1.7. Holomorphic Semigroups.

Among the many semigroups which occur in applications one class is very common, the holomorphic semigroups. Roughly speaking these are the semigroups $t \geq 0 \mapsto S_t \in \mathcal{L}(\mathcal{B})$ which can be continued holomorphically into a sector of the complex plane containing the positive axis. Among these semigroups one can also identify a subclass analogous to the M-bounded semigroups, i.e., the semigroups satisfying a bound of the form $\|S_t\| \leq M$. This subclass consists of holomorphic semigroups which are uniformly bounded within appropriate subsectors of the sector of holomorphy. For example if H is a positive self-adjoint operator on the Hilbert space $\mathcal{H}$ and $S_t = \exp\{-tH\}$ is the corresponding semigroup then $a \in \mathcal{H} \mapsto S_t a \in \mathcal{H}$ extends to a vector valued function holomorphic in the right half plane satisfying

$$\|S_z a\| = \|S_{\mathrm{Re}\ z} a\| \leq \|a\|$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}\ z \geq 0$. Thus S is a bounded holomorphic semigroup with the right half plane as region of holomorphy.

The general definition of these semigroups is as follows.

DEFINITION 1.7.1. *A* C_0*-semigroup* S *on the Banach space* $\mathcal{B}$ *is called a holomorphic semigroup if for some* $\theta \in \langle 0, \pi/2]$ *one has the following properties:*

1. $t \geq 0 \mapsto S_t$ *is the restriction to the positive real axis of a holomorphic operator function*

$z \in \Delta_\theta \mapsto S_z \in \mathcal{L}(\mathcal{B})$ *where* $\Delta_\theta = \{z \; ; \; |\mathrm{Arg}\, z| < \theta\}$,

2. $$S_{z_1} S_{z_2} = S_{z_1+z_2}$$

for all $z_1, z_2 \in \Delta_\theta$,

3. $$\lim_{z \in \Delta_\theta, z \to 0} \|S_z a - a\| = 0$$

for all $a \in \mathcal{B}$.

If additionally S *is uniformly bounded in* Δ_{θ_1} *for each* $0 < \theta_1 < \theta$ *then* S *is called a bounded holomorphic semigroup.*

There are a variety of ways of characterizing holomorphic semigroups and the following theorem presents two characterizations in terms of the derivative of $t \mapsto S_t$ and the derivatives of the powers $(I+\alpha H)^{-n}$ of the resolvent $(I+\alpha H)^{-1}$.

THEOREM 1.7.2. *Let* $S_t = \exp\{-tH\}$ *be a* C_0*-semigroup on the Banach space* $\mathcal{B}$. *The following conditions are equivalent:*

1. S *is a (bounded) holomorphic semigroup,*

2. *there is a* $C > 0$ *such that*

$$\|HS_t\| < Ct^{-1}$$

for all $0 < t \le 1$ *(for all* $t \ge 0$*)* ,

3. *there is a* $C > 0$ *such that*

$$\|H(I+\alpha H)^{-(n+1)}\| \le C(\alpha n)^{-1}$$

for $0 < \alpha \le 1$, $n\alpha \le 1$, *and* $n = 1, 2, \ldots$

(for $\alpha > 0$ *and* $n = 1, 2, \ldots$ *.)*

N.B. In the above formulation the parenthetic conditions should be read simultaneously to give a characterization of bounded holomorphic semigroups. Their omission covers the general case.

Proof. $1 \Rightarrow 2$. Assume S has a holomorphic extension to $\Delta_\theta = \{z \; ; \; |\text{Arg } z| < \theta\}$. Since S is continuous it follows from the principle of uniform boundedness that there exists an M_1 such that $\|S_z\| \le M_1$ for all $z \in \Delta_{\theta_1} \cap \{z \; ; \; |z| \le 2\}$ where $0 < \theta_1 < \theta$. But by Cauchy's integral representation

$$HS_t = \frac{-d}{dt} S_t = (2\pi i)^{-1} \int_{C_1} dz \frac{S_z}{(z-t)^2}$$

with $C_1 = \{z \; ; \; |z - t| = \text{Sin } \theta_1 t\}$. Consequently

$$\|HS_t\| \le \frac{M_1}{\text{Sin } \theta_1} \frac{1}{t}$$

for all $0 < t \le 1$. Moreover if $\|S_z\|$ is uniformly bounded in Δ_{θ_1} the same argument establishes the estimate for all $t > 0$.

$2 \Rightarrow 3$. Since S is a C_0-semigroup there exist constants $M \ge 1$ and $\omega \ge 0$ such that

(*) $$\|HS_t\| < \frac{C_1 e^{\omega_1 t}}{t} .$$

But

$$H(I+\alpha H)^{-(n+1)} = (n!)^{-1} \int_0^\infty dt\ t^n e^{-t} HS_{\alpha t}$$

and hence

$$\|H(I+\alpha H)^{-(n+1)}\| \le (n!)^{-1} \int_0^\infty dt\ t^{n-1}\alpha^{-1} C_1 e^{-t(1-\alpha\omega_1)}$$

$$= \left(\frac{C_1}{n\alpha}\right)\left(\frac{1}{1-\alpha\omega_1}\right)^n , \qquad 0 < \alpha\omega_1 < 1$$

$$\le \left(\frac{C_1}{n\alpha}\right)\left(\frac{1}{1-\omega_1/n}\right)^n$$

$$\le \left(\frac{C_1}{n\alpha}\right) \frac{1}{1-\omega_1} .$$

Where the second inequality follows from $n\alpha \le 1$ and the third follows because $x \mapsto \left(1-\omega_1/x\right)^{-x}$ is decreasing.

Note that in the bounded case (*) is valid with $\omega_1 = 0$ and then the required bound follows for all $\alpha > 0$.

3 ⇒ 2. It follows directly from Condition 3 and Remark 1.3.3 that

$$\|HS_t\| = \lim_{n\to\infty} \|H\left(I + \frac{t}{n} H\right)^{-n}\| \le Ct^{-1} .$$

2 ⇒ 1. This implication can be established by a variety of arguments which begin with a power series definition. We will

briefly sketch the sequence of ideas.

First let $z = t + is$ with $|s| < t/Ce$ and $0 < t \leq 1$. Then one can define S_z by the norm convergent power series

$$S_z = \sum_{n\geq 0} \frac{(-is)^n}{n!} \left(HS_{t/n}\right)^n .$$

Second one calculates that $S_z D(H) \subseteq D(H)$ and

$$\frac{d}{dz} S_z a = -HS_z a = -S_z Ha$$

for all $a \in D(H)$. Thus

$$\|\left(S_z - I\right)a\| \leq |z| \; \|Ha\|$$

and consequently

$$\lim_{z\to 0} \|\left(S_z - I\right)a\| = 0$$

for all $a \in D(H)$. But then the same conclusion is valid for all $a \in \mathcal{B}$ because $D(H)$ is norm dense.

Third if $0 < t \leq 1$, $a \in D(H)$, and z_1, z_2, $z_1 + z_2$ are in the domain of definition of S_z, the foregoing identification of the derivative gives

$$\frac{d}{dt}\left\{S_{tz_1} S_{tz_2} - S_{t(z_1+z_2)}\right\}a = 0 .$$

Thus integrating and using strong continuity at the origin one finds

$$\left(S_{z_1}S_{z_2} - S_{z_1+z_2}\right)a = 0 \ .$$

But D(H) is norm dense and hence

$$S_{z_1}S_{z_2} = S_{z_1+z_2} \ .$$

Finally one must extend the definition of S_z to the region $\Delta_\theta = \{z \ ; \ \text{Re } z > 0 \ |\text{Arg } z| < \theta\}$ where $\text{Tan } \theta = 1/Ce$. This is achieved by first remarking that each $z \in \Delta_\theta$ can be decomposed in the form $z = z_1 + z_2 + \ldots + z_n$ with $z_i \in \Delta_\theta$ and $\text{Re } z_i \leq 1$. Then one defines

$$S = S_{z_1} S_{z_2} \ldots S_{z_n} \ .$$

There is, however, a problem of consistency since the decomposition of z is clearly not unique. But consistency is easily established by use of the semigroup property in the restricted region. The semigroup property for the larger region then follows by definition.

In the bounded case this last argument is superfluous because S_z can be defined for all $z \in \Delta_\theta$ by the power series expansion and this also establishes that $\|S_z\|$ is uniformly bounded in Δ_{θ_1} for each $0 < \theta_1 < \theta$. □

There are alternative characterizations of holomorphic semigroups in terms of spectral properties of the generator and resolvent bounds. Typically one has the following criterion for a bounded holomorphic semigroup.

THEOREM 1.7.3. *Let* $S_t = \exp\{-tH\}$ *be a* C_0*-semigroup on the Banach space* $\mathcal{B}$ *.*

The following conditions are equivalent:

1. S *is a bounded holomorphic semigroup,*

2. *there is a* $\theta > 0$ *such that*

$$\sigma(H) \subseteq \overline{\Delta}_{\frac{\pi}{2}-\theta} = \{z \ ; \ \ |\mathrm{Arg}\ z| \le \tfrac{\pi}{2} - \theta\}$$

where $\sigma(H)$ *denotes the spectrum of* H *. Moreover*

$$\|(zI-H)^{-1}\| \le M_1 \big/ d_{\theta_1}(z)$$

for all $z \in \mathbb{C}\backslash\overline{\Delta}_{\frac{\pi}{2}-\theta_1}$ *, where* $0 \le \theta_1 < \theta$ *,*

$$d_{\theta_1}(z) = \inf\{|w-z| \ ; \ \ w \in \Delta_{\frac{\pi}{2}-\theta_1}\}$$

and M_1 *can depend on* θ_1 *.*

Proof. $1 \Rightarrow 2$. Suppose $z \mapsto S_z$ is holomorphic in the sector $\Delta_\theta = \{z \ ; \ |\mathrm{Arg}\ z| < \theta\}$. Next consider the C_0-semigroups $S_t^w = \exp\{-twH\}$ where $w = \exp\{i\alpha\}$ and $0 \le |\alpha| < \theta$. The generator of S^w is wH and hence $\sigma(wH) \subseteq \{z \ ; \ \mathrm{Re}\ z \ge 0\}$, by Proposition 1.2.1. Therefore $\sigma(H) \subseteq \{z \ ; \ |\mathrm{Arg}\ z| \le \frac{\pi}{2} - \theta\}$. Moreover, since there is an M_1 such that $\|S_t^w\| \le M_1$ for

$w \in \Delta_{\theta_1}$ where $0 \leq \theta_1 < \theta$, one must have

$$\|(\lambda I-wH)^{-1}\| = \|\int_0^\infty dt\ e^{\lambda t} S_t^w\| \leq M_1 \Big/ |\text{Re}\ \lambda|$$

whenever $\text{Re}\ \lambda < 0$. Consequently

$$\|(zI-H)^{-1}\| \leq M_1 \Big/ d_{\theta_1}(z) \ .$$

2 ⇒ 1. The detailed proof of this implication is rather protracted, although completely straightforward. Again we only sketch the outlines.

First let Γ be a wedge shaped contour lying in the resolvent set $r(H)$ of H with asymptotes $\text{Arg}\ z = \pm\left(\frac{\pi}{2}-\theta_2\right)$ where $0 \leq \theta_2 < \theta_1$ and for $z \in \Delta_\theta$ define S by

$$S_z = (2\pi i)^{-1} \int_\Gamma d\lambda\ e^{\lambda z}(\lambda I-H)^{-1} \ .$$

By Cauchy's theorem the integral is independent of the particular contour chosen and one can use this freedom of choice, together with the resolvent bounds, to deduce that $z \in \Delta_\theta \mapsto \|S_z\|$ is uniformly bounded.

Second one calculates that S satisfies the semigroup property $S_{z_1} S_{z_2} = S_{z_1+z_2}$ by choosing Γ_2 outside Γ_1 and noting that

$$S_{z_1}S_{z_2} = (2\pi i)^{-2}\int_{\Gamma_1}\int_{\Gamma_2} d\lambda_1 d\lambda_2\, e^{\lambda_1 z_1+\lambda_2 z_2}(\lambda_1 I-H)^{-1}(\lambda_2 I-H)^{-1}$$

$$= (2\pi i)^{-2}\int_{\Gamma}\int_{\Gamma} d\lambda_1 d\lambda_2\, \frac{e^{\lambda_1 z_1+\lambda_2 z_2}}{\lambda_2-\lambda_1}\left\{(\lambda_1 I-H)^{-1} - (\lambda_2 I-H)^{-1}\right\}$$

$$= (2\pi i)^{-1}\int_{\Gamma} d\lambda\, e^{\lambda(z_1+z_2)}(\lambda I-H)^{-1} .$$

Here we have used the obvious resolvent identity, Cauchy's theorem, and Fubini's theorem.

Third one notes that if $a \in D(H)$

$$(I-S_z)a = (2\pi i)^{-1}\int_{\Gamma} d\lambda\, e^{\lambda z}\left\{\lambda^{-1}I - (\lambda I-H)^{-1}\right\}a$$

$$= -(2\pi i)^{-1}\int_{\Gamma} d\lambda\, e^{\lambda z}\lambda^{-1}(\lambda I-H)^{-1}Ha$$

$$\xrightarrow[z\to 0]{} 0$$

when the last conclusion follows from the resolvent bound and the Lebesgue dominated convergence theorem.

Finally one identifies H as the generator of S by careful calculation of the derivative of S. This again requires Cauchy's theorem. □

One simple explicit example of a bounded holomorphic semigroup is the semigroup S generated by the Laplacian on $L^p(\mathbb{R}^\nu)$. This semigroup is holomorphic in the sector $\Delta_{\pi/2}$ and its action is given by

$$(S_z a)(x) = (4\pi z)^{-\nu/2}\int d^\nu y\, e^{-(x-y)^2/4z}a(y) .$$

Note that if $p = 2$ then

$$\|S_2 a\|_2 = \|S_{\mathrm{Re}\, z} a\|_2 \le \|a\|_2$$

since $S_z = \exp\{-zH\}$ where H is self-adjoint. Moreover S has a boundary value as $\mathrm{Re}\, z \to 0$ because

$$\lim_{s\to 0} \|S_{s+it} a - e^{-itH} a\| = 0 .$$

But if $p = 1$

$$\|S_z\| = \int d^\nu y \, |(4\pi z)^{-\nu/2} e^{-y^2/4z}| = (|z| \; \mathrm{Re}\, z)^{\nu/2}$$

for $\mathrm{Re}\, z > 0$, and a similar result is true for $p = \infty$. Thus in these latter cases $\|S_z\| \to \infty$ as z approaches the imaginary axis, away from the origin, and S does not have a boundary value.

Exercises 1.7.1.

1. Let S be a self-adjoint contraction semigroup on a Hilbert space $\mathcal{H}$. Prove that S is holomorphic for $\mathrm{Re}\, z > 0$ and that $\|S_z\| \le 1$ in this sector.