13. APPROXIMATION OF BOUNDED OPERATORS

In this section we consider some modes of approximating an operator $T \in BL(X)$ by a sequence (T_n) of operators in $BL(X)$. We study the relationships among these modes. Our interest lies in the approximation of $\sigma(T)$ by $\sigma(T_n)$.

If $T_n x \to Tx$, $x \in X$, i.e., $\|T_n x - Tx\| \to 0$ for every $x \in X$, we say that (T_n) <u>is a pointwise approximation of</u> T, and denote this fact by $T_n \xrightarrow{p} T$.

The pointwise approximation has, in general, no implication for the approximation of the spectrum: (i) For $n = 1,2,\dots$, there may exist an eigenvalue λ_n of T_n such that (λ_n) converges to an element of the resolvent set of T. For example, let $X = \ell^2$, and for $x = [x(1), x(2),\dots]^t$ in X, let

$$T_n x = [0,\dots,0,\ x(n+1),\ x(n+2),\dots]^t ,$$

where the zeros occur in the first n places. Then $T_n \xrightarrow{p} T = 0$, $\sigma(T_n) = \{0,1\}$, while $\sigma(T) = \{0\}$. (ii) There may exist an eigenvalue λ of T such that no subsequence of (λ_n), where $\lambda_n \in \sigma(T_n)$, converges to λ. For example, let $X = \ell^2$, and

$$T_n x = [x(2),\dots,\ x(n),\ 0,\ 0,\ \dots]^t , \quad Tx = [x(2),\ x(3),\ \dots]^t .$$

Then $T_n \xrightarrow{p} T$, $\sigma(T_n) = \{0\}$, while $\sigma(T) = \{z \in \mathbb{C} : |z| \le 1\}$ and every $z \in \mathbb{C}$ with $|z| < 1$ is an eigenvalue of T.

The above two examples point out the lack of upper semicontinuity and lower semicontinuity of the spectrum with respect to the pointwise approximation. (Cf. (9.11) and the discussion there.) However, for self-adjoint operators on a Hilbert space, we do have lower semicontinuity of the spectrum with respect to the pointwise convergence.

PROPOSITION 13.1 Let T and T_n, $n = 1,2,\dots$, be self-adjoint operators on a Hilbert space $X \neq \{0\}$. Let $T_n \xrightarrow{p} T$ and $\lambda \in \sigma(T)$. Then for every $\epsilon > 0$, there is $n_0(\epsilon)$ such that if $n \geq n_0(\epsilon)$, then $|\lambda_n - \lambda| < \epsilon$ for some $\lambda_n \in \sigma(T_n)$.

Proof Since T is self-adjoint, we see by Theorem 8.7(a) that λ is a real number. Let $\epsilon > 0$, and $\mu = \lambda + i\epsilon$, so that $\mu \in \rho(T)$. By (8.13), we have

$$\|R(\mu)\| = \frac{1}{\operatorname{dist}(\mu, \sigma(T))} = \frac{1}{\epsilon} .$$

Let $x \in X$ with $\|x\| = 1$ and $\|R(\mu)x\| \geq 1/2\epsilon$. Again, since T_n is self-adjoint, we have $\mu \in \rho(T_n)$ and

$$\|R_n(\mu)\| = \frac{1}{\operatorname{dist}(\mu, \sigma(T_n))} \leq \frac{1}{\epsilon} ,$$

with $R_n(\mu) = (T_n - \mu I)^{-1}$. Now, by (9.2),

$$R_n(\mu) - R(\mu) = R_n(\mu)(T - T_n)R(\mu) ,$$

where $\|R_n(\mu)\| \leq 1/\epsilon$ and $\|(T-T_n)R(\mu)x\| \to 0$. This shows that $R_n(\mu)x \to R(\mu)x$ and hence there is an integer $n_0(\epsilon)$ such that $\|R_n(\mu)x\| \geq 1/3\epsilon$ for all $n \geq n_0(\epsilon)$. Since $\|x\| = 1$, we have

$$\frac{1}{\operatorname{dist}(\mu, \sigma(T_n))} = \|R_n(\mu)\| \geq 1/3\epsilon ,$$

i.e., $\operatorname{dist}(\mu, \sigma(T_n)) \leq 3\epsilon$ for all $n \geq n_0(\epsilon)$. Thus, there is $\lambda_n \in \sigma(T_n)$ such that

$$|\lambda - \lambda_n| \leq |\lambda - \mu| + |\mu - \lambda_n| \leq \epsilon + 3\epsilon = 4\epsilon .$$

This would prove the proposition if we had started with $\epsilon/4$ instead of ϵ. //

We now consider stronger modes of approximating T .

If $\|T_n - T\| \to 0$, then (T_n) is said to be a norm approximation of T ; we denote this fact by $T_n \xrightarrow{\|\ \|} T$. It is clear that $T_n \xrightarrow{\|\ \|} T$ implies $T_n \xrightarrow{p} T$, but the converse is not true in general.

The discussion around (9.11) in Section 9 shows that the spectrum is upper semicontinuous with respect to a norm approximation, but it is not lower semicontinuous in general. However, if λ is an isolated spectral value of T , $\sigma(T)$ is both upper and lower semicontinuous in a neighbourhood of λ with respect to a norm approximation. This will follow as a special case of Remark 14.6(iii).

If X is a finite dimensional space, then for each $T \in BL(X)$, we have $\sigma(T) = \sigma_d(T)$. Also, $T_n \xrightarrow{p} T$ if and only if $T_n \xrightarrow{\|\ \|} T$. Thus, we have the continuity of the spectrum of T when X is finite dimensional. For other special cases of continuous change of the spectrum, see Problem 9.2.

Although a norm approximation is good enough to give the continuity of the spectrum in a neighbourhood of an isolated spectral value, many useful approximation procedures (T_n) of T do *not* converge in the norm, as we shall see in Sections 15 and 16. For this reason, we study yet another important mode of approximation.

We say that (T_n) is a collectively compact approximation of T if $T_n \xrightarrow{p} T$ and there is an integer n_0 such that the set

$$\bigcup_{n=n_0}^{\infty} \{(T_n - T)x \ : \ x \in X \ , \ \|x\| \leq 1\} \tag{13.1}$$

has a compact closure (or equivalently, is totally bounded) in X , i.e., if for $k = 1,2,\ldots$, $x_k \in X$ with $\|x_k\| \leq 1$, and $j_k \geq n_0$, then the sequence $((T_{j_k} - T)x_k)$ has a convergent subsequence in X . We denote this fact by $T_n \xrightarrow{cc} T$.

Note that if $T \in BL(X)$ is itself a compact operator, then $T_n \xrightarrow{cc} T$ if and only if $T_n \xrightarrow{p} T$ and there is an integer n_0 such that the set

$$\bigcup_{n=n_0}^{\infty} \{T_n x : x \in X \ , \ \|x\| \leq 1\}$$

has a compact closure in X. This follows because the compactness of T means that the set $\{Tx : x \in X \ , \ \|x\| \leq 1\}$ itself has a compact closure. Notice, on the other hand, that if $T_n \xrightarrow{cc} T$, and if infinitely many T_n's are compact, then $T = T_n + (T-T_n)$ is compact.

We now prove a useful characterization of a collectively compact approximation.

PROPOSITION 13.2 Let $T_n \xrightarrow{p} T$. Then $T_n \xrightarrow{cc} T$ if and only if

(a) there is an integer n_0 such that $T_n - T$ is a compact operator for all $n \geq n_0$.

(b) whenever $\|x_k\| \leq 1$ for $k = 1,2,\ldots$, and $1 \leq n_1 < n_2 < \ldots$ are integers, $((T_{n_k} - T)x_k)$ has a convergent subsequence in X.

Proof Let $T_n \xrightarrow{cc} T$, and let the set

$$W = \bigcup_{n=n_0}^{\infty} \{(T_n - T)x : x \in X \ , \ \|x\| \leq 1\}$$

have a compact closure. Then for each $n \geq n_0$, the set $\{(T_n - T)x : x \in X \ , \ \|x\| \leq 1\}$ is contained in W and hence has a compact closure. Thus, the condition (a) is satisfied. Next, choose k_0 such that $k \geq k_0$ implies $n_k \geq n_0$. Again, the set $\{(T-T_{n_k})x_k : k \geq k_0\}$ is contained in W and hence has a compact closure. Thus, the condition (b) also follows.

Assume now that the conditions (a) and (b) hold. For $n = 1,2,\dots$, let $\|x_n\| \le 1$, and let $j_n \ge n_0$ be positive integers. Note that (j_n) is not necessarily an increasing sequence of integers. We show that the sequence $((T_{j_n}-T)x_n)$ has a convergent subsequence.

Case 1: The set $\{j_n : n = 1,2,\dots\}$ is finite. Then there is a positive integer $m \ge n_0$ and integers m_k such that $1 \le m_1 < m_2 < \dots$ and $j_{m_k} = m$ for each $k = 1,2,\dots$. Now,

$$(T_{j_{m_k}}-T)x_{m_k} = (T_m-T)x_{m_k},$$

where (T_m-T) is a compact operator by the condition (a). Hence $((T_m-T)x_{m_k})$ has a convergent subsequence.

Case 2: The set $\{j_n : n = 1,2,\dots\}$ is infinite. We can then find positive integers m_k such that $m_1 < m_2 < \dots$ and $j_{m_1} < j_{m_2} < \dots$. For $k = 1,2,\dots$, let $j_{m_k} = n_k$ and $x_{m_k} = y_k$. Then

$$(T_{j_{m_k}}-T)x_{m_k} = (T_{n_k}-T)y_k,$$

where $\|y_k\| \le 1$ and $n_1 < n_2 < \dots$. Now by the condition (b), the sequence $((T_{n_k}-T)y_k)$ has a convergent subsequence. //

We recall the *uniform boundedness principle* which says that if (T_n) is a sequence in $BL(X)$ and $\|T_nx\| \le \alpha_x < \infty$ for each $x \in X$, then $\|T_n\| \le \alpha < \infty$ for some $\alpha \ge 0$ and all n. A consequence of this principle is that if $T_n \xrightarrow{p} T$, and if W is a totally bounded subset of X, then $T_nx \to Tx$ uniformly for $x \in W$. (See [L], 9.1 and 9.3.)

We now study approximation by the composition of a pointwise convergent sequence and a sequence which converges in a collectively compact manner.

PROPOSITION 13.3 Let A_n , A , B_n , B be all in $BL(X)$, $A_n \xrightarrow{p} A$ and $B_n \xrightarrow{cc} B$. Then

(13.2) $$(B_n-B)A_n \xrightarrow{cc} 0 ,$$

(13.3) $$\|(A_n-A)(B_n-B)\| \to 0 .$$

If, in addition, B is compact, then

(13.4) $$A_nB_n \xrightarrow{cc} AB , \quad B_nA_n \xrightarrow{cc} BA ,$$

(13.5) $$\|(A-A_n)B\| \to 0 , \quad \|(A-A_n)B_n\| \to 0 .$$

Proof Since $A_n \xrightarrow{p} A$ and $B_n \xrightarrow{p} B$, we have $\|A_n\| \leq \alpha$ and $\|B_n\| \leq \beta$ for all n . Then it follows easily that $A_nB_n \xrightarrow{p} AB$, $B_nA_n \xrightarrow{p} BA$, and $(B_n-B)A_n \xrightarrow{p} 0$. As $B_n \xrightarrow{cc} B$, let n_0 be such that the set

$$W = \bigcup_{n=n_0}^{\infty} \{(B_n-B)x : x \in X , \|x\| \leq 1\} ,$$

is totally bounded. Now,

$$\bigcup_{n=n_0}^{\infty} \{(B_n-B)A_nx : x \in X , \|x\| \leq 1\} \subset \{\alpha y : y \in W\} .$$

Hence $(B_n-B)A_n \xrightarrow{cc} 0$. This proves (13.2). Next, the pointwise convergence $A_nx \to Ax$ is uniform for x in the totally bounded set W . This implies $\|(A_n-A)(B_n-B)\| \to 0$, i.e., (13.3) holds.

Let, now, B be compact. Then AB and BA are both compact. To show $A_nB_n \xrightarrow{cc} AB$, we prove that if $\|x_k\| \leq 1$ for $k = 1,2,\ldots$, and $1 \leq n_1 < n_2 < \ldots$, then $(A_{n_k}B_{n_k}x_k)$ has a convergent subsequence (Proposition 13.2). But since $B_n \xrightarrow{cc} B$ and B is compact, $(B_{n_k}x_k)$ has a subsequence $(B_{n_{k_j}}x_{k_j})$ which converges to some y in X. Then

$$\|A_{n_{k_j}}B_{n_{k_j}}x_{k_j} - Ay\| \leq \|A_{n_{k_j}}\| \, \|B_{n_{k_j}}x_{k_j} - y\| + \|A_{n_{k_j}}y - Ay\| ,$$

so that $(A_{n_{k_j}}B_{n_{k_j}}x_{k_j})$ converges to Ay. Next, to conclude $B_nA_n \xrightarrow{cc} BA$, we note that the set

$$\bigcup_{n=n_0}^{\infty} \{B_nA_nx : x \in X , \|x\| \leq 1\}$$

is contained in the set $\bigcup_{n=n_0}^{\infty} \{\alpha B_nx : x \in X , \|x\| \leq 1\}$ which is totally bounded since the set E is totally bounded, and B is compact. Finally, since $A_n \xrightarrow{p} A$ and B is compact, the pointwise convergence $A_nx \to Ax$ is uniform on the totally bounded set $\{Bx : x \in X , \|x\| \leq 1\}$. Hence $\|(A-A_n)B\| \to 0$. Also, $\|(A-A_n)B_n\| \to 0$ by (13.3). //

A nice criterion is available for the collectively compact approximation by a sequence of projection operators. It will prove to be very useful in the next section.

THEOREM 13.4 (Anselone) Let $P_n \xrightarrow{p} P$, where $P_1, P_2, \ldots$ are projections in $BL(X)$. If $P_n \xrightarrow{cc} P$, then $\operatorname{rank} P_n = \operatorname{rank} P$ for all large n. Conversely, if for all large n, $\operatorname{rank} P_n = \operatorname{rank} P < \infty$, then

$P_n \xrightarrow{cc} P$, and we have

(13.6) $\qquad \|(P-P_n)P\| \to 0$ and $\|(P-P_n)P_n\| \to 0$.

Proof First note that since $P_n \xrightarrow{p} P$ and each P_n is a projection, so is P . Let $P_n \xrightarrow{cc} P$, and $\bigcup_{n=n_0}^{\infty} \{(P_n-P)x : x \in X , \|x\| \leq 1\}$ be totally bounded. If neither P nor any of P_{n_0} , $P_{n_0+1}, \ldots$ is compact, then by Corollary 3.9, rank P_n = rank $P = \infty$ for all $n \geq n_0$. Next, let either P or some P_n $(n \geq n_0)$ be compact. Then, in fact, P is compact since $P = (P-P_n) + P_n$. Now, by (13.5), we have

$$\|(P-P_n)P\| \to 0 \quad \text{and} \quad \|(P-P_n)P_n\| \to 0 .$$

Hence for all large n ,

$$r_\sigma(P(P_n-P)) = r_\sigma((P_n-P)P) < 1 \quad \text{and} \quad r_\sigma(P_n(P_n-P)) = r_\sigma((P_n-P)P_n) < 1 ,$$

by (5.12) and (5.10). It follows by Proposition 9.6 that $P_n(X)$ and $P(X)$ are linearly homeomorphic, and hence have the same (finite) dimension for all large n .

Conversely, assume that for all large n , we have rank P_n = rank $P = m < \infty$, so that P , P_n and hence $P_n - P$ are compact. To show $P_n \xrightarrow{cc} P$, it is sufficient, by Proposition 13.2, to prove that if for $k = 1,2,\ldots$, $\|u_k\| \leq 1$, $1 \leq n_1 < n_2 < \ldots$, then $(P_{n_k}u_k)$ has a convergent subsequence in X .

Let $x_1,\ldots,x_m$ be an ordered basis of $P(X)$, and let $x_1^*,\ldots,x_m^*$ be in X^* which are adjoint to $x_1,\ldots,x_m$. Let for $i = 1,\ldots,m$,

$$x_{n,i} = P_n x_i .$$

Since $P_n \xrightarrow{p} P$, we see that for $i,j = 1,\ldots,m$,

$$\langle x_{n,i}, x_j^* \rangle \to \langle x_i, x_j^* \rangle = \delta_{i,j} .$$

Hence for all large n , the matrix

$$A_n = [a_{n,i,j}] , \quad \text{where} \quad a_{n,i,j} = \langle x_{n,i}, x_j^* \rangle , \; i,j = 1,\ldots,m ,$$

is invertible. By Remark 3.4, it follows that the set $\{x_{n,1},\ldots,x_{n,m}\}$ is linearly independent in $P_n(X)$, and since $\operatorname{rank} P_n = \operatorname{rank} P = m$, we see that it forms a basis of $P_n(X)$. Also, if $B_n = [b_{n,i,j}]$ is the inverse of A_n , then the elements

$$x_{n,j}^* = \sum_{k=1}^{m} \bar{b}_{n,k,j} x_k^* , \quad j = 1,\ldots,m$$

in X^* are adjoint to $x_{n,1},\ldots,x_{n,m}$: $\langle x_{n,i}, x_{n,j}^* \rangle = \delta_{i,j}$. Since for fixed i,j , we have $a_{n,i,j} \to \delta_{i,j}$, as $n \to \infty$, we note that $b_{n,i,j} \to \delta_{i,j}$ as well. This shows that for all large n

$$|b_{n,i,j}| \le \beta , \; i,j = 1,\ldots,m ,$$

and hence

$$\|x_{n,j}^*\| \le \alpha , \; j = 1,\ldots,m ,$$

where α and β are constants. Now, for $k = 1,2,\ldots$, P_{n_k} is of finite rank, so that by (3.8) we have

$$P_{n_k} u_k = \sum_{j=1}^{m} \langle P_{n_k} u_k, x_{n_k,j}^* \rangle x_{n_k,j} .$$

If we let

$$v_k = \sum_{j=1}^{m} \langle P_{n_k} u_k, x_{n_k,j}^* \rangle x_j ,$$

then we see that $P_{n_k} u_k = P_{n_k} v_k$, and

$$\|v_k\| \le m \,\alpha \, \sup\{\|P_k\| : k = 1,2,\ldots\} \max\{\|x_j\| : j = 1,\ldots,m\} .$$

Now, since (v_k) is a bounded sequence in the compact subset $P(X)$ of X , it has a subsequence which converges to some v in $P(X)$. For the sake of ease in notation, we denote this subsequence by (v_k) itself. Then

$$\begin{aligned} \|P_{n_k} v_k - Pv\| &\le \|P_{n_k} v_k - P_{n_k} v\| + \|P_{n_k} v - Pv\| \\ &\le \|P_{n_k}\| \, \|v_k - v\| + \|P_{n_k} v - Pv\| . \end{aligned}$$

Since $P_{n_k} v \to Pv$, this implies $P_{n_k} u_k = P_{n_k} v_k \to Pv = v$ as $k \to \infty$, and completes the proof. //

We now investigate relationships between norm approximation and collectively compact approximation.

THEOREM 13.5 (a) Let $T_n \xrightarrow{\|\ \|} T$, and assume that $T_n - T$ is compact for all large n . Then $T_n \xrightarrow{cc} T$.

(b) Let $T_n \xrightarrow{cc} T$. Then $\|(T_n-T)^2\| \to 0$. If, in addition, $T_n^* \xrightarrow{p} T^*$, then $T_n \xrightarrow{\|\ \|} T$.

Proof (a) Proposition 13.2 shows that we need only prove the following: If $\|x_k\| \le 1$ and $1 \le n_1 < n_2 < \ldots$, then $((T_{n_k}-T)x_k)$ has a convergent subsequence. But

$$\|(T_{n_k}-T)x_k\| \le \|T_{n_k}-T\|$$

which tends to zero as $k \to \infty$, and we are through.

(b) Letting $A_n = B_n = T_n$ and $A = B = T$ in (13.3), we see that $\|(T_n-T)^2\| \to 0$.

Now, let $T_n^* \xrightarrow{p} T^*$, and assume for a moment that (T_n) is not a norm approximation of T . Then there exist $\delta > 0$, $x_k \in X$ with $\|x_k\| \leq 1$ and integers $1 \leq n_1 < n_2 < \ldots$ such that

$$\|(T_{n_k} - T)x_k\| \geq \delta .$$

Since $T_n \xrightarrow{cc} T$, we may assume by passing to a subsequence, if necessary, that $((T_{n_k} - T)x_k)$ converges to some y in X . Then $\|y\| \geq \delta$. Let $y^* \in X^*$ be such that $\langle y^*, y\rangle = \|y\|$ by the Hahn-Banach theorem (Proposition 1.1). Then

$$\begin{aligned} 0 < \delta \leq \|y\| = \langle y^*, y\rangle &= \lim\langle y^*, (T_{n_k} - T)x_k\rangle \\ &= \lim\langle (T_{n_k} - T)^* y^*, x_k\rangle \\ &\leq \overline{\lim} \|T_{n_k}^* y^* - T^* y^*\| \\ &= 0 . \end{aligned}$$

This contradiction shows that we must, in fact, have $T_n \xrightarrow{\|\ \|} T$. //

We give examples to show that the converse statements of parts (a) and (b) of the above theorem are false.

First, let $x = \ell^2$, $T = 0$, and

$$T_n x = \langle x, e_n\rangle e_1 , \quad x \in \ell^2 , \ n = 1, 2, \ldots ,$$

where e_j is the j-th standard basis vector in ℓ^2 . Thus, T_n is the projection on the span of e_1 along the orthogonal complement of e_n . Since $\langle x, e_n\rangle \to 0$ as $n \to \infty$ for every $x \in \ell^2$, we see that $T_n \xrightarrow{p} T$. Also, for each n , the bounded operator $T_n - T = T_n$ is compact because it is of rank 1 . Now,

$$\bigcup_{n=1}^{\infty} \{(T_n - T)x : x \in \ell^2 , \|x\| \leq 1\} = \{te_1 : t \in \mathbb{C}, |t| \leq 1\} ,$$

which is a one dimensional closed and bounded set, and hence it is compact. Thus, $T_n \xrightarrow{cc} T$. But since $T_n e_n = e_1$, we see that $\|T_n - T\| = \|T_n\| \geq 1$ for each n, showing that (T_n) is not a norm approximation of T.

Secondly, let $X = \ell^2$, $T = 0$, and

$$T_n x = x/n \ , \ x \in \ell^2 \ , \ n = 1,2,\dots \ .$$

Then it can be easily seen that each T_n is self-adjoint and $T_n \xrightarrow{\|\ \|} T$. However, for every n_0, we have

$$\bigcup_{n=n_0}^{\infty} \{(T_n - T)x : x \in \ell^2 \ , \ \|x\| \leq 1\} = \bigcup_{n=n_0}^{\infty} \{\frac{x}{n} : x \in \ell^2 \ , \ \|x\| \leq 1\} \ .$$

This set contains the set $\{x/n_0 : x \in \ell^2 \ , \ \|x\| \leq 1\}$, whose closure is not compact since ℓ^2 is infinite dimensional, showing that (T_n) is not a collectively compact approximation of T.

Thus, we remark that while a norm approximation and a collectively compact approximation are both stronger modes of approximating T than a pointwise approximation, neither is stronger than the other. Since both these modes occur in practical situations (as the examples in Sections 15 and 16 will show), we wish to study the implication of these modes for the approximation of the spectrum, especially, its discrete part. In the next section we shall introduce another mode of approximation which will allow us to unify the study of a norm approximation and a collectively compact approximation.

Problems

13.1 Let $X = \ell^2$, and $\lambda_n \in \mathbb{C}$ with $|\lambda_n| \leq \alpha$, $n = 1,2,\ldots$. Let

$$T_n[x(1), x(2), \ldots]^t = [x(1), 0, \ldots, 0, \lambda_n x(n), 0, 0, \ldots]^t ,$$
$$T[x(1), x(2), \ldots]^t = [x(1), 0, 0, \ldots]^t .$$

Then $\sigma(T_n) = \{0, 1, \lambda_n\}$, $\sigma(T) = \{0,1\}$, while $T_n \xrightarrow{p} T$. (i) Let $\lambda_n \to c$, $c \neq 0$ or 1. Then $\lambda_n \in \sigma(T_n)$, but $\lim_{n\to\infty} \lambda_n \in \rho(T)$. (ii) Let $\lambda_n = 1$ for all n. Then λ_n is a double eigenvalue of T_n, but $\lim_{n\to\infty} \lambda_n$ is a simple eigenvalue of T.

13.2 Let $T_n \xrightarrow{p} T$. Then $T_n \xrightarrow{cc} T$ if and only if there is n_0 such that $T - T_n$ is compact for all $n \geq n_0$, and whenever $k = 1,2,\ldots$, $\|x_k\| \leq 1$, the set $\{(T-T_k)x_k : n = 1,2,\ldots\}$ has a compact closure in X.

13.3 If $T_n \xrightarrow{p} T$ and the set $\bigcup_{n=1}^{\infty} \{T_n x : x \in X, \|x\| \leq 1\}$ has a compact closure in X, then T is compact. (Higgins has proved that if each T_n is compact and the set $\{(T-T_k)x_k : k = 1,2,\ldots\}$ has a compact closure in X whenever $x_k \in X$ with $\|x_k\| \leq 1$, then T is compact.)

13.4 Let $A_n \xrightarrow{p} A$ and $B_n \xrightarrow{cc} B$, where B is compact. Then $A_n B_n \xrightarrow{\|\ \|} AB$.

13.5 Let X be a Hilbert space and T, $T_n \in BL(X)$. Assume that for each $n = 1,2,\ldots$, $T - T_n$ is compact and self-adjoint. Then $T_n \xrightarrow{\|\ \|} T$ if and only if $T_n \xrightarrow{cc} T$.

13.6 Let $T_n \xrightarrow{p} T$. Then for all sufficiently large n, $\dim R(T) \leq \dim R(T_n)$.