambda_0 \in \rho(T_0)$.

THEOREM 17.5 Let $0 \neq \lambda_0$, $\underset{\sim}{u}$, φ_0 , $\underset{\sim}{v}$ and φ_0^* be as in Corollary 17.2. Consider an iteration scheme

$$(17.13) \qquad \begin{aligned} \lambda_j &= \langle T\varphi_{j-1}, \varphi_0^* \rangle , \\ \varphi_j &= \xi_j + S_0 \eta_j , \end{aligned} \qquad j = 1,2,\ldots,$$

where ξ_j , $\eta_j \in X$ may depend on T , $\lambda_0,\ldots,\lambda_j$, φ_0 , $\varphi_1,\ldots,\varphi_{j-1}$, and where $P_0\eta_j = 0$. Then for $j = 1,2,\ldots$.

$$(17.14) \qquad \lambda_j = \sum_{i=1}^{n} \langle T\varphi_{j-1}, x_i^* \rangle \overline{v(i)} ,$$

$$(17.15) \qquad \varphi_j = \xi_j + \frac{1}{\lambda_0} \left[-\eta_j + \sum_{i=1}^{n} \alpha_j(i) x_i \right] ,$$

where $\underset{\sim}{\alpha}_j = [\underset{\sim}{\alpha}_j(1),\ldots,\underset{\sim}{\alpha}_j(n)]^t$ is the unique solution of

$$(A-\lambda_0 I)\underset{\sim}{\alpha}_j = \underset{\sim}{\beta}_j , \quad \underset{\sim}{v}^H \underset{\sim}{\alpha}_j = 0 ,$$

with

$$\underset{\sim}{\beta}_j = [\langle \eta_j, x_1^* \rangle, \ldots, \langle \eta_j, x_n^* \rangle]^t .$$

Moreover, we have for $j = 1,2,\ldots$ and $i = 1,\ldots,n$,

$$(17.16) \qquad \langle \varphi_j, x_i^* \rangle = \langle \xi_j, x_i^* \rangle + \alpha_j(i) .$$

Proof Since $\varphi_0^* = \sum_{i=1}^{n} v(i) x_i^*$, the expression (17.14) for λ_j follows immediately. To prove (17.15), we note that by Proposition 17.4,

$$S_0\eta_j = \frac{1}{\lambda_0} \left[-\eta_j + \sum_{i=1}^{n} \alpha_j(i) x_i \right] ,$$

where $\underset{\sim}{\alpha}_j = [\alpha_j(1),\ldots,\alpha_j(n)]^t$ satisfies

$$(A-\lambda_0 I)\underset{\sim}{\alpha}_j = \underset{\sim}{\beta}_j , \quad \underset{\sim}{v}^H \underset{\sim}{\alpha}_j = 0 ,$$

with $\underset{\sim}{\beta}_j = [\langle \eta_j, x_1^* \rangle, \ldots, \langle \eta_j, x_n^* \rangle]^t$. Hence (17.15) holds. Lastly, by letting $x = S_0\eta_j$ in (17.12), we see that $\langle S_0\eta_j, x_i^*\rangle = \alpha_j(i)$. Hence the relation (17.16) follows. //

Remark 17.6 The above theorem can be applied to the Rayleigh-Schrödinger iteration scheme (11.18) with

$$\xi_j = \varphi_{j-1}\ , \quad \eta_j = -(T-\lambda_1 I)\varphi_{j-1} + \sum_{k=2}^{j} (\lambda_k-\lambda_{k-1})\varphi_{j-k}$$

and to the fixed point iteration scheme (11.19) with

$$\xi_j = \varphi_{j-1}\ , \ \eta_j = -T\varphi_{j-1} + \lambda_j\varphi_{j-1}\ .$$

In both the cases, we have for $j = 1,2,\ldots$ and $i = 1,\ldots,n$,

$$\begin{aligned}\langle\varphi_j, x_i^*\rangle &= \langle\varphi_{j-1}, x_i^*\rangle + \alpha_j(i)\\ &= \langle\varphi_0, x_i^*\rangle + \alpha_1(i) + \ldots + \alpha_j(i)\\ &= \lambda_0 u(i) + \alpha_1(i) + \ldots + \alpha_j(i)\ ,\end{aligned}$$

because of (17.16) and (17.8). If we let

$$\alpha_0(i) = \lambda_0 u(i)\ , \quad i = 1,\ldots,n\ ,$$

then we have

$$\langle\varphi_j, x_i^*\rangle = \sum_{k=0}^{j} \alpha_k(i)\ , \quad j = 0,1,2,\ldots\ . \tag{17.17}$$

This relation can be used in calculating the right hand sides $\beta_j = [\langle\eta_j, x_1^*\rangle,\ldots,\langle\eta_j, x_n^*\rangle]^t$, $j = 1,2,\ldots,$ as follows. For the iteration scheme (11.18), we have

$$\begin{aligned}\beta_j(i) &= \langle\eta_j, x_i^*\rangle\\ &= -\langle T\varphi_{j-1}, x_i^*\rangle + \lambda_1 \sum_{k=0}^{j-1} \alpha_k(i) + \sum_{p=2}^{j} (\lambda_p-\lambda_{p-1}) \sum_{k=0}^{j-p} \alpha_k(i)\\ &= -\langle T\varphi_{j-1}, x_i^*\rangle + \sum_{k=0}^{j-1} \lambda_{j-k}\alpha_k(i)\ .\end{aligned} \tag{17.18}$$

For the iteration scheme (11.19), we have

$$(17.19)\qquad \beta_j(i) = \langle \eta_j, x_i^* \rangle = -\langle T\varphi_{j-1}, x_i^* \rangle + \lambda_j \sum_{k=0}^{j-1} \alpha_k(i) .$$

We can also apply Theorem 17.5 to the modified fixed point iteration scheme (11.31) with

$$\xi_j = \frac{T\varphi_{j-1}}{\lambda_j} , \quad \eta_j = \frac{1}{\lambda_j} \left[-T^2\varphi_{j-1} + \frac{\langle T^2\varphi_{j-1}, \varphi_0^* \rangle}{\lambda_j} T\varphi_{j-1} \right] .$$

In this case, we have

$$(17.20)\qquad \beta_j(i) = \langle \eta_j, x_i^* \rangle = \frac{1}{\lambda_j} \left[-\langle T^2\varphi_{j-1}, x_i^* \rangle + \frac{\mu_j}{\lambda_j} \langle T\varphi_{j-1}, x_i^* \rangle \right] ,$$

where

$$(17.21)\qquad \mu_j = \langle T^2\varphi_{j-1}, \varphi_0^* \rangle = \sum_{i=1}^{n} \langle T^2\varphi_{j-1}, x_i^* \rangle \, \overline{v(i)} .$$

Thus, we also need to calculate $T^2\varphi_{j-1}$ and $\langle T^2\varphi_{j-1}, x_i^* \rangle$.

Remark 17.7 In case we have an iteration scheme

$$\lambda_j = \langle T\varphi_{j-1}, \varphi_0^* \rangle , \quad \varphi_j = \xi_j + S_0\tilde{\eta}_j , \qquad j = 1,2,\ldots$$

where $\tilde{\eta}_j$ may not satisfy $P_0\tilde{\eta}_j = 0$, we can let

$$\eta_j = \tilde{\eta}_j - P_0\tilde{\eta}_j ,$$

and observe that since $S_0P_0 = 0$, we have

$$\varphi_j = \xi_j + S_0\eta_j , \quad P_0\eta_j = 0 .$$

We can then apply Theorem 17.5. This is the situation for Ahue's iteration scheme (11.35), where

$$\varphi_j = \frac{T\varphi_{j-1}}{\lambda_j} + \frac{S_0}{\lambda_j}\left[-T^2\varphi_{j-1} + \lambda_j T\varphi_{j-1} \right] , \quad j = 1,2,\ldots .$$

Let $\xi_j = \dfrac{T\varphi_{j-1}}{\lambda_j}$ and $\tilde{\eta}_j = \dfrac{1}{\lambda_j}\left[-T^2\varphi_{j-1} + \lambda_j T\varphi_{j-1}\right]$. Then

$$(17.22)\quad \eta_j = \frac{1}{\lambda_j}\left[-T^2\varphi_{j-1} + \lambda_j T\varphi_{j-1}\right] - \frac{1}{\lambda_j}\left[-\langle T^2\varphi_{j-1},\varphi_0^*\rangle + \lambda_j\langle T\varphi_{j-1},\varphi_0^*\rangle\right]\varphi_0$$
$$= \frac{1}{\lambda_j}\left[-T^2\varphi_{j-1} + \lambda_j T\varphi_{j-1} + (\mu_j-\lambda_j^2)\varphi_0\right] ,$$

where μ_j is given by (17.21). In this case, we have

$$(17.23)\quad \beta_j(i) = \langle \eta_j, x_i^*\rangle$$
$$= \frac{1}{\lambda_j}\left[-\langle T^2\varphi_{j-1},x_i^*\rangle + \lambda_j\langle T\varphi_{j-1},x_i^*\rangle + \lambda_0(\mu_j-\lambda_j^2)u(i)\right] .$$

We now write down algorithms for implementing various iteration schemes considered so far; the iterates approximate eigenelements of a bounded operator T on X . Let

$$T_0 x = \sum_{i=1}^{n} \langle x, x_i^*\rangle x_i \ , \quad x \in X ,$$

be a fixed finite rank operator on X , and consider the matrix

$$A = [\langle x_j, x_i^*\rangle] \ , \quad i,j = 1,\dots,n .$$

ALGORITHM 17.8 For $j = 0,1,2,\dots,$ the iterates

$$\lambda_j = \langle T\varphi_{j-1},\varphi_0^*\rangle ,$$
$$\varphi_j = \varphi_{j-1} + S_0\left[-(T-\lambda_1 I)\varphi_{j-1} + \sum_{k=2}^{j}(\lambda_k-\lambda_{k-1})\varphi_{j-k}\right]$$

of the Rayleigh-Schrödinger scheme (11.18) can be found as follows.

Step 1 (i) Solve the eigenvalue problem for A . If λ_0 is a nonzero simple eigenvalue of A , find a corresponding eigenvector $\underset{\sim}{u} = [u(1),\dots,u(n)]^t$.

(ii) Find the eigenvector $\underset{\sim}{v} = [v(1),\dots,v(n)]^t$ of A^H corresponding to $\bar{\lambda}_0$ such that $\underset{\sim}{v}^H\underset{\sim}{u} = \dfrac{1}{\lambda_0}$.

Step 2 Put $\underset{\sim}{\alpha}_0 = \lambda_0 \underset{\sim}{u}$, $\varphi_0 = u(1)x_1 + \ldots + u(n)x_n$, and for $j = 1,2,\ldots,$ repeat the following:

(i) Calculate $\langle T\varphi_{j-1}, x_i^* \rangle$, $i = 1,\ldots,n$, and put

$$\lambda_j = \sum_{i=1}^{n} \langle T\varphi_{j-1}, x_i^* \rangle \overline{v(i)} .$$

(ii) Let $\underset{\sim}{\beta}_j = [\beta_j(1),\ldots,\beta_j(n)]^t$ with

$$\beta_j(i) = -\langle T\varphi_{j-1}, x_i^* \rangle + \sum_{k=0}^{j-1} \lambda_{j-k}\alpha_k(i) , \quad i = 1,\ldots,n .$$

Find $\underset{\sim}{\alpha}_j = [\alpha_j(1),\ldots,\alpha_j(n)]^t$ such that

$$\underset{\sim}{v}^H \underset{\sim}{\alpha}_j = 0 , \quad (A-\lambda_0 I)\underset{\sim}{\alpha}_j = \underset{\sim}{\beta}_j .$$

(iii) Calculate $T\varphi_{j-1}$ and put

$$\varphi_j = \frac{1}{\lambda_0}\left[\sum_{i=1}^{n} \alpha_j(i)x_i + \sum_{k=1}^{j} (\lambda_{k-1}-\lambda_k)\varphi_{j-k} + T\varphi_{j-1}\right] .$$

ALGORITHM 17.9 For $j = 0,1,2,\ldots,$ the iterates

$$\lambda_j = \langle T\varphi_{j-1}, \varphi_0^* \rangle ,$$
$$\varphi_j = \varphi_{j-1} + S_0\left[-T\varphi_{j-1} + \lambda_j\varphi_{j-1}\right]$$

of the fixed point scheme (11.19) can be found as in Step 1 and Step 2 of Algorithm 17.8 except for the following changes:

In Step 2 (ii), for $i = 1,\ldots,n$, let

$$\beta_j(i) = -\langle T\varphi_{j-1}, x_i^* \rangle + \lambda_j \sum_{k=0}^{j-1} \alpha_k(i)$$

and in Step 2 (iii), let

$$\varphi_j = \frac{1}{\lambda_0}\left[\sum_{i=1}^{n} \alpha_j(i)x_i + (\lambda_0-\lambda_j)\varphi_{j-1} + T\varphi_{j-1}\right] .$$

ALGORITHM 17.10 For $j = 0,1,2,\ldots,$ the iterates

$$\lambda_j = \langle T\varphi_{j-1}, \varphi_0^* \rangle ,$$
$$\varphi_j = \frac{T\varphi_{j-1}}{\lambda_j} + \frac{S_0}{\lambda_j}\left[-T^2\varphi_{j-1} + \frac{\langle T^2\varphi_{j-1}, \varphi_0^* \rangle}{\lambda_j} T\varphi_{j-1}\right]$$

of the modified fixed point scheme (11.31) can be found as in Step 1 and Step 2 of Algorithm 17.8 except for the following changes:

In Step 2 (i), calculate additionally $\langle T^2\varphi_{j-1}, x_i^*\rangle$, $i = 1,\dots,n$, and put

$$\mu_j = \sum_{i=1}^{n} \langle T^2\varphi_{j-1}, x_i^*\rangle \overline{v(i)} .$$

In Step 2 (ii), for $i = 1,\dots,n$, let

$$\beta_j(i) = \frac{1}{\lambda_j}\left[-\langle T^2\varphi_{j-1}, x_i^*\rangle + \frac{\mu_j}{\lambda_j}\langle T\varphi_{j-1}, x_i^*\rangle\right] .$$

In Step 2 (iii), calculate additionally $T^2\varphi_{j-1}$, and put

$$\varphi_j = \frac{1}{\lambda_0\lambda_j}\left[\lambda_j \sum_{i=1}^{n} \alpha_j(i)x_i + \left[\lambda_0 - \frac{\mu_j}{\lambda_j}\right]T\varphi_{j-1} + T^2\varphi_{j-1}\right] .$$

ALGORITHM 17.11 For $j = 0,1,2,\dots,$ the iterates

$$\lambda_j = \langle T\varphi_{j-1}, \varphi_0^*\rangle ,$$
$$\varphi_j = \frac{T\varphi_{j-1}}{\lambda_j} + \frac{S_0}{\lambda_j}[-T^2\varphi_{j-1} + \lambda_j T\varphi_{j-1}]$$

of the Ahués scheme (11.35) can be found as in Step 1 and Step 2 of Algorithm 17.8 except for the following changes:

In Step 2 (i), calculate additionally $\langle T^2\varphi_{j-1}, x_i^*\rangle$, $i = 1,\dots,n$, and put

$$\mu_j = \sum_{i=1}^{n} \langle T^2\varphi_{j-1}, x_i^*\rangle \overline{v(i)} .$$

In Step 2 (ii), for $i = 1,\dots,n$, let

$$\beta_j(i) = \frac{1}{\lambda_j}\left[-\langle T^2\varphi_{j-1}, x_i^*\rangle + \lambda_j\langle T\varphi_{j-1}, x_i^*\rangle + \lambda_0(\mu_j - \lambda_j^2)u(i)\right] .$$

In Step 2 (iii), calculate additionally $T^2\varphi_{j-1}$ and put

$$\varphi_j = \frac{1}{\lambda_0\lambda_j}\Big[\lambda_j \sum_{i=1}^{n} \alpha_j(i)x_i + (\lambda_j^2-\mu_j)\varphi_0 + (\lambda_0-\lambda_j)T\varphi_{j-1} + T^2\varphi_{j-1}\Big] .$$

In writing the above algorithms, we have made use of Theorem 17.3 and the expressions (17.18), (17.19), (17.20) and (17.23) for $\beta_j(i)$, $i = 1,\ldots,n$, $j = 1,2\ldots$.

REMARK 17.12 We now make some remarks regarding the choice of a finite rank operator T_0 and its simple nonzero eigenvalue λ_0. We have seen in Section 14 that if λ is a simple nonzero eigenvalue of $T \in BL(X)$, separated by a simple closed rectifiable curve Γ from zero as well as from the rest of the spectrum of T, and if (T_n) is a resolvent operator approximation of T, then for all large n, T_n has a simple nonzero eigenvalue λ_n inside Γ, and it is the only spectral value of T_n inside Γ. Let φ_n (resp., φ_n^*) be an eigenvector of T_n (resp., T_n^*) corresponding to λ_n (resp., $\bar{\lambda}_n$) such that $\|\varphi_n\| = 1 = \|\varphi_n^*\|$. Then if n_0 is sufficiently large, and we make the choice $T_0 = T_{n_0}$, $\lambda_0 = \lambda_{n_0}$, $\varphi_0 = \varphi_{n_0}$, $\varphi_0^* = \varphi_{n_0}^*$, all the iteration schemes considered in Section 11 converge and yield the eigenvalue λ and a corresponding eigenvector φ of T which satisfies $\langle\varphi,\varphi_{n_0}^*\rangle = 1$. Moreover, if λ is the dominant spectral value of T, then for all large n, λ_n is the dominant spectral value of T_n; if λ has the second largest absolute value among the elements of $\sigma(T)$, then the same holds for λ_n as far as the elements of $\sigma(T_n)$ are concerned. Hence the choice of λ_{n_0} from among the elements of $\sigma(T_{n_0})$ is dictated by which eigenvalue λ of T we wish to approximate.

Thus, T_0 can be chosen to be a member T_{n_0} of a sequence (T_n) of finite rank operators, which is a resolvent operator approximation of

T on $\rho(T)$. If T_n is of rank n , then one attempts to keep n_0 small, since in the first step of the algorithms, we need to find the eigenvalues of an $n_0 \times n_0$ matrix. However, there is no practically verifiable criterion for deciding the smallest integer n_0 which implies convergence of an iteration scheme, and one has to proceed by a trial and error method. The numerical experiments in Section 19 show that even $n_0 = 2$ works in some cases; in general, though, it is safer to choose $n_0 \geq 4$. The norm approximation and the collectively compact approximation provide useful examples of resolvent operator approximation. We list below some important choices for T_0 in these categories.

Let π_0 be a bounded projection of finite rank given by

$$\pi_0 x = \sum_{i=1}^{n} \langle x, e_i^* \rangle e_i ,$$

where $e_i \in X$, $e_i^* \in X^*$ with $\langle e_j, e_i^* \rangle = \delta_{i,j}$ for $i,j = 1,\ldots,n$. Let (cf. (15.1))

$$T_0^P = \pi_0 T , \quad T_0^S = T\pi_0 , \quad T_0^G = \pi_0 T \pi_0 .$$

In case T is a Fredholm integral operator

$$Tx(s) = \int_a^b k(s,t)x(t)dt , \quad x \in X \text{ and } s \in [a,b] ,$$

where $X = L^2([a,b])$ or $C([a,b])$, let (cf. (16.3))

$$T_0^D = \sum_{i=1}^{n} \left[\int_a^b y_i(t)s(t)dt \right] x_i , \quad x \in X ,$$

where $x_i, y_i \in X$, $i,j = 1,\ldots,n$; also for $X = C([a,b])$ and a quadrature formula

$$f_0(x) = \sum_{i=1}^{n} w_i x(t_i) \ , \quad x \in X \ ,$$

with $w_i \in \mathbb{C}$, and $a \leq t_1 < t_2 < \ldots < t_n \leq b$, let (cf. (16.6))

$$T_0^N x(s) = \sum_{i=1}^{n} w_i k(s,t_i)x(t_i) \ ;$$

if the projection π_0 is an interpolatory projection with nodes at t_i , $i = 1,\ldots,n$, i.e., $\langle x,e_i^* \rangle = x(t_i)$, so that

$$\pi_0 x = \sum_{i=1}^{n} x(t_i)e_i \ , \quad x \in X \ ,$$

with $e_j(t_i) = \delta_{i,j}$, then let (cf. (16.7))

$$T_0^F x = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} w_j k(t_i,t_j)x(t_j) \right] e_i \ .$$

The expressions for x_i and x_i^* appearing in

$$T_0 x = \sum_{i=1}^{n} \langle x,x_i^* \rangle x_i \ , \quad x \in X \ ,$$

for the above choices for T_0 as well as the matrix $A = [\langle x_j,x_i^* \rangle]$ are tabulated below.

$$
\begin{array}{lllll}
T_0 & & x_i & \langle x, x_i^* \rangle & A = [\langle x_j, x_i^* \rangle] \\
\downarrow & & \downarrow & \downarrow & \downarrow \\
T_0^P & & e_i & \langle x, T^* e_i^* \rangle & [\langle Te_j, e_i^* \rangle] \\
T_0^S & & Te_i & \langle x, e_i^* \rangle & [\langle Te_j, e_i^* \rangle] \\
\multirow{2}{*}{T_0^G} & \left\{\begin{array}{l} 1 \\ \\ 2 \end{array}\right. & \begin{array}{l} e_i \\ \\ \sum_{j=1}^{n} \langle Te_i, e_j^* \rangle e_j \end{array} & \begin{array}{l} \sum_{j=1}^{n} \langle x, e_j^* \rangle \langle e_j, T^* e_i^* \rangle \\ \\ \langle x, e_i^* \rangle \end{array} & \begin{array}{l} [\langle Te_j, e_i^* \rangle] \\ \\ [\langle Te_j, e_i^* \rangle] \end{array} \\
T_0^D & & x_i & \int_a^b y_i(t) x(t) dt & [\int_a^b y_i(t) x_j(t) dt] \\
T_0^N & & w_i k(\cdot, t_i) & x(t_i) & [w_j k(t_i, t_j)] \\
T_0^F & \left\{\begin{array}{l} 1 \\ \\ 2 \end{array}\right. & \begin{array}{l} e_i \\ \\ w_i \sum_{j=1}^{n} k(t_j, t_i) e_j \end{array} & \begin{array}{l} \sum_{j=1}^{n} w_j k(t_i, t_j) x(t_j) \\ \\ x(t_i) \end{array} & \begin{array}{l} [w_j k(t_i, t_j)] \\ \\ [w_j k(t_i, t_j)] \end{array}
\end{array}
$$

Table 17.1

We conclude this section by pointing out some interesting relationships among the iterates for the projection, Sloan and Galerkin methods for approximating $T \in BL(X)$.

PROPOSITION 17.13 Consider a projection $\pi_0 \in BL(X)$. Let λ_0 be a nonzero simple eigenvalue of $T_0^G = \pi_0 T \pi_0$, and let φ_0 (resp., φ_0^*) be an eigenvector of T_0^G (resp., $(T_0^G)^*$) corresponding to λ_0 (resp., $\bar{\lambda}_0$) such that $\langle \varphi_0, \varphi_0^* \rangle = 1$. Then

(a) λ_0 is a simple eigenvalue of $T_0^P = \pi_0 T$, and of $T_0^S = T\pi_0$; the elements

$$(17.24) \qquad \varphi_0^P = \varphi_0 \ , \ \varphi_0^{*P} = T^* \varphi_0^* / \bar{\lambda}_0 \ , \ \varphi_0^S = T\varphi_0 / \lambda_0 \ , \ \varphi_0^{*S} = \varphi_0^*$$

are eigenvectors of T_0^P , $(T_0^P)^*$, T_0^S , $(T_0^S)^*$ corresponding to λ_0 , $\bar{\lambda}_0$, λ_0 , $\bar{\lambda}_0$, respectively, such that

$$\langle \varphi_0^P, \varphi_0^{*P} \rangle = 1 = \langle \varphi_0^S, \varphi_0^{*S} \rangle \ .$$

(b) For $j = 1,2,\ldots,$ let

$$\lambda_j^P \ , \ \varphi_j^P \quad \text{and} \quad \lambda_j^S \ , \ \varphi_j^S$$

denote the corresponding iterates in any one of the iteration schemes (11.18), (11.19), (11.31) or (11.35). Then for $j = 0,1,2,\ldots,$

$$(17.25) \qquad \lambda_j^S = \lambda_j^P \quad \text{and} \quad \varphi_j^S = \frac{1}{\lambda_0} T\varphi_j^P \ .$$

Proof We note from Table 17.1 that for all the three operator T_0^G , T_0^P and T_0^S , the corresponding matrix $A = [\langle x_j, x_i^* \rangle]$ is the same, namely, $[\langle Te_j, e_i^* \rangle]$, $i,j = 1,\ldots,n$. Hence it follows from Remark 17.3 and Corollary 17.2 that λ_0 is a simple eigenvalue of A , and hence of T_0^P and T_0^S .

Since $\pi_0 T_0^G \varphi_0 = \lambda_0 \varphi_0$ and $\pi_0^*(T_0^G)^* \varphi_0^* = \bar{\lambda}_0 \varphi_0^*$, we see that $\varphi_0 \in \pi_0(X)$ and $\varphi_0^* \in \pi_0^*(X^*)$, i.e., $\pi_0 \varphi_0 = \varphi_0$ and $\pi_0^* \varphi_0^* = \varphi_0^*$. Now,

$$
\begin{aligned}
T_0^P \varphi_0 &= \pi_0 T \varphi_0 = \pi_0 T \pi_0 \varphi_0 = T_0^G \varphi_0 = \lambda_0 \varphi_0 \,, \\
(T_0^P)^*(T^* \varphi_0^*) &= T^* \pi_0^*(T^* \pi_0^* \varphi_0^*) = T^*(T_0^G)^* \varphi_0^* = \bar{\lambda}_0 T^* \varphi_0^* \,, \\
\langle \varphi_0, T^* \varphi_0^* \rangle &= \langle \pi_0 \varphi_0, T^* \pi_0^* \varphi_0^* \rangle = \langle \varphi_0, \pi_0^* T^* \pi_0^* \varphi_0^* \rangle = \lambda_0 \langle \varphi_0, \varphi_0^* \rangle = \lambda_0 \,, \\
T_0^S(T\varphi_0) &= T\pi_0(T\pi_0 \varphi_0) = T T_0^G \varphi_0 = \lambda_0 T \varphi_0 \,, \\
(T_0^S)^*(\varphi_0^*) &= \pi_0^* T^*(\pi_0^* \varphi_0^*) = (T_0^G)^* \varphi_0^* = \bar{\lambda}_0 \varphi_0^* \,, \\
\langle T\varphi_0, \varphi_0^* \rangle &= \langle \varphi_0, T^* \varphi_0^* \rangle = \lambda_0 \,.
\end{aligned}
$$

Hence the results in part (a) follow.

(b) Let P_0^P (resp., P_0^S) denote the spectral projection associated with T_0^P (resp., T_0^S) and λ_0. Then for $x \in X$,

$$
\begin{aligned}
P_0^P x &= \langle x, T^* \varphi_0 \rangle \varphi_0 / \lambda_0 = \langle Tx, \varphi_0^* \rangle \varphi_0 / \lambda_0 \,, \\
P_0^S x &= \langle x, \varphi_0^* \rangle T\varphi_0 / \lambda_0 \,.
\end{aligned}
\tag{17.26}
$$

Hence $P_0^S T = T P_0^P$. Next, we show that

$$
S_0^S T = T S_0^P \,, \tag{17.27}
$$

where S_0^P (resp., S_0^S) is the reduced resolvent associated with T_0^P (resp., T_0^S) and λ_0. First, let $x \in X$ with $P_0^P x = 0$. Then by (17.26), $P_0^S Tx = 0$, and since

$$
T_0^S T = T T_0^P \,,
$$

we have

$$
(T_0^S - \lambda_0 I) T S_0^P x = T(T_0^P - \lambda_0 I) S_0^P x = T(I - P_0^P) x = Tx \,.
$$

Thus, $y = T S_0^P x$ is the unique element of X which satisfies

$$
(T_0^S - \lambda_0 I) y = Tx \,, \quad P_0^S y = 0 \,,
$$

where $P_0^S Tx = 0$. Hence $y = S_0^S Tx$, i.e., $TS_0^P x = S_0^S Tx$. Next, if $x \in X$ with $P_0^P x = x$, then again by (17.26), we see that $P_0^S Tx = Tx$. Hence

$$TS_0^P x = TS_0^P P_0^P x = 0 = S_0^S P_0^S Tx = S_0^S Tx .$$

Thus, (17.27) holds by considering $x = P_0 x + (I-P_0)x$ for all $x \in X$.

Consider an iteration scheme

$$\begin{aligned} \varphi_j &= \xi_j + S_0 \eta_j , \\ \lambda_j &= \langle T\varphi_{j-1}, \varphi_0^* \rangle , \end{aligned} \qquad j = 1,2,\ldots$$

where the initial terms φ_0 and λ_0 are eigenelements of T_0 . It follows from (17.27) that if for some $j = 1,2,\ldots,$

$$\xi_j^S = T\xi_j^P/\lambda_0 \quad \text{and} \quad \eta_j^S = T\eta_j^P/\lambda_0 , \tag{17.28}$$

then

$$\varphi_j^S = \xi_j^S + S_0^S \eta_j^S = T(\xi_j^P + S_0^P \eta_j^P)/\lambda_0 = T\varphi_j^P/\lambda_0 ,$$

and hence

$$\lambda_{j+1}^S = \langle T\varphi_j^S, \varphi_0^{*S} \rangle = \frac{1}{\lambda_0} \langle T^2 \varphi_j^P, \varphi_0^* \rangle = \frac{1}{\lambda_0} \langle T\varphi_j^P, T^* \varphi_0^* \rangle = \langle T\varphi_j^P, \varphi_0^{*P} \rangle = \lambda_{j+1}^P$$

with obvious notations. Using this result it can be proved by induction on j , that for each of the iteration schemes (11.18), (11.19), and (11.35), the relations in (17.28) hold. Hence the desired result (17.25) holds for these iteration schemes. For the iteration scheme (11.31) one needs to note additionally that if $\varphi_j^S = T\varphi_j^P/\lambda_0$, then

$$\begin{aligned} \mu_{j+1}^S &\equiv \langle T^2 \varphi_j^S, \varphi_0^{*S} \rangle = \frac{1}{\lambda_0} \langle T^3 \varphi_j^P, \varphi_0^* \rangle = \frac{1}{\lambda_0} \langle T^2 \varphi_j^P, T^* \varphi_0^* \rangle = \langle T^2 \varphi_{j-1}^P, \varphi_0^{*P} \rangle \\ &\equiv \mu_{j+1}^P . \quad // \end{aligned}$$

PROPOSITION 17.14 Under the hypotheses and notations of Proposition 17.13, the following relations hold for the Rayleigh-Schrödinger iteration scheme (11.18) and the fixed point iteration scheme (11.19):

$$\lambda_1^G = \lambda_0 , \tag{17.29}$$

$$\varphi_1^G = T\varphi_0/\lambda_0 = \varphi_0^S , \tag{17.30}$$

$$\lambda_2^G = \langle T^2\varphi_0, \varphi_0^*\rangle/\lambda_0 = \lambda_1^P . \tag{17.31}$$

Proof By definition,

$$\lambda_1^G = \langle T\varphi_0, \varphi_0^*\rangle = \langle T_0^G\varphi_0, \varphi_0^*\rangle + \langle (T-T_0^G)\varphi_0, \varphi_0^*\rangle .$$

But since $\varphi_0^* = (T_0^G)^* \varphi_0^*/\lambda_0$ and

$$T_0^G(T-T_0^G)\varphi_0 = T_0^G(T-T_0^G)\pi_0\varphi_0 = (T_0^G)^2\varphi_0-(T_0^G)^2\varphi_0 = 0 ,$$

we have

$$\langle (T-T_0^G)\varphi_0, \varphi_0^*\rangle = \langle T_0^G(T-T_0^G)\varphi_0, \varphi_0^*\rangle/\lambda_0 = 0 .$$

Hence

$$\lambda_1^G = \langle T_0^G\varphi_0, \varphi_0^*\rangle = \lambda_0\langle \varphi_0, \varphi_0^*\rangle = \lambda_0 .$$

This proves (17.29). Next, by definition and by (17.29),

$$\varphi_1^G = \varphi_0 + S_0^G[-(T-\lambda_1^G I)\varphi_0] = \varphi_0 - S_0^G(T\varphi_0-\lambda_0\varphi_0)$$

for both the iteration schemes (11.18) and (11.19). We claim that

$$S_0^G(T\varphi_0-\lambda_0\varphi_0) = -\frac{1}{\lambda_0}(T\varphi_0-\lambda_0\varphi_0) . \tag{17.32}$$

Now, by (17.29),

$$\langle T\varphi_0-\lambda_0\varphi_0, \varphi_0^*\rangle = \lambda_1^G - \lambda_0 = 0 ,$$

and

$$(T_0^G-\lambda_0 I)(T\varphi_0-\lambda_0\varphi_0) = T_0^G T\varphi_0 - \lambda_0 T\varphi_0 - \lambda_0 T_0^G\varphi_0 + \lambda_0^2\varphi_0$$
$$= (T_0^G)^2\varphi_0 - \lambda_0 T\varphi_0 = -\lambda_0(T\varphi_0-\lambda_0\varphi_0) .$$

This proves (17.32). Hence

$$\varphi_1^G = \varphi_0 + \frac{1}{\lambda_0}(T\varphi_0-\lambda_0\varphi_0) = \frac{T\varphi_0}{\lambda_0} = \varphi_0^S ,$$

proving (17.30). Finally, by definition and by (17.30),

$$\lambda_2^G = \langle T\varphi_1^G,\varphi_0^*\rangle = \langle\varphi_1^G,T^*\varphi_0^*\rangle = \langle T\varphi_0/\lambda_0,T^*\varphi_0^*\rangle$$
$$= \langle T\varphi_0,T^*\varphi_0^*/\bar{\lambda}_0\rangle = \langle T\varphi_0,(\varphi_0^P)^*\rangle = \lambda_1^P . \qquad //$$

REMARK 17.15 The relation (17.25) shows that if one knows the projection iterates λ_j^P and φ_j^P for one of the schemes (11.18), (11.19), (11.31) and (11.35), then the Sloan iterates $\lambda_j^S = \lambda_j^P$ and $\varphi_j^S = T\varphi_j^P/\lambda_0$ are available easily; there being no need to implement the algorithm again.

The relations (17.29), (17.30) and (17.31) provide useful checks when we perform computations with the projection, Sloan and Galerkin methods.

The relation (17.29), namely $\lambda_1^G = \lambda_0$, says that the generalized Rayleigh quotient λ_1^G of T at (φ_0,φ_0^*) equals the initial eigenvalue λ_0 . In other words λ_1^G does *not* improve upon λ_0 as an approximation of an eigenvalue λ of T . Also, the relation (17.30)

$$\varphi_1^G = T\varphi_0/\lambda_0 = T\varphi_0/\langle T\varphi_0,\varphi_0^*\rangle$$

tells us that the first Galerkin iterate φ_1^G in schemes (11.18) and (11.19) is, in fact, the <u>iterated Galerkin eigenvector</u> proposed by Sloan in [SL], and that it coincides with the first eigenvector iterate of the power method with φ_0 and φ_0^* as the initial terms. As such,

$$\lambda_2^G = \langle T^2\varphi_0,\varphi_0^*\rangle/\langle T\varphi_0,\varphi_0^*\rangle$$

coincides with the second eigenvalue iterate of the power method. (See (11.36) and (11.37)).

Problems

For $x \in X$, let $T_0 x = \sum_{i=1}^{n} \langle x, x_i^* \rangle x_i$, where $x_1, \dots, x_n$ are in X and $x_1^*, \dots, x_n^*$ are in X^*. Let $A = [\langle x_j, x_i^* \rangle]$, $1 \le i,j \le n$.

17.1 With the assumptions and notations of Corollary 17.2,

$$\langle x_j, \varphi_0^* \rangle = \lambda_0 \overline{v(j)}, \quad j = 1, \dots, n .$$

17.2 Let $0 \neq \lambda_0 \in \mathbb{C}$. The algebraic and the geometric multiplicities of λ_0 as an eigenvalue of T_0 and of A are the same. The orders of the poles at λ_0 of the resolvent operators of T_0 and A are equal.

17.3 Let λ_0 be a nonzero semisimple eigenvalue of A. Let $\{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_m\}$ (resp., $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_m\}$) be a basis of the eigenspace of A (resp., A^*) corresponding to λ_0 (resp., $\bar{\lambda}_0$) such that $\underset{\sim}{v}_i^H \underset{\sim}{u}_j = \delta_{i,j}/\lambda_0$, $i,j = 1, \dots, m$. Then analogues of Corollary 17.2 and Proposition 17.4 hold.

17.4 In Table 17.1, for T_0^N we can consider

$$x_i = k(\cdot, t_i), \quad \langle x, x_i^* \rangle = w_i x(t_i)$$

and for T_0^F we can consider

$$x_i = \sum_{j=1}^{n} k(t_j, t_i) e_j, \quad \langle x, x_i^* \rangle = w_i x(t_i) .$$

In both these cases, $[\langle x_j, x_i^*\rangle] = [w_i k(t_i, t_j)] = A'$, say. If $A = [w_j k(t_i, t_j)]$, then the nonzero eigenvalues of A and A' are the same, the corresponding algebraic and geometric multiplicities coincide, as do the orders of the corresponding poles of the resolvent operators of A and A'. If $\underset{\sim}{u}$ is an eigenvector of A corresponding to $\lambda_0 \neq 0$, then $\underset{\sim}{u}' = [w_1 u(1), \ldots, w_n u(n)]^t$ is an eigenvector of A' corresponding to λ_0.

17.5 For the Rayleigh-Schrödinger scheme (11.18), the relations in (17.25) can be proved by considering the families of operators

$$T^P(t) = T_0^P + t(T - T_0^P) \quad \text{and} \quad T^S(t) = T_0^S + t(T - T_0^S)$$

for $t \in \mathbb{C}$ with $|t|$ small enough. (Hint: (10.4) and (10.5))

17.6 Let λ_0 be a nonzero eigenvalue of T_0^F and let φ_0 (resp., φ_0^*) be an eigenvector of T_0^F (resp., $(T_0^F)^*$) corresponding to λ_0 (resp., $\bar{\lambda}_0$) such that $\langle \varphi_0, \varphi_0^* \rangle = 1$. Then λ_0 is a simple eigenvalue of T_0^N, and $\varphi_0^N = T_0^N \varphi_0 / \lambda_0$ (resp., $\varphi_0^{*N} = \varphi_0^*$) is an eigenvector of T_0^N (resp., $(T_0^N)^*$) corresponding to λ_0 (resp., $\bar{\lambda}_0$) such that $\langle \varphi_0^N, \varphi_0^{*N} \rangle = 1$.