

ON GENERIC DIFFERENTIABILITY OF LOCALLY LIPSCHITZ FUNCTIONS ON  
BANACH SPACE

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**ABSTRACT**

Recently David Preiss proved a remarkable theorem about dense differentiability of locally Lipschitz functions on Banach spaces. Using his result and adopting a technique of Petar Kenderov used to prove generic differentiability of convex functions we establish similar generic differentiability properties for locally Lipschitz functions. We apply our results to determine further differentiability properties of distance functions.

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## 1. Introduction

In the 1970s considerable attention was focussed on the problem of determining classes of Banach spaces where the continuous convex functions have differentiability properties similar to those on Euclidean spaces. An Asplund (weak Asplund) space is defined as a Banach space where every continuous convex function is Fréchet (Gâteaux) differentiable on a dense  $G_\delta$  subset of its open convex domain, and in particular the class of Asplund spaces has been characterised in several ways which demonstrate its significance.

Recently, David Preiss [12] has shown that any real locally Lipschitz function on an open subset of an Asplund space also has the property that it is Fréchet differentiable on a dense subset of its domain. His proof is very technical but his result is very powerful and has many applications. However, the number of applications would be multiplied if conditions could be determined under which the set of points of differentiability is a generic set.

In this paper we use Preiss' work and modify techniques used by Petar Kenderov [9] associated with the earlier weak Asplund and Asplund investigations, to find classes of such spaces on which certain locally Lipschitz functions are generically Gâteaux and Fréchet differentiable on dense subsets of their domains. The results have immediate application in determining differentiability properties of distance functions on Banach spaces.

## 2. Differentiability of locally Lipschitz functions

Given a real Banach space  $X$ , a real function  $\phi$  on an open subset  $A$  of  $X$  is said to be a *locally Lipschitz* function on  $A$  if for each  $x \in A$  there exists a  $K > 0$  and a  $\delta > 0$  such that

$$|\phi(y) - \phi(z)| \leq K \|y - z\| \quad \text{for all } y, z \in B(x; \delta).$$

We define the *local Lipschitz constant* for  $\phi$  at  $x$  by

$$\lambda(x) \equiv \lim_{\delta \rightarrow 0} \omega(x; \delta)$$

where

$$\omega(x; \delta) \equiv \sup \left\{ \frac{|\phi(y) - \phi(z)|}{\|y - z\|} : y, z \in B(x; \delta), y \neq z \right\}$$

Such a function  $\phi$  is said to be *Gâteaux differentiable* at  $x \in A$  if there exists a continuous linear functional  $\phi'(x)$  on  $X$  where, given  $\varepsilon > 0$  and  $\|y\| = 1$  there exists a  $\delta(\varepsilon, x, y) > 0$  such that

$$\left| \frac{\phi(x+ty) - \phi(x)}{t} - \phi'(x)(y) \right| < \varepsilon \quad \text{when } 0 < |t| < \delta .$$

The function  $\phi$  is said to be *Fréchet differentiable* at  $x$  if there exists  $\delta(\varepsilon, x) > 0$  such that the inequality holds for all  $\|y\| = 1$ .

Our work in this paper is dependent on the following theorem which is the main result of Preiss' paper, [12, Theorem 2.4].

### 2.1 Preiss' Theorem

*Let  $X$  be a Banach space with an equivalent norm Gâteaux (Fréchet) differentiable away from the the origin. Then every locally Lipschitz function  $\phi$  defined on an open subset  $A$  of  $X$  is Gâteaux (Fréchet) differentiable on a dense subset  $D$  of  $A$ .*

*More generally in the second case, when  $X$  is an Asplund space then every locally Lipschitz function  $\phi$  defined on an open subset  $A$  of  $X$  is Fréchet differentiable on a dense subset  $D$  of  $A$ .*

*In all these cases and for the appropriate derivative, for every open ball  $B$  in  $A$  and for every  $y, z \in B$*

$$\inf\{\phi'(x)(y-z) : x \in B \cap D\} \leq \phi(y) - \phi(z) \leq \sup\{\phi'(x)(y-z) : x \in B \cap D\} .$$

This last inequality in Preiss' Theorem has the following interesting implication

### 2.2 Corollary

*In such a space  $X$ , a locally Lipschitz function  $\phi$  on an open subset  $A$  has the property that for any  $x \in A$  and  $\varepsilon > 0$  where  $B(x; \varepsilon) \subseteq A$ , there exist  $y, z \in B(x; \varepsilon)$  such that*

$$\sup \{ \|\phi'(u)\| : u \in B(x; \varepsilon) \cap D \}$$

$$\geq \frac{|\phi(y) - \phi(z)|}{\|y-z\|} > \lambda(x) - \varepsilon$$

*where  $\lambda(x)$  is the local Lipschitz constant of  $\phi$  at  $x$ .*

A useful tool for discussing the differentiability of a locally Lipschitz function  $\phi$  on an open subset  $A$  of a normed linear space  $X$  is the *Clarke generalised subdifferential* of  $\phi$  at  $x$  defined by

$$\partial\phi(x) \equiv \left\{ f \in X^* : f(y) \leq \limsup_{\substack{z \rightarrow x \\ t \rightarrow 0^+}} \frac{\phi(z+ty) - \phi(z)}{t} \text{ for all } y \in X \right\}$$

If  $\partial\phi(x)$  is singleton then  $\phi$  is Gâteaux differentiable at  $x$  but the converse is not true in general.

Whenever  $\phi$  has the property that  $\partial\phi(x)$  is singleton we say that  $\phi$  is *strictly differentiable* at  $x$ , [5, p.33]. The *subdifferential mapping*  $x \rightarrow \partial\phi(x)$  has the useful property that it is weak\* upper semi-continuous.

We are now ready to present our main result.

### 2.3. Theorem

*On a Banach space  $X$  with rotund dual  $X^*$  (and with norm Fréchet differentiable away from the origin) a locally Lipschitz function  $\phi$  on an open subset  $A$ , with the property that wherever  $\phi$  is Gâteaux (Fréchet) differentiable it is strictly differentiable, is Gâteaux differentiable on a dense  $G_\delta$  subset of  $A$  and at each point  $x$  of this subset,  $\|\phi'(x)\| = \lambda(x)$ , the local Lipschitz constant of  $\phi$  at  $x$ .*

#### Proof

a. Consider the real function  $\psi$  on  $A$  defined by

$$\psi(x) = \inf \{ \|f\| : f \in \partial\phi(x) \}.$$

(i) We show that  $\psi$  is lower semi-continuous on  $A$ .

For  $x \in A$ , consider a sequence  $\{x_n\}$  in  $A$  convergent to  $x$ . For each  $n$  choose

$f_n \in \partial\phi(x_n)$  such that  $\psi(x_n) > \|f_n\| - 1/n$ . Since the subdifferential mapping  $x \rightarrow \partial\phi(x)$  is weak\* upper semi-continuous and  $\partial\phi(x)$  is weak\*compact,  $\{f_n\}$  has a weak\* cluster point

$f \in \partial\phi(x)$ . From the weak\* lower semi-continuity of the norm on  $X^*$  we deduce that

$$\liminf \psi(x_n) \geq \liminf \|f_n\| \geq \|f\| \geq \psi(x).$$

(ii) Since  $\psi$  is lower semi-continuous on an open subset  $A$  of a Banach space  $X$ , it follows that  $\psi$  is continuous on a dense  $G_\delta$  subset  $A_\psi$  of  $A$ .

b. Consider the real function  $\lambda$  on  $A$  where  $\lambda(x)$  is the local Lipschitz constant of  $\phi$  at  $x$ . We show that  $\lambda$  is upper semi-continuous on  $A$ . For  $x \in A$ , consider a sequence  $\{x_n\}$  in  $A$  convergent to  $x$  and sequences  $\{y_n\}, \{z_n\}$  where for each  $n, y_n, z_n \in B(x_n; 1/n), y_n \neq z_n$  are such that

$$\frac{|\phi(y_n) - \phi(z_n)|}{\|y_n - z_n\|} > \lambda(x_n) - \frac{1}{n}.$$

$$\text{Then } \lambda(x) \geq \limsup_{n \rightarrow \infty} \frac{|\phi(y_n) - \phi(z_n)|}{\|y_n - z_n\|} \geq \limsup_{n \rightarrow \infty} \lambda(x_n).$$

c. We show that at every point  $x \in A_\Psi$ ,

$$\lambda(x) = \|f\| = \psi(x) \quad \text{for all } f \in \partial\phi(x).$$

From Preiss' Theorem, since  $X$  has norm Gâteaux (Fréchet) differentiable away from the origin, we have that  $\phi$  is Gâteaux (Fréchet) differentiable on a dense set  $D$  in  $A$ . From the Corollary to Preiss' Theorem, for each  $n$  we can choose  $x_n \in D \cap B(x; 1/n)$  such that

$$\|\phi'(x_n)\| > \lambda(x) - \frac{1}{n}.$$

But then the sequence  $\{x_n\}$  is convergent to  $x$  and

$$\lambda(x) \geq \limsup \lambda(x_n) \geq \limsup \|\phi'(x_n)\| \geq \liminf \|\phi'(x_n)\| \geq \lambda(x).$$

But since  $\phi$  is strictly differentiable on  $D$ ,

$$\lambda(x) = \lim \|\phi'(x_n)\| = \lim \psi(x_n) = \psi(x).$$

However  $\psi(x) \leq \|f\| \leq \lambda(x)$  for all  $f \in \partial\phi(x)$

$$\text{so } \lambda(x) = \|f\| = \psi(x) \quad \text{for all } f \in \partial\phi(x).$$

d. Since  $X$  has rotund dual  $X^*$ , we have for every  $x \in A_\Psi$  that  $\partial\phi(x)$  is singleton.

Therefore,  $\phi$  is Gâteaux differentiable on  $A_\Psi$  and  $\|\phi'(x)\| = \lambda(x)$  for all  $x \in A_\Psi$ . //

For this theorem to hold we require that the locally Lipschitz function satisfy the property that wherever it is Gâteaux differentiable it is strictly differentiable. We note that this condition is satisfied if the Clarke and Michel-Penot generalised subdifferentials coincide for the function, [2, p.515].

We should notice that this theorem holds more generally in the second case when  $X$  is an Asplund space.

This result has a general version for the family of Banach spaces isomorphic to those specified in the theorem. This corollary for locally Lipschitz functions is a generalisation of Kenderov's Theorem [9] for convex functions.

#### 2.4. Corollary

*On a Banach space  $X$  which can be equivalently renormed to have dual  $X^*$  rotund, a locally Lipschitz function  $\phi$  on an open subset  $A$ , with the property that wherever  $\phi$  is Gâteaux differentiable it is strictly differentiable, is Gâteaux differentiable on a dense  $G_\delta$  subset of  $A$ .*

The Preiss Theorem enables us to extend the theorem to produce a Fréchet differentiable result. Again this corollary for locally Lipschitz functions is a generalisation of the Namioka-Phelps Theorem [11] for convex functions.

#### 2.5 Corollary

*On a Banach space  $X$  which can be equivalently renormed to have a dual which is rotund and where the norm and weak\* topologies coincide on the dual unit sphere, a locally Lipschitz function  $\phi$  on a open subset  $A$ , with the property that wherever  $\phi$  is Fréchet differentiable it is strictly differentiable, is Fréchet differentiable on a dense  $G_\delta$  subset of  $A$ .*

#### Proof

Consider  $X$  so renormed. Then  $X$  has rotund dual  $X^*$  and the norm of  $X$  is Fréchet differentiable away from the origin. From the Theorem,  $\phi$  is Gâteaux differentiable on a dense  $G_\delta$  subset  $A_\psi$  of  $A$  and at each  $x \in A_\psi$ ,  $\|\phi'(x)\| = \lambda(x)$ . We show that at each  $x \in A_\psi$ ,  $\phi'(x)$  is a Fréchet derivative of  $\phi$ .

From Preiss' Theorem, given  $\varepsilon > 0$  and  $\|y\| < \varepsilon$

$$\begin{aligned} & \inf \{ \|\phi'(u)(y) - \phi'(x)\| : u \in B(x; \varepsilon) \cap D \} \\ & \leq \|\phi(x+y) - \phi(x) - \phi'(x)(y)\| \\ & \leq \sup \{ \|\phi'(u)(y) - \phi'(x)(y)\| : u \in B(x; \varepsilon) \cap D \}. \end{aligned}$$

Then  $|\phi(x+y) - \phi(x) - \phi'(x)(y)|$

$$\begin{aligned} & \leq \sup \{ \|\phi'(u)(y) - \phi'(x)(y)\| : u \in B(x; \varepsilon) \cap D \} \\ & \leq \|y\| \sup \{ \|\phi'(u) - \phi'(x)\| : u \in B(x; \varepsilon) \cap D \} \dots\dots\dots \dagger \end{aligned}$$

Since  $\phi$  is also strictly differentiable at  $x$ , as before it follows from the weak\* upper semi-continuity of the subdifferential mapping  $x \rightarrow \partial\phi(x)$  that for  $u \in D$  converging to  $x$ , we have  $\phi'(u)$  converges weak\* to  $\phi'(x)$ . But from the weak\* lower semi-continuity of the dual norm

$$\liminf \|\phi'(u)\| \geq \|\phi'(x)\| = \lambda(x).$$

But also from the definition of a local Lipschitz constant we have

$$\limsup \|\phi'(u)\| \leq \lambda(x)$$

and so  $\|\phi'(u)\|$  converges to  $\|\phi'(x)\| = \lambda(x)$ .

From the properties of the renormed space we have  $\phi'(u)$  is norm convergent to  $\phi'(x)$ . Then from inequality † we deduce that  $\phi$  is Fréchet differentiable on  $A_\Psi$ . //

A normed linear space  $X$  is said to have *locally uniformly rotund* dual  $X^*$  if for any sequence  $\{f_n\}$  in  $X^*$  where  $\|f_n\| = 1 = \|f\|$  and  $\|f_n + f\| \rightarrow 2$  we have  $\|f_n - f\| \rightarrow 0$ . It is known that on such a space the norm and weak\* topologies coincide on the dual unit sphere.

Recently, Fabian [7] has shown that any weakly compactly generated Asplund space can be equivalently renormed to have locally uniformly rotund dual, so such spaces which include reflexive Banach spaces satisfy the requirements of this corollary.

### 3. Applications to distance functions

We now explore the implications of these results for the differentiability of distance functions.

Given a non-empty closed set  $K$  in a normed linear space  $X$ , the distance function  $d$  generated by  $K$  is defined by

$$d(x) = d(x, K)$$

If there exists a point  $p(x) \in K$  such that  $d(x, K) = \|x - p(x)\|$  we say that  $p(x)$  is a *point of best approximation* to  $x$  in  $K$ . We apply our theory to  $d$  as a Lipschitz 1 function on  $X \setminus K$ .

A normed linear space  $X$  is said to have *uniformly Gâteaux differentiable norm* if given  $\varepsilon > 0$  and  $\|x\| = \|y\| = 1$  there exists a continuous linear functional  $f_x$  on  $X$  and a  $\delta(\varepsilon, y) > 0$  such that

$$\left| \frac{\|x + ty\| - \|x\|}{t} - f_x(y) \right| < \varepsilon \text{ when } 0 < |t| < \delta \text{ and for all } \|x\| = 1.$$

Such a space  $X$  has rotund dual  $X^*$ , [6, p.148]. It is also known that on such a space  $X$  every distance function  $d$  on  $X \setminus K$  has the property that wherever it is Gâteaux differentiable it is strictly differentiable, [2, p.525].

Theorem 2.3 has the following immediate application to distance functions.

### 3.1. Theorem

*On a Banach space  $X$  with uniformly Gâteaux differentiable norm, the distance function  $d$  generated by a non-empty closed set  $K$  is Gâteaux differentiable on a dense  $G_\delta$  subset  $D$  of  $X \setminus K$  and  $\|d'(x)\| = 1$  for all  $x \in D$ .*

This is an improvement on [8, Theorem 2], [2, p.526].

From Corollary 2.5 we can make the following deduction.

### 3.2. Theorem

*On a reflexive Banach space  $X$  with uniformly Gâteaux differentiable norm, the distance function generated by a non-empty closed set  $K$  is Fréchet differentiable on a dense  $G_\delta$  subset of  $X \setminus K$ .*

The advantage of having conditions holding generically is made even more apparent when we link our results to those of Lau [10] on points of best approximation.

### 3.3. Corollary

*In a reflexive Banach space  $X$  with uniformly Gâteaux and Kadec norm, given a non-empty closed set  $K$  there exists a dense  $G_\delta$  subset of  $X \setminus K$  each of whose points has points of best approximation in  $K$  and at whose points the distance function  $d$  generated by  $K$  is Fréchet differentiable.*

### Proof

Lau has shown that in a reflexive Banach space  $X$  with Kadec norm there exists a dense  $G_\delta$  subset  $P$  in  $X \setminus K$  where each point of  $P$  has a best approximating point in  $K$ . A reflexive Banach space can be equivalently renormed so that its dual is locally uniformly rotund. Now the property that wherever a locally Lipschitz function is Gâteaux differentiable it is strictly differentiable is an isomorphic property. In the renormed space the distance function  $d$  remains



a locally Lipschitz function and by Corollary 2.5 it is Fréchet differentiable on a dense  $G\delta$  subset  $A_\Psi$  of  $X \setminus K$ . But then in the original space the distance function  $d$  is Fréchet differentiable on the dense  $G\delta$  subset  $A_\Psi$  of  $X \setminus K$ . We conclude that  $d$  has the required properties on the dense  $G\delta$  subset  $A_\Psi \cap P$  of  $X \setminus K$ . //

But again we can link this result to [1, Theorem 6.6] which generalizes that of Stechkin [13] for points of unique best approximation.

### 3.4. Corollary

*In a Banach space with uniformly Gâteaux differentiable norm and with Fréchet differentiable dual norm, given a non-empty closed set  $K$  there exists a dense  $G\delta$  subset of  $X \setminus K$  each of whose points has a unique point of best approximation in  $K$  and at whose points the distance function  $d$  generated by  $K$  is Fréchet differentiable.*

For distance functions on Hilbert space we have not gained any information which was not already known [3, p.379]. This is because in such a space the square of the distance function is the difference of two convex functions. However, using a special case of the smooth variational principle due to Borwein and Preiss [4, p.525] it is possible to deduce Lau's result for Hilbert space with a short proof which also shows how points with best approximation and their best approximating points in the set are related to general points off the set.

### 3.5. The Borwein-Preiss Theorem

*Consider a proper extended real lower semi-continuous function  $\theta$  bounded below on a Hilbert space  $H$ . Given  $\varepsilon > 0$  and  $x_0 \in H$  such that*

$$\theta(x_0) < \inf \theta + \varepsilon$$

*for any  $\lambda > 0$  there exists a  $v \in H$  such that*

$$\|x_0 - v\| < \lambda$$

*and a  $w \in H$  such that*

$$\|x_0 - w\| < \lambda \quad \text{and}$$

$$\theta(x) + (\varepsilon/\lambda^2) \|x - w\|^2 \geq \theta(v) + (\varepsilon/\lambda^2) \|x - v\|^2 \quad \text{for all } x \in H.$$

### 3.6. Theorem

Consider a non-empty closed set  $K$  in a Hilbert space  $H$  and  $y_0 \in H \setminus K$  and  $\delta > 0$ .

For sufficiently small  $0 < \varepsilon < d^2(y_0)$  and  $x_0 \in K \cap B(y_0; d(y_0) + \varepsilon/4d(y_0))$ , given  $\lambda > 0$  there exists a  $y \in B(y_0; \delta)$  with best approximation  $p(y) \in K \cap B(x_0; \lambda)$  such that for some  $w \in B(x_0; \lambda)$  we have  $y = \frac{y_0 + (\varepsilon/\lambda^2)w}{1 + \varepsilon/\lambda^2}$ .

#### Proof

Consider the function

$$\theta(x) = \begin{cases} \|x - y_0\|^2 & \text{when } x \in K \\ +\infty & \text{when } x \in H \setminus K \end{cases}$$

Since  $\|x_0 - y_0\| < d(y_0) + \varepsilon/4d(y_0)$  and  $0 < \varepsilon < d^2(y_0)$  we have

$$\theta(x_0) = \|x_0 - y_0\|^2 < d^2(y_0) + \varepsilon = \inf \theta + \varepsilon.$$

Applying the Borwein–Preiss Theorem to  $\theta$  we see that there exists a  $v \in B(x_0; \lambda)$  and a

$w \in B(x_0; \lambda)$  such that for all  $x \in K$

$$\|x - y_0\|^2 + (\varepsilon/\lambda^2) \|x - w\|^2 \geq \|v - y_0\|^2 + (\varepsilon/\lambda^2) \|v - w\|^2$$

so

$$(1 + \varepsilon/\lambda^2) \|x\|^2 - 2(x, y_0 + (\varepsilon/\lambda^2)w) \geq (1 + \varepsilon/\lambda^2) \|v\|^2 - 2(v, y_0 + (\varepsilon/\lambda^2)w)$$

and

$$\left\| x - \frac{y_0 + (\varepsilon/\lambda^2)w}{1 + \varepsilon/\lambda^2} \right\|^2 \geq \left\| v - \frac{y_0 + (\varepsilon/\lambda^2)w}{1 + \varepsilon/\lambda^2} \right\|^2.$$

Since  $\theta(v) < \infty$  we have that  $v \in K$  and so we deduce that  $v = p(y)$  where  $y \equiv \frac{y_0 + (\varepsilon/\lambda^2)w}{1 + \varepsilon/\lambda^2}$ .

$$\text{Now } \|y_0 - y\| = \frac{\varepsilon}{\lambda^2 + \varepsilon} \|y_0 - w\| < \frac{\varepsilon}{\lambda^2 + \varepsilon} (d(y_0) + \lambda + \varepsilon/4d(y_0))$$

so by suitable choices of  $\varepsilon$  and  $\lambda$  we have  $\|y - y_0\| < \delta$ . //

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