A PROBLEM ON NEAREST POINTS IN BANACH SPACES

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If E is a real Banach space and C is a closed non-empty subset of E then the distance function d_C is defined by

$$d_C(x) := \inf\{ \|x - z\| : z \in C\},$$

and any z in C with $d_C(x) = \|x - z\|$ is a nearest point in C to x. If $z \in C$ and there is some $x \in E\backslash C$ with z as its nearest point we call z a nearest point. If the set of points in E\C possessing nearest points in C is generic (contains a dense G_δ) we call C almost proximinal. We say that E is (sequentially) Kadec if we have $\lim_{n\to\infty}\|x_n - x\| = 0$ for every sequence $\{x_n\}$ in E converging weakly to x with $\lim_{n\to\infty}\|x_n\| = \|x\|$. [Each L_p space $(1 < p < \infty)$ has this property, as does any locally uniformly convex Banach space.] The following results are described fully in Borwein and Fitzpatrick [Bo-F].

Theorem 1. (Swiss cheese lemma). Let $\{U_\alpha : \alpha \in A\}$ be a collection of mutually disjoint open convex subsets of a reflexive Banach space. Then
$C := E\backslash\cup\{U_\alpha : \alpha \in A\}$ is almost proximinal if it is non-empty.

Theorem 2. (Lau-Konjagin) In a Banach space E the following conditions are equivalent.

(A) E is reflexive and Kadec.

(B) For each closed non-empty subset C of E, the set of points in E\C with nearest points in C is dense in E\C.

(C) Each closed non-empty subset C of E is almost proximinal.

If C is almost proximinal it follows that the nearest points of C are dense in the boundary of C. There are reflexive Banach spaces E which do not have the Kadec property but such that, nevertheless, for each closed non-empty subset C of E the set of nearest points in C to points of E\C is dense in the boundary of C.

Theorem 3. Let X be a reflexive Kadec space, Y a finite dimensional normed space and $|||\cdot|||$ a Riesz (lattice) norm on $\mathbb{R}^2$. Let $E := X \oplus Y$ in the norm $\|(x,y)\| := |||(\|x\|,\|y\|)|||$. For each closed non-empty subset C of E the nearest points in C to points not in C are dense in the boundary of C.

For unbounded subsets of non-reflexive subspaces there are no general results on nearest points. For bounded closed sets we have:

Theorem 4. Let E be a Banach space with the Radon-Nikodym property and let C be a closed bounded non-empty subset of E. Then C is contained in the closed convex hull of its nearest points to points in $E\backslash C$. In particular C possesses nearest points.

The main open question is: <u>Are nearest points dense in the boundary of every closed subset of every reflexive space?</u> Indeed: <u>Can a proper non-empty closed set in a reflexive space fail to have any nearest points?</u>

References.

[Bo-F] J. M. Borwein and S. Fitzpatrick, "Existence of nearest points in Banach spaces," preprint (1988).

[Ko] S. V. Konjagin, "On approximation properties of closed sets in Banach spaces and the characterization of strongly convex spaces," *Soviet Math. Dokl.* **21**(1980), 418-422.

[La] K.-S. Lau, "Almost Chebyshev subsets in reflexive Banach spaces," *Indiana Univ. Math. J.* **27**(1978), 791-795.