1.2. Semigroups and Generators.

Let $\mathcal{B}$ be a complex Banach space and $\mathcal{B}^*$ its dual. We denote elements of $\mathcal{B}$ by a, b, c, ... and elements of $\mathcal{B}^*$ by f, g, h, ... Moreover we use (f, a) to denote the value of f on a and $\|\cdot\|$ to denote the norm on $\mathcal{B}$ and also the dual norm on $\mathcal{B}^*$, i.e.,

$$\|f\| = \sup\{|f(a)| \ ; \ \|a\| \le 1\} .$$

A *semigroup* S on $\mathcal{B}$ is defined to be a family $S \ ; \ t \in \mathbb{R}_+ \mapsto S_t \in \mathcal{L}(\mathcal{B})$ of bounded linear operators on $\mathcal{B}$ which satisfy

1. $S_s S_t = S_{s+t}$, $\quad s, t \ge 0$,

2. $S_0 = I$

where I denotes the identity operator on $\mathcal{B}$.

This notion of semigroup is not of great interest unless one imposes some further hypothesis of continuity. There are a variety of possible forms of continuity. Let us first consider continuity at the origin.

The strongest possible requirement would be uniform continuity, i.e.,

$$\lim_{t\to 0+} \|S_t - I\| = 0 ,$$

where the operator norm is defined in the usual manner

$$\|S_t - I\| = \sup\{\|S_t a - a\| \; ; \; \|a\| \leq 1\} .$$

But this is a very restrictive assumption. It can be established that a semigroup is uniformly continuous at the origin if, and only if, there exists a bounded operator H such that

$$S_t = I + \sum_{n\geq 1} \frac{(-t)^n}{n!} H^n = \exp\{-tH\}$$

(see Exercise 1.2.1). This is of limited interest in applications. Nevertheless we occasionally use uniformly continuous matrix semigroups for illustrative purposes.

A weaker continuity requirement is strong continuity at the origin, i.e.,

$$\lim_{t\to 0+} \|(S_t - I)a\| = 0$$

for all $a \in \mathcal{B}$. Semigroups with this property are usually called C_0-*semigroups* and we adopt this notation throughout the sequel. The heat semigroup on $C_0(\mathbb{R})$ is a semigroup of this type. Note that if S is a C_0-semigroup then it follows from the principle of uniform boundedness (see Exercise 1.2.2) that

$$\|S_t\| \leq Me^{\omega t}$$

for some $M \geq 1$ and some finite $\omega \geq \inf_{t>0} \left(t^{-1} \log\|S_t\|\right)$. In particular this implies that strong continuity of S at the origin is equivalent to strong continuity at all $t \geq 0$. This follows from the easy estimate

$$\|(S_{s+t}-S_s)a\| \le \|S_s\| \; \|(S_t-I)a\|$$
$$\le Me^{\omega s}\|(S_t-I)a\| \; .$$

Moreover it establishes that the analysis of a general C_0-semigroup can be reduced to the analysis of an M-bounded C_0-semigroup, i.e., a semigroup satisfying

$$\|S_t\| \le M \; .$$

This reduction is effected by replacing S_t by $S_t e^{-\omega t}$. The case $M = 1$ is of particular importance.

A C_0-semigroup S for which each S_t is contractive, i.e.,

$$\|S_t\| \le 1 \; , \qquad t \ge 0 \; ,$$

is called a C_0-*semigroup of contractions*. The foregoing discussion of boundedness properties indicates that the theory of contractive semigroups is very close to the general theory. Nevertheless there are some significant differences which lead to complications if $M > 1$ and there are a number of techniques which are only applicable to the contractive case $M = 1$, $\omega = 0$. Consequently for simplicity of exposition and diversity of method we restrict the ensuing discussion to contraction semigroups.

Before proceeding to the detailed discussion of C_0-semigroups we note that there are other weaker forms of continuity which are of interest. One continuity hypothesis, which is natural from the mathematical point of view, is weak continuity at the origin. By this we mean

$$(*) \qquad \lim_{t\to 0+} (f, S_t a) = (f, a)$$

for all $a \in \mathcal{B}$, and all $f \in \mathcal{B}^*$. But here an unexpected simplification occurs; *every weakly continuous semigroup is automatically strongly continuous* (see Exercise 1.2.3). Alternatively one could make the weaker hypothesis that (*) is valid for all $a \in \mathcal{B}$ and all f in some 'large' subspace of $\mathcal{B}^*$. In particular if $\mathcal{B}$ has a predual, i.e., if $\mathcal{B}$ is the dual of a Banach space $\mathcal{B}_*$, then one could suppose that (*) holds for all $a \in \mathcal{B}$ and all $f \in \mathcal{B}_*$. This hypothesis is referred to as weak*-continuity and a semigroup that satisfies it is called a C_0^*-*semigroup*. This notation is appropriate because it follows by duality that each C_0^*-semigroup on $\mathcal{B}$ is the dual of a C_0-semigroup acting on the predual $\mathcal{B}_*$. Hence many facets of the theory of C_0^*-semigroups can be deduced by duality from the C_0-case. The group of translations acting on $L^\infty(\mathbb{R}; dx)$ is an example of a C_0^*-group which is not a C_0-group; it is the dual of the C_0-group of translations acting on $L^1(\mathbb{R}; dx)$. We consider the basic theory of C_0^*-semigroups of contractions in Section 1.6.

The most important concept in the theory of continuous semigroups is that of the (infinitesimal) generator. This generator is defined as the (right) derivative of the semigroup at the origin where the sense in which the derivative is taken is dictated by the continuity hypothesis. In particular the generator of a C_0-semigroup is defined as the strong derivative. The detailed definition is as follows.

If S is a C_0-semigroup on the Banach space $\mathcal{B}$ the *(infinitesimal) generator* of S is defined as the linear operator H on $\mathcal{B}$ whose domain $D(H)$ consists of those $a \in \mathcal{B}$ for which there exists a $b \in \mathcal{B}$ with the property that

$$\lim_{t\to 0+} \left\| \frac{(I-S_t)}{t} a - b \right\| = 0 .$$

If $a \in D(H)$ the action of H is defined by

$$Ha = b .$$

Note that the semigroup property of S automatically implies $S_t D(H) \subseteq D(H)$ and

$$HS_t a = S_t Ha$$

for all $a \in D(H)$ and $t \geq 0$. Moreover one has the differential equation

$$\frac{dS_t}{dt} a = -HS_t a = -S_t Ha$$

for each $a \in D(H)$, where the strong derivative dS_t/dt is defined by

$$\frac{dS_t}{dt} a = \lim_{h\to 0} \frac{(S_{t+h}-S_t)}{h} a$$

whenever the limit exists. It also follows that S and H are connected by the integral equation

$$S_t a - a = -\int_0^t ds\, S_s H a = -\int_0^t ds\, H S_s a$$

for each $a \in D(H)$. The integrals, both here and throughout the sequel, are understood as $\mathcal{B}$-valued Riemann integrals.

We now derive the basic properties of generators and their resolvents.

Recall that the *resolvent set* $r(H)$ of an operator H on $\mathcal{B}$ is the set $\lambda \in \mathbb{C}$ for which $\lambda I - H$ has a bounded inverse, the spectrum $\sigma(H)$ of H is the complement of $r(H)$ in $\mathbb{C}$, and if $\lambda \in r(H)$ then $(\lambda I - H)^{-1}$ is called the *resolvent* of H .

PROPOSITION 1.2.1. *Let* S *be a* C_0*-semigroup of contractions on the Banach space* $\mathcal{B}$ *with generator* H .

It follows that

1. H *is norm closed, norm densely defined,*

2. *If* $\mathrm{Re}\lambda < 0$ *the range* $R(\lambda I - H)$ *of* $\lambda I - H$ *satisfies*

$$R(\lambda I - H) = \mathcal{B}$$

and for $a \in D(H)$

$$\|(\lambda I - H)a\| \geq |\mathrm{Re}\lambda|\ \|A\| ,$$

3. *If* $\mathrm{Re}\lambda < 0$ *the resolvent of* H *is given by the*

Laplace transform

$$(\lambda I-H)^{-1}a = -\int_0^\infty ds\ e^{\lambda s}S_s a\ , \qquad a \in \mathcal{B}\ .$$

In particular $\sigma(H) \subseteq \{\lambda\ ;\ \mathrm{Re}\lambda \geq 0\}$.

Proof. Since $\mathrm{Re}\lambda < 0$ we may define a bounded operator $R_\lambda(H)$ on $\mathcal{B}$ by

$$R_\lambda(H)a = -\int_0^\infty ds\ e^{\lambda s}S_s a\ , \qquad a \in \mathcal{B}\ .$$

Explicitly one has

$$\|R_\lambda(H)a\| \leq \int_0^\infty ds\ e^{-s|\mathrm{Re}\lambda|}\|S_s a\|$$

$$\leq \int_0^\infty ds\ e^{-s|\mathrm{Re}\lambda|}\|a\| = |\mathrm{Re}\lambda|^{-1}\|a\|\ .$$

But for each $a \in \mathcal{B}$ one also has

$$t^{-1}(I-S_t)R_\lambda(H)a = -t^{-1}\int_0^\infty ds\ e^{\lambda s}(S_s-S_{s+t})a$$

$$= -t^{-1}\int_0^\infty ds\ e^{\lambda s}(1-e^{-\lambda t})S_s a - t^{-1}\int_0^t ds\ e^{\lambda(s-t)}S_s a$$

$$\xrightarrow[t\to 0+]{} \lambda R_\lambda(H)a - a$$

where both integrals converge in norm. This last conclusion uses the strong continuity of S and the Lebesgue dominated theorem. It follows that $R_\lambda(H)a \in D(H)$ and

$$(\lambda I-H)R_\lambda(H)a = a\ .$$

In particular

$$R(\lambda I-H) = \mathcal{B} .$$

But since

$$S_t R_\lambda(H) = R_\lambda(H) S_t$$

and $R_\lambda(H)$ is bounded one finds that

$$(\lambda I-H)R_\lambda(H)a = R_\lambda(H)(\lambda I-H)a = a$$

for $a \in D(H)$. Hence $\lambda \in r(H)$ and

$$(\lambda I-H)^{-1} = R_\lambda(H) .$$

But boundedness of $(\lambda I-H)^{-1}$ implies that $\lambda I - H$, and hence H , is norm closed. Moreover the explicit estimate for $\|R_\lambda(H)a\|$ derived at the beginning of the proof immediately gives the desired lower bound on $\|(\lambda I-H)a\|$.

Finally $a_n = -nR_n(H)a \in D(H)$ for all $a \in \mathcal{B}$ and $n \geq 1$. But

$$\begin{aligned} a_n - a &= n \int_0^\infty ds\, e^{-ns}\left(S_s - I\right)a \\ &= \int_0^\infty ds\, e^{-s}\left(S_{\frac{s}{n}} - I\right)a \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

by another application of strong continuity and the Lebesgue dominated convergence theorem. Thus $D(H)$ is norm dense. □

This result has two simple implications which we often use in the sequel without further comment. First the proposition implies that for each $\alpha > 0$ generators satisfy

$$(*) \qquad \|(I+\alpha H)a\| \geq \|a\| \ , \qquad a \in D(H) \ .$$

But it immediately follows that *the generator* H *of a* C_0*-semigroup* S *has no proper extension satisfying* (*), i.e., generators are in this sense maximal. To deduce this suppose $\hat{H}$ extends H and also satisfies (*) then for $a \in D(\hat{H})$ set $b = (I+\alpha\hat{H})a$. But there is an $a' \in D(H)$ such that $b = (I+\alpha H)a'$, by Condition 2 of Proposition 1.2.1, and hence $(I+\alpha\hat{H})(a-a') = 0$, because $\hat{H}$ extends H. Thus $a = a'$ by (*) and $\hat{H} = H$. The second implication gives a characterization of a core of H. Recall that a subset D of the domain D(X) of an operator X is called a core of X if for each $a \in D(X)$ there is a sequence $a_n \in D$ such that $\|a_n - a\| \to 0$ and $\|Xa_n - Xa\| \to 0$ as $n \to \infty$. In particular if X is closed then D is a core if, and only if, the norm closure $X|_D$, of X restricted to D, is equal to X. It follows that *a subset* $D \subseteq D(H)$ *is a core for the generator* H *if, and only if,* $(\lambda I-H)D$ *is norm dense in* $\mathcal{B}$ *for some* λ *with* $\mathrm{Re}\lambda < 0$, *or for all* λ *with* $\mathrm{Re}\lambda < 0$. Clearly if D is a core $\overline{R(\lambda I-H)D} = \mathcal{B}$ by Proposition 1.2.1. Conversely if $\hat{H}$ denotes the closure of $H|_D$ and $R(\lambda I-\hat{H}) = \mathcal{B}$ one again concludes that $A = H$ by use of (*).

A slight variation of the argument used to prove Proposition 1.2.1 also provides the following slightly less evident criterion for a core of a generator.

COROLLARY 1.2.2. *Let* S *be a* C_0*-semigroup of contractions on the Banach space* $\mathcal{B}$ *with generator* H *and let* D *be a subset of the domain* D(H) *of* H *which is norm dense and invariant under* S, *i.e.,* $S_t a \in D$ *for all* $a \in D$ *and* $t \geq 0$.

It follows that D *is a core for* H.

Proof. Let $\hat{H}$ denote the closure of $H|_D$. By the above remarks it suffices to prove that $R(\lambda I-\hat{H}) = \mathcal{B}$ for some λ with $\mathrm{Re}\lambda < 0$. But for $a \in D$ one can choose Riemann approximants

$$\sum_N(a) = -\sum_{i=1}^{N} e^{\lambda s_i} S_{s_i} a\left(s_{i+1}-s_i\right)$$

$$\sum_N((\lambda I-H)a) = -\sum_{i=1}^{N} e^{\lambda s_i} S_{s_i} (\lambda I-H)a\left(s_{i+1}-s_i\right)$$

which converge simultaneously to $(\lambda I-H)^{-1}a$ and a. Now $\sum_N(a) \in D$ because of the invariance of D under S and

$$(\lambda I-H)\sum_N(a) = \sum_N((\lambda I-H)a) .$$

Thus $\sum_N(a) \to (\lambda I-H)^{-1}a$ and $(\lambda I-H)\sum_N(a) \to a$. Therefore $D \subseteq R(\lambda I-H)$. But $(\lambda I-H)^{-1}$ is bounded and hence $R(\lambda I-\hat{H})$ is norm closed. Thus $R(\lambda I-\hat{H}) = D$ by the norm density of D. $\square$

Exercises.

1.2.1. Prove that if a semigroup S is uniformly continuous then there exists a bounded operator H such that

$$S_t = \sum_{n\geq 0} \frac{(-t)^n}{n!} H^n .$$

Hint: For small $s > 0$ the operator

$$H_s = s^{-1}\int_0^s dt\, S_t$$

is invertible, with bounded inverse, and

$$(I-S_t)H_s/t = (I-S_s)H_t/s \ .$$

1.2.2. Prove that a weakly continuous semigroup S must satisfy

$$\|S_t\| \le Me^{\omega t}$$

for some $M \ge 1$ and some finite $\omega \ge \inf_{t\ge 0}\left(t^{-1} \log\|S_t\|\right)$.

Hint: Use the uniform boundedness principle for small t and the semigroup property for large t .

1.2.3. Verify that if $\mathrm{Re}\ z > 0$ then

$$t \mapsto (S_t f)(x) = (4\pi t z)^{-\nu/2} \int d^\nu y \ e^{-(x-y)^2/4tz} f(y)$$

defines a C_0-semigroup on $C_0(\mathbb{R}^\nu)$ satisfying

$$\|S_t\|_\infty = \left\{|z|/\mathrm{Re}\ z\right\}^{\nu/2} \ .$$

1.2.4. Prove that weak and strong continuity of a semigroup S are equivalent.

Hint: The weak generator H_w of a weakly continuous semigroup S is defined by

$$(f,\ H_w a) = \lim_{t\to 0+} \left(f,\ (I-S_t)a\right)/t$$

with $D(H_w)$ the set of a for which the limit exists for all $f \in \mathcal{B}^*$. Adapt the argument used in the proof of Proposition 1.2.1 to deduce that $D(H_w)$ is weakly dense and hence, by the Hahn-Banach theorem, strongly dense. Finally use

$$(f,\ a-S_t a) = \int_0^t ds\ (f,\ S_s H_w a)$$

to prove strong continuity for all $a \in D(H_w)$.

1.2.5. Prove that the generator H and weak generator H_w of a C_0-semigroup S coincide.

Hint: Adapt the proof of Proposition 1.2.1 to deduce that $H \supseteq H_w$.

1.2.6. If H is the generator of a C_0-semigroup prove that

$$D_\infty(H) = \bigcap_{n\geq 1} D(H^n)$$

is norm dense.

Hint: For each a define a_n by

$$a_n = \int_0^\infty dt\ f(t) S_{t/n}\, a$$

where f is a positive, infinitely often differentiable, function with compact support in $\langle 0, \infty\rangle$ and with total integral one. Then $a_n \in D_\infty(H)$ and $\|a_n - a\| \to 0$ as $n \to \infty$.

1.2.7. Let S denote the heat semigroup on $L^p(\mathbb{R}^\nu)$,

$$(S_t f)(x) = (4\pi t)^{-\nu/2} \int d^\nu y \; e^{-(x-y)^2/4t} f(y) \; .$$

Prove that the generator of S is the closure of the restriction of the Laplacian

$$-\nabla^2 = -\sum_{i=1} \frac{\partial^2}{\partial x_i^2}$$

to the infinitely often differentiable functions in $L^p(\mathbb{R}^\nu)$.

Hint: Use Corollary 1.2.2.