CHAPTER 4

REGULARITY OF WEAKLY HARMONIC MAPS

Regularity, existence, and uniqueness of solutions of the Dirichlet problem, if the image is contained in a convex ball

4.1 THE CONCEPT OF WEAK SOLUTIONS

We first want to discuss the concept of stationary points of the energy integral or of weak solutions of the corresponding Euler-Lagrange equations. In the present chapter, the image Y will always be covered by a single coordinate chart so that we can define the Sobolev space $W_2^1(\Omega,Y)$ unambiguously with the help of this chart, without having to use the Nash embedding theorem as in 1.3.

Ω will be an open bounded set in some Riemannian manifold with boundary $\partial\Omega$.

In the sequel, we shall use some of the notations of [EL4].

If $u \in W_2^1(\Omega,Y)$, then du is an almost everywhere on Ω defined 1-form with values in $u^{-1}\,TY$. The energy of u is

$$E(u) = \frac{1}{2} \int_\Omega \langle du, du \rangle \, d\Omega \; ,$$

where the scalar product is taken in $T^*\Omega \otimes u^{-1}\,TY$.

We let $\phi \in C_0(\bar{\Omega}, u^{-1}\,TY)$ be a section along u which vanishes on $\partial\Omega$. This means $\phi(x) \in T_{u(x)}Y$. We want to construct a variation of u with tangent field ϕ .

Since we assume that Y is covered by a single coordinate chart, we can simply represent everything in those coordinates and denote the representations in these coordinates by $\tilde{}$ and define

$$\tilde{u}_t(x) = \tilde{u}(x) + t\tilde{\phi}(x) .$$

These coordinates also identify each tangent space T_uY with $\mathbb{R}^n$ ($n = \dim Y$) . Hence $\tilde{\phi}$ is a map from Ω into $\mathbb{R}^n$. This allows us to define $d\tilde{\phi}$ and hence via this identification also $d\phi$. (Note that it is not obvious how to define $d\phi$ intrinsically, since $\phi(x) \in T_{u(x)}Y$, and as u is not necessarily continuous, the base point of ϕ may vary in a noncontinuous way.) We then suppose that

$$\int_\Omega \langle d\phi, d\phi\rangle < \infty \tag{4.1.1}$$

and show that the Euler-Lagrange equations, if u is a critical point of E ,

$$\int\left\{\gamma^{\alpha\beta}\frac{\partial u^i}{\partial x^\alpha}\frac{\partial\psi^i}{\partial x^\beta} - \gamma^{\alpha\beta}\,\Gamma^i_{kj}\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta}\psi^i\right\}\sqrt{\gamma}\,dx = 0 \qquad \text{for } \psi \in W^1_2 \cap L^\infty(\Omega,\mathbb{R}^n) \; (\psi|\partial\Omega=0) \tag{4.1.2}$$

are equivalent to

$$\int \langle du, d\phi\rangle = 0 \qquad \text{for all bounded } \phi \text{ satisfying (4.1.1) and } \phi|\partial\Omega = 0 . \tag{4.1.3}$$

Proof Let
$$\phi = \phi^i(x)\frac{\partial}{\partial u^i} .$$

Then
$$d\phi = \nabla_{\frac{\partial}{\partial x^\alpha}}\left(\phi^i\frac{\partial}{\partial u^i}\right)dx^\alpha = \frac{\partial\phi^i}{\partial x^\alpha}\frac{\partial}{\partial u^i} + \phi^i\,\Gamma^k_{ij}\frac{\partial u^j}{\partial x^\alpha}\frac{\partial}{\partial u^k} .$$

Hence

$$\langle du, d\phi\rangle = g_{ij}\,\gamma^{\alpha\beta}\frac{\partial u^i}{\partial x^\alpha}\frac{\partial\phi^j}{\partial x^\beta} + \gamma^{\alpha\beta}\,g_{ik}\,\phi^\ell\,\Gamma^k_{\ell j}\frac{\partial u^j}{\partial x^\beta}\frac{\partial u^i}{\partial x^\alpha} . \tag{4.1.4}$$

On the other hand, we choose $\psi^i = g_{ij}\phi^j$ as a test vector in (4.1.2). Then the integrand of (4.1.2) becomes

$$\gamma^{\alpha\beta}g_{ij}\frac{\partial u^i}{\partial x^\alpha}\frac{\partial\phi^j}{\partial x^\beta} + \gamma^{\alpha\beta}g_{kj,\ell}\frac{\partial u^\ell}{\partial x^\beta}\frac{\partial u^k}{\partial x^\alpha}\phi^j - \gamma^{\alpha\beta}g_{ij}\,\Gamma^i_{k\ell}\frac{\partial u^k}{\partial x^\alpha}\frac{\partial u^\ell}{\partial x^\beta}\phi^j$$

$$= \gamma^{\alpha\beta}g_{ij}\frac{\partial u^i}{\partial x^\alpha}\frac{\partial\phi^j}{\partial x^\beta} + \gamma^{\alpha\beta}\,\tfrac{1}{2}(g_{kj,\ell} + g_{k\ell,j} - g_{j\ell,k})\frac{\partial u^\ell}{\partial x^\beta}\frac{\partial u^k}{\partial x^\alpha}\phi^j$$

which after changing some indices, is the same as (4.1.4).

q.e.d.

Remark If one wants to define $d\phi$ also if the image is not necessarily contained in a single coordinate chart, one can use the Nash embedding theorem as in section 1.3.

In the following sections, we want to provide conditions which ensure that a weak solution $u \in H_2^1 \cap L^\infty$ of (4.1.2) or (4.1.3) is continuous (which then in turn will also imply higher regularity of u).

We have already seen in 1.4 that $\frac{x}{|x|} : D^n \to S^{n-1}$ for $n \geq 3$ is a discontinuous weak solution. One might think that the discontinuity in this case is caused by the global topology of the image. We can however take the totally geodesic embedding $i : S^{n-1} \to S^n$ which maps S^{n-1} onto the equator of S^n. By Lemma 1.7.2, $i \circ \frac{x}{|x|}$ then is harmonic for $x \neq 0$ and hence weakly harmonic by the argument of 1.4. The image of $i \circ \frac{x}{|x|}$, however, is contained in a closed hemisphere, so that there is no longer a topological obstruction to regularity, and the discontinuity has to be caused by the geometry of the image.

As pointed out, in this case the image is contained in a geodesic ball of radius $\frac{\pi}{2}$ in S^n. In the following sections, we shall see that the radius $\frac{\pi}{2}$ is precisely the limiting case for regularity, i.e. that any weakly harmonic map with image contained in a geodesic ball of radius $< \frac{\pi}{2}$ actually is regular. (We shall of course consider more general image manifolds than only spheres.)

Finally, we remark that in many cases $i \circ \frac{x}{|x|}$ even minimizes energy w.r.t. its boundary values, as was demonstrated by Jäger-Kaul [JäK3] and Baldes [Ba].

In the following sections, we assume w.l.o.g. that the dimension n of the domain Ω is at least 3, because otherwise we can simply look at the map

$\tilde{u} : \Omega \times S^1 \to Y$, $\tilde{u}(x,t) = u(x)$ which satisfies the same assumptions as u .

4.2 A LEMMA OF GIAQUINTA-GIUSTI-HILDEBRANDT

The following lemma is due to Giaquinta-Hildebrandt [GH].

LEMMA 4.2.1 *Suppose* $u : \Omega \to Y$ *is weakly harmonic,* $f : Y \to \mathbb{R}$ *is strictly convex on* $u(\Omega)$. *Then for every ball* $B(x_0, 2R_0) \subset \Omega$

$$\int_{B(x_0,R_0)} d(x,x_0)^{2-n} \, |du|^2 \le c_1 < \infty . \tag{4.2.1}$$

Furthermore, for any $\varepsilon > 0$ *and* $R_0 > 0$ *we can calculate* $R_1 > 0$ *, independent of* x_0 *and* u *with the property that for some* R *,* $R_1 \le R \le R_0$

$$R^{2-n} \int_{B(x_0,R)} |du|^2 \le \varepsilon . \tag{4.2.2}$$

c_1 *and* R_1 *depend on the supremum of* f *and on a lower bound* $\lambda > 0$ *for the eigenvalues of its Hessian and on the geometry of* Ω *(curvature bounds, injectivity radius, dimension).*

Proof One idea is taken from [JK1], p.11, the other from [GG1], p.

We put $h = f \circ u$. By (1.7.2)

$$\Delta h \ge \lambda |du|^2 . \tag{4.2.3}$$

Let $r(x) = d(x,x_0)$ and $g_\rho(x) = \min\left\{ r(x)^{2-n} - \rho^{2-n}, \left(\frac{\rho}{2}\right)^{2-n} - \rho^{2-n} \right\}$ on $B(x_0,\rho)$. Then

$$\begin{aligned}
(4.2.4) \quad \lambda \int_{B(x_0,\rho)} g_\rho(x) \, |du|^2 &\le \int_{B(x_0,\rho)} g_\rho(x) \, \Delta h(x) \qquad \text{by (4.2.3)} \\
&= -\int_{B(x_0,\rho)\setminus B(x_0,\rho/2)} \langle \operatorname{grad} g_\rho, \operatorname{grad} h \rangle \\
&= \int_{B(x_0,\rho)\setminus B(x_0,\rho/2)} h \, \Delta g_\rho - \int_{\partial(B(x_0,\rho)\setminus B(x_0,\rho/2))} h \, \langle \operatorname{grad} g_\rho, \vec{dO} \rangle \\
&\le c_2 \rho^2 + \frac{n-2}{\rho^{n-1}} \int_{\partial B(x_0,\rho)} h - \frac{n-2}{(\rho/2)^{n-1}} \int_{\partial B(x_0,\rho/2)} h
\end{aligned}$$

by Lemma 2.7.1, if ρ satisfies the assumptions of this lemma.

Now on $B(x_0,R_0\cdot 2^{-i})\backslash B(x_0,R_0\cdot 2^{-i-1})$, $r(x)^{2-n} \le (R_0\cdot 2^{-i-1})^{2-n}$ and thus

$$(4.2.5)\quad \int_{B(x_0,R_0)} r(x)^{2-n}\ |du|^2 \le \sum_{i=0}^{\infty} 2^{n-2}(R_0\cdot 2^{-i})^{2-n} \int_{B(x_0,R_0\cdot 2^{-i})} |du|^2 .$$

Since $2^{2-n}((\frac{1}{2})^{2-n} - 1)\ r^{2-n} = r^{2-n} - (2r)^{2-n}$, from (4.2.5), defining $g_i = g_{R_0\cdot 2^{-i+1}}$

$$\int_{B(x_0,R_0)} r(x)^{2-n}\ |du|^2 \le c_n \sum_{i=0}^{\infty} \int_{B(x_0,R_0\cdot 2^{-i+1})} g_i\ |du|^2$$

where c_n depends only on n.

(4.2.4) then implies

$$(4.2.6)\quad \int_{B(x_0,R_0)} r(x)^{2-n}\ |du|^2 \le \frac{2c_2}{\lambda} c_n\ R_0^2 + \frac{c_n}{\lambda} \sum_{i=0}^{\infty}$$

$$\left\{(R_0\cdot 2^{-i+1})^{1-n} \int_{\partial B(x_0,R_0\cdot 2^{-i+1})} h - (R_0\cdot 2^{-i})^{1-n} \int_{\partial B(x_0,R_0\cdot 2^{-i})} h\right\}$$

$$=: c_3\ R_0^2 + c_4 \sum_{i=0}^{\infty} (\mu_{i-1} - \mu_i) .$$

Hence

$$(4.2.7)\quad \int_{B(x_0,R_0)} r(x)^{2-n}\ |du|^2 \le c_3\ R_0^2 + c_4\ \mu_0 .$$

This implies (4.2.1), noting that $\mu_0 \le \sup\limits_{u(\Omega)} f \cdot \mathrm{vol}\ \partial B(x_0,2R_0)R_0^{1-n}$. From (4.2.4)

$$(R_0\cdot 2^{-i})^{2-n} \int_{B(x_0,R_0\cdot 2^{-i})} |du|^2 \le c_5((R_0\cdot 2^{-i})^2 + \mu_{i-1} - \mu_i) .$$

We first choose i_0 so large that $c_5\left(R_0\cdot 2^{-i_0}\right)^2 \le \frac{\varepsilon}{2}$. For every $m \in \mathbb{N}$, we can find j, $i_0 \le j \le m + i_0$, with

$$\mu_j - \mu_{j+1} \le \frac{1}{m} \mu_{i_0} \le \frac{1}{m} (\mu_0+c_6R_0^2)$$

(for the last inequality, note that h is subharmonic and see the proof of (2.7.5)).

Hence choosing $m \geq \dfrac{2c_6(\mu_0 + c_6 R_0^2)}{\varepsilon}$ and $R_1 = R_0 \cdot 2^{-i_0 - m}$, $R = R_0 \cdot 2^{-j}$, (4.2.2) follows.

q.e.d.

4.3 CHOICE OF A TEST FUNCTION

Suppose $B(x_0, 2R) \subset \Omega$ for some $R > 0$. Let $\eta \in \mathrm{Lip}(B(x_0, 2R))$ be the standard localizer, i.e. $\eta \equiv 1$ on $B(x_0, R)$, $|\nabla\eta| \leq \frac{c}{R}$, supp $\eta \subset\subset B(x_0, 2R)$. Suppose there exists a strictly convex function on $u(B(x_0, 2R))$, i.e. the assumptions of Lemma 4.2.1 are satisfied.

Suppose f is a C^2-function on $u(B(x_0, 2R))$, and g is a Lipschitz function on $B(x_0, 2R)$, so we can choose $\nabla f \cdot \eta \cdot g$ as a test vector ϕ in (4.1.3).

If e_α is an orthonormal frame on Ω , ω^α the dual coframe, then $du = u_{e_\alpha} \omega^\alpha$, and (4.1.3) yields

$$(4.3.1) \qquad 0 = \int_{B(x_0, 2R)} \eta\, g \langle d(\nabla f), u_{e_\alpha} \omega^\alpha \rangle + \int_{B(x_0, 2R)} g\, \eta_{e_\alpha} f(u)_{e_\alpha} + \int_{B(x_0, 2R)} \eta\, g_{e_\alpha} f(u)_{e_\alpha} .$$

Now

$$\begin{aligned} \langle d(\nabla f), u_{e_\alpha} \omega^\alpha \rangle_{T^*\Omega \otimes u^{-1}TY} &= \langle d(\nabla f) e_\alpha, u_{e_\alpha} \rangle_{u^{-1}TY} \\ &= \langle \nabla^{u^{-1}TY}_{e_\alpha} \nabla f, u_{e_\alpha} \rangle_{u^{-1}TY} \quad \text{by definition of } d \\ &= \langle \nabla^{Y}_{u_*(e_\alpha)} \nabla f, u_*(e_\alpha) \rangle_{TY} \\ &= D^2 f(du, du) \end{aligned}$$

where $D^2 f$ is the Hessian of f .

Hence from (4.3.1)

$$(4.3.2)\quad \int_{B(x_0,2R)} g_{e_\alpha}(\eta f(u))_{e_\alpha} = -\int_{B(x_0,2R)} \eta g\, D^2 f(du,du) - \int_{B(x_0,2R)} g\eta_{e_\alpha} f(u)_{e_\alpha} + \int_{B(x_0,2R)} f(u)\, g_{e_\alpha} \eta_{e_\alpha} .$$

Remark If one is not familiar with the notation employed in the derivation of (4.3.2), one can alternatively insert the test vector ψ given by $\psi^i = \eta\cdot g \frac{\partial f}{\partial u^i}$ in (4.1.2) and carry out the calculations in local coordinates.

For $y \in B(x_0,R/2)$, $x \in B(x_0,2R)$, we now put

$$g(x) = g^\nu(x,y) = \min(d(x,y)^{2-n}, \nu) \qquad \text{for } \nu \in \mathbb{N} .$$

Writing $D(x_0,\nu,R) = \{x \in B(x_0,R) : d(x,y)^{2-n} < \nu\}$, (4.3.2) yields

$$(4.3.3)\quad \int_{D(x_0,\nu,R)} g^\nu(\cdot,y)_{e_\alpha}(\eta\, f(u))_{e_\alpha} = -\int_{B(x_0,2R)} \eta g^\nu(\cdot,y)\, D^2 f(du,du) - \int_{B(x_0,2R)} g^\nu(\cdot,y)\eta_{e_\alpha}\, f(u)_{e_\alpha} + \int_{D(x_0,\nu,R)} f(u)\, g^\nu(\cdot,y)_{e_\alpha}\, \eta_{e_\alpha} .$$

We write (4.3.3) as

$$I_\nu = II_\nu + III_\nu + IV_\nu .$$

Then with $D'(x_0,\nu,R) = \{x \in B(x_0,2R) : d(x,y)^{2-n} \geq \nu\}$

$$(4.3.4)\quad I_\nu = \int_{D(x_0,\nu,R)} \Delta(d(\cdot,y)^{2-n})\eta f(u) - \int_{\partial D'(x_0,\nu,R)} \eta f(u) \langle \operatorname{grad} g(\cdot,y), \vec{dO}\rangle ,$$

since η has compact support in $B(x_0,2R)$.

By (2.1.4), for sufficiently small R (depending on the injectivity radius and an upper curvature bound on Ω)

$$(4.3.5)\quad \int_{D(x_0,\nu,R)} \Delta(d(\cdot,y)^{2-n})\, \eta f(u) \leq c_7 R^2 < \varepsilon , \qquad \text{if } R \leq R_1(\varepsilon) ,$$

where c_7 depends on $n = \dim \Omega$, a curvature bound on Ω , and on $\sup f$.

If we choose for $f\circ u$ its Lebesgue representative, then we can find a subsequence of the ν's for which

$$(4.3.6)\quad \lim_{\nu\to\infty} \int_{\partial D'(x_0,\nu,R)} \eta f(u) \langle \operatorname{grad} g(\cdot,y), \vec{dO}\rangle = -(n-2)\omega_n f(u(y))$$

(note that $\eta(y) = 1$, since $y \in B(x_0,R/2))$.

Furthermore

$$(4.3.7)\qquad II_\nu = -\int_{B(x_0,R)} - \int_{T(x_0,R)} ,$$

where $$T(x_0,R) := B(x_0,2R) \setminus B(x_0,R) .$$

Since $y \in B(x_0,R/2)$, we infer from Lemma 4.2.1

$$(4.3.8)\qquad \int_{T(x_0,R)} \eta g^\nu(\cdot,y)\, D^2 f(du,du) < \varepsilon(n-2)\omega_n$$

for prescribed $\varepsilon > 0$ and some R , $R_2(\varepsilon) \le R \le R_1(\varepsilon)$, where $R_2 = R_2(\varepsilon) > 0$ can be calculated explicitly in terms of ε . It depends on the Hessian of f , but is independent of ν and y and u .

Since $\eta_{e_\alpha} \equiv 0$ outside $T(x_0,R)$

$$(4.3.9)\quad III_\nu \le \frac{c_8}{R}\int_{T(x_0,R)} g^\nu(\cdot,y)\,|du| \le \frac{c_9}{R}\left(\int_{T(x_0,R)} g^\nu(\cdot,y)\right)^{\frac{1}{2}} \left(\int_{T(x_0,R)} g^\nu(\cdot,y)\,|du|^2\right)^{\frac{1}{2}} \le (n-2)\omega_n\varepsilon ,$$

again for some suitable R which we can choose to be the same one as in (4.3.8). Here, the quantities depend on $|\nabla f|$.

In order to estimate IV_ν, let u_R be the mean value of u on $T(x_0,R)$. u_R can be defined with the help of our coordinates. We write

$$u_R = \fint_{T(x_0,R)} u .$$

We now write $f(u) = f(u_R) + (f(u) - f(u_R))$. Similar as in (4.3.5) and

(4.3.6), we obtain

$$(4.3.10)\qquad \lim_{\nu\to\infty} IV_\nu \le (n-2)\omega_n\, f(u_R) + c_{10}\, R^2 + \int_{T(x_0,R)} (f(u) - f(u_R))(d(\cdot,y)^{2-n})_{e_\alpha}\, \eta_{e_\alpha}\, .$$

Furthermore

$$\int_{T(x_0,R)} (f(u) - f(u_R))\; d(\cdot,y)^{2-n}_{e_\alpha}\, \eta_{e_\alpha} \le \frac{c_{11}}{R^n}\int_{T(x_0,R)} |f(u) - f(u_R)|$$

$$\le \frac{c_{12}}{R^n} R^{n/2} \sup|\nabla f|\; (\int_{T(x_0,R)} d(u,u_R)^2)^{\frac{1}{2}}$$

$$\le c_{13}\, R^{-n/2}\, (c_{14}\, R^2 \int_{T(x_0,R)} |du|^2)^{\frac{1}{2}}$$

by the Poincaré inequality, where c_{13} and c_{14} are independent of R.

Combined with (4.3.10), the preceding inequality yields

$$(4.3.11)\quad \lim_{\nu\to\infty} IV_\nu \le (n-2)\omega_n\, f(u_R) + c_{10}\, R^2 + c_{15}(R^{2-n}\int_{T(x_0,R)} |du|^2\,)^{\frac{1}{2}}$$

$$\le (n-2)\omega_n\, f(u_R) + \varepsilon(n-2)\omega_n$$

(w.l.o.g. we can assume that (4.3.11) again is satisfied for the same R as in (4.3.8) and (4.3.9)).

From (4.3.3)-(4.3.11), we obtain for $y \in B(x_0,R/2)$, using Lebesgue's Theorem on dominated convergence

$$(4.3.12)\quad f(u(y)) \le f(u_R) + 4\varepsilon - \{(n-2)\omega_n\}^{-1}\int_{B(x_0,R)} d(\cdot,y)^{2-n}\, D^2 f(du,du)$$

for some $R \ge R_3(\varepsilon)$ where $R_3(\varepsilon) > 0$ is independent of u and x_0.

4.4 AN ITERATION ARGUMENT. CONTINUITY OF WEAK SOLUTIONS

In this section, we want to use an iteration argument based on (4.3.12) to prove continuity of a weakly harmonic map with image in a convex ball. This result appeared explicitly for the first time in [HJW], but the method of proof in a somewhat different setting was already developed in [HW2]. The present proof (4.2-4.4) uses ideas of Wiegner, Hildebrandt, Widman, Kaul, Jost, Giaquinta, and Karcher, cf. [Wi], [HW2], [HKW3], [HJW], [GH], and [JK].

THEOREM 4.4.1 *Suppose* $u : \Omega \to B(p,M)$ *is weakly harmonic, that* $-\omega^2 \le K \le \kappa^2$ *are curvature bounds on* $B(p,M) \subset Y$, $M < \min\left(\frac{\pi}{2\kappa}, i(p)\right)$, *where* $i(p)$ *is the injectivity radius of* p, *and* $x_0 \in \Omega$.

Then for each $\tau > 0$ *one can calculate* $\rho > 0$ *with*

$$\operatorname*{osc}_{B(x_0,\rho)} u < \tau .$$

ρ *depends only on* $\tau, d(x_0, \partial\Omega)$, *curvature bounds and the injectivity radius of* Ω, $\dim \Omega$, $\dim Y$, M, ω, κ.

In particular, u *is continuous in* Ω.

Proof Let

$$h_0 = \min\left(\frac{\pi}{2M\kappa} - 1,\ 1\right) .$$

Then there exists ε', $0 < \varepsilon' < 1$, with

$$h' = \frac{\pi}{2\kappa} - \{(1-h_0)^2 M^2 + \varepsilon'\}^{\frac{1}{2}} > 0 .$$

Let

$$h = \min\left(\frac{2h'\kappa}{\pi},\ h_0\right)$$

and

$$0 < \varepsilon'' < \tau^2 .$$

Let ε in (4.3.12) be taken as

$$\varepsilon = \frac{1}{8} h(2-h) \min\left(\varepsilon',\ \frac{\varepsilon''}{8}\right)$$

and s be the smallest positive integer with

$$(1-h)^{2s} < \frac{\varepsilon''}{8M^2} .$$

The assumptions of Lemma 4.2.1 are satisfied, because $r^2(q) = d^2(q,p)$ is strictly convex on $B(p,M)$ by Lemma 2.3.2.

We start with $R_0 = \frac{1}{2}d(x_0, \partial\Omega)$, $p_0 = p$. On $B(p,M)$, we initially take normal coordinates centred at $p = p_0$. They cover $B(p,M)$, since $B(p,M)$ by assumption is disjoint to the cut locus of p.

Let $\bar{u}_{R_0}$ be the mean value of u on $T(x_0,R_0) = B(x_0,2R_0) - B(x_0,R_0)$ taken with respect to these coordinates:

$$\bar{u}_{R_0} = \fint_{T(x_0,R_0)} u(x)\, dx .$$

Let c_0 be the unique geodesic arc from p_0 to $\bar{u}_{R_0}$, and let p_1 be the point on c_0 with

$$d(p_1,p_0) = h_0\, d(\bar{u}_{R_0},p_0) .$$

Now for $q \in B(p,M)$

$$\begin{aligned} d(q,p_1) &\le d(q,p_0) + d(p_1,p_0) \\ &\le M + h_0 M \\ &\le \frac{\pi}{2\kappa} \qquad \text{by choice of } h_0 . \end{aligned}$$

Hence, by Lemma 2.3.2, $d^2(\cdot,p_1)$ is convex on $B(p,M)$. Thus, for $y \in B(x_0,R_1)$, where $2R_1$ is the radius $R \le R_0$ of (4.3.12), (4.3.12) implies for $f = d^2(\cdot,p_1)$

$$\begin{aligned} \text{(4.4.1)} \quad d^2(u(y),p_1) &\le d^2(\bar{u}_{R_0},p_1) + 4\varepsilon \\ &\le (1-h_0)^2 \sup_{x\in B(x_0,2R_0)} d^2(u(x),p_0) + 4\varepsilon \end{aligned}$$

by choice of p_1.

Let $j \in \mathbb{N}$.

Suppose now that we have found points $p_i \in B(p,M)$ and radii R_i for $i \leq j-1$ with the property that for $y \in B(x_0,R_i)$

$$d^2(u(y),p_i) \leq (1-h_0)^2 M^2 + \varepsilon' \tag{4.4.2}$$

and

$$d^2(u(y),p_i) \leq (1-h)^2 \sup_{B(x_0,2R_{i-1})} d^2(u(x),p_{i-1}) + 4\varepsilon . \tag{4.4.3}$$

We then want to prove (4.4.2) and (4.4.3) for $i = j$ and suitably chosen p_j and R_j .

First of all, by (4.4.2)

$$d(u(y),\ p_{j-1}) \leq \frac{\pi}{2K} - h' \qquad \text{for} \quad y \in B(x_0,R_{j-1}) .$$

If we choose normal coordinates on $B(p,M)$ centred at p_{j-1} which is possible by Prop. 2.4.1, and take $\bar{u}_{R_{j-1}}$ as being the mean value of u over $T(x_0,R_{j-1})$ with respect to these coordinates, then again by Prop. 2.4.1, there is a unique geodesic arc c_{j-1} in $B(p,M)$ from p_{j-1} to $\bar{u}_{R_{j-1}}$. We choose p_j as that point on c_{j-1} with

$$d(p_j,p_{j-1}) = h\ d(\bar{u}_{R_{j-1}},p_{j-1}) .$$

Then for $y \in B(x_0,R_{j-1})$

$$\begin{aligned} d(u(y),p_j) &\leq d(u(y),\ p_{j-1}) + d(p_j,p_{j-1}) \\ &\leq ((1-h_0)^2 M^2 + \varepsilon')^{\frac{1}{2}} + hM \qquad \text{by (4.4.2)} \\ &\leq \frac{\pi}{2K} - h' + hM \\ &\leq \frac{\pi}{2K} . \end{aligned}$$

Hence, $d^2(\cdot,p_j)$ is convex on $u(B(x_0,R_{j-1}))$, and from (4.3.12) for $y \in B(x_0,R_j)$, taking $2R_j = R \leq R_{j-1}$ in (4.3.12)

$$\begin{aligned} d^2(u(y),p_j) &\le d^2(p_j,\bar{u}_{R_{j-1}}) + 4\varepsilon \\ &\le (1-h)^2\, d^2(\bar{u}_{R_{j-1}},p_{j-1}) + 4\varepsilon \\ &\le (1-h)^2 \sup_{x\in B(x_0,2R_{j-1})} d^2(u(x),p_{j-1}) + 4\varepsilon\ . \end{aligned}$$

Thus (4.4.3) is also satisfied for $i = j$.

Iterating (4.4.3), we obtain

$$(4.4.4)\qquad \sup_{y\in B(x_0,R_j)} d^2(u(y),p_j) \le (1-h)^{2j} \sup d^2(u(y),p_0) + 4\varepsilon \frac{1}{1-(1-h)^2}\ .$$

For $j > 0$, $\qquad \dfrac{1}{1-(1-h)^2} + (1-h)^{2j} \le \dfrac{2}{h(2-h)}$,

and thus from (4.4.4) and (4.4.1), since $d^2(u(x),p_0) \le M^2$,

$$(4.4.5)\qquad \sup_{y\in B(x_0,R_j)} d^2(u(y),p_j) \le (1-h)^{2j}\,(1-h_0)^2\, M^2 + \min\left(\varepsilon', \frac{\varepsilon''}{8}\right)\ .$$

In particular, (4.4.2) holds for $i = j$. Moreover, (4.4.5) implies

$$(\operatorname*{osc}_{B(x_0,R_j)} u)^2 \le 4 \sup_{y\in B(x_0,R_j)} d^2(u(y),p_j) \le 4(1-h)^{2j} M^2 + \frac{\varepsilon''}{2}$$

and hence

$$\operatorname*{osc}_{B(x_0,R_s)} u < \sqrt{\varepsilon''} < \tau\ .$$

R_s can be computed explicitly, since the radius $R_3(\varepsilon)$ in (4.3.12) can be computed from the geometric quantities of the statement of the theorem by Lemma 4.2.1. Note in particular, that the strictly convex function required in Lemma 4.2.1 is $d^2(\cdot,p)$ and that all choices of f in (4.3.12) are likewise given by squared distance functions. Hence their gradients and Hessians are controlled by the geometry of the image through Lemma 2.3.2.

q.e.d.

4.5 HÖLDER CONTINUITY OF WEAK SOLUTIONS

We now want to prove Hölder continuity of u.

THEOREM 4.5.1 *Suppose that the assumptions of Thm. 4.4.1 hold. Let* $B(x_1,2d) \subset \Omega$ *be a ball which is disjoint from the cut locus of its centre. Furthermore, suppose that* $-\sigma^2 \le K \le \tau^2$ *for the curvature on* $B(x_1,2d)$ *and* $d < \frac{\pi}{4\tau}$. *Then for all* $x,y \in B(x_1,d)$

$$d(u(x),u(y)) \le c\, d(x,y)^\beta$$

where $\beta \in (0,1)$ *and* c *depend only on* $\dim \Omega$, $\dim Y$, σ, τ, ω, κ, d, *and* M.

Proof By Thm. 4.4.1, we can find ρ, $0 < \rho < d$, with

$$(4.5.1) \qquad \operatorname*{osc}_{B(x,\rho)} u < M \quad \text{for all } x \in B(x_1,d) .$$

We choose an arbitrary $x_0 \in B(x_1,d)$ and R with $0 < R \le \frac{\rho}{2}$, and define again $T(x_0,2r) = B(x_0,2r) \setminus B(x_0,r)$ and moreover

$$T^*(x_0,2r) = B\left(x_0, \frac{7R}{4}\right) \setminus B\left(x_0, \frac{5R}{4}\right) .$$

We let q be the point, where

$$H(q) = \int_{T(x_0,2R)} d^2(u(x),q)\, dx$$

achieves its minimum. (That we can find a unique such q, follows from (4.5.1) and (2.3.4)). Then

$$\int_{T(x_0,2R)} \nabla_q\, d^2(u(x),p) = 0$$

$$\Leftrightarrow \int_{T(x_0,2R)} \exp_q^{-1} u(x) = 0 .$$

That means that if we choose normal coordinates centred at q and denote the corresponding coordinate representation by v, then

$$(4.5.2) \qquad \int_{T(x_0,2R)} v(x) = 0$$

and hence by the Poincaré inequality

$$(4.5.3) \qquad \int_{T(x_0,2R)} v^2 \le c_{15} R^2 \int_{T(x_0,2R)} |\nabla v|^2$$

where c_{15} like the following constants $c_{16},\ldots$ is independent of R.

η from 4.3 will now be required to satisfy $\eta \equiv 1$ on $B\left(x_0, \frac{5R}{4}\right)$ and $\eta \equiv 0$ on $\Omega\backslash B\left(x_0, \frac{7R}{4}\right)$.

In (4.3.3) we now take $f(u) = d^2(u,q)$ and $y = x_0$. Then from (4.3.4)-(4.3.6)

$$(4.5.4) \qquad \lim_{\nu\to\infty} I_\nu \ge (n-2)\omega_n\, d^2(u(x_0),p) - c_{16} R^2 .$$

Furthermore

$$D^2 f(du,du) \ge 2\kappa M \operatorname{ctg} \kappa M\, |du|^2$$

by Lemma 2.3.2 and (4.5.1) and hence

$$(4.5.5) \quad C_{17} \int_{B(x_0,2R)} \eta g^\nu(\cdot,x_0)\, D^2 f(du,du) \ge \int_{B(x_0,2R)} \eta g^\nu(\cdot,x_0)\, |du|^2 .$$

By choice of η, the integral III_ν extends only over $T(x_0,2R)$, and taking $\nu > R^{2-n}$, noting $f(u)_{e_\alpha} = 2vv_{e_\alpha}$ $(f(u(x)) = v^2(x))$,

$$(4.5.6) \qquad |III_\nu| \le c_{18}\Big(R^{2-n} \int_{T(x_0,2R)} |\nabla v|^2 + R^{-n} \int_{T(x_0,2R)} |v|^2\Big)$$

$$\le c_{19} R^{2-n} \int_{T(x_0,2R)} |\nabla v|^2 \qquad \text{by (4.5.3)} .$$

Now

$$(4.5.7) \quad |IV_\nu| \le c_{20}\Big(R^{-2}\delta^{-1} \int_{T(x_0,2R)} |v|^2 + \delta \int_{T^*(x_0,2R)} |v|^2\, |\nabla d(\cdot,x_0)^{2-n}|^2$$

for each $\delta > 0$

$$\int_{T(x_0,2R)} \phi^2 v^2\, d(\cdot,x_0)^{2-n}\, \Delta d(\cdot,x_0)^{2-n} = \int \phi^2 v^2\, |\nabla d(\cdot,x_0)^{2-n}|^2$$

$$+ \int 2\phi\nabla\phi v^2\, d(\cdot,x_0)^{2-n}\, \nabla d(\cdot,x_0)^{2-n} + \int \phi^2 2v\nabla v\, d(\cdot,x_0)^{2-n}\, \nabla d(\cdot,x_0)^{2-n}$$

if $\operatorname{supp} \phi \subset T(x_0, 2R)$, $\phi \equiv 1$ on $T^*(x_0, 2R)$, $|d\phi| \le \frac{c}{R}$.

Using Lemma 2.7.1 and Hölder's inequality, this implies

$$(4.5.8) \quad \int_{T^*(x_0,2R)} v^2 |\nabla d(\cdot, x_0)^{2-n}|^2 \le c_{21} R^{2(2-n)} \int_{T(x_0,2R)} v^2$$

$$+ c_{22} \left(R^{-2} \int_{T(x_0,2R)} |v|^2 \, |d(\cdot,x_0)^{2-n}|^2 + \int_{T(x_0,2R)} d(\cdot,x_0)^{2(2-n)} \, |\nabla v|^2 \right) .$$

Choosing $\delta = R^{n-2}$ in (4.5.7) and using (4.5.8) and (4.5.3),

$$(4.5.9) \qquad |IV_\nu| \le c_{23} R^{2-n} \int |dv|^2 .$$

From (4.5.3), (4.5.4), (4.5.5), (4.5.6), and (4.5.9) and letting $\nu \to \infty$

$$(4.5.10) \quad \int_{B(x_0,R)} d(\cdot,x_0)^{2-n} \, |du|^2 \le c_{24} R^{2-n} \int_{T(x_0,2R)} |dv|^2 + c_{26} R^2$$

$$\le c_{25} \int_{T(x_0,2R)} d(\cdot,x_0)^{2-n} \, |du|^2 + c_{26} R^2 .$$

(Note that $\int |dv|^2 = \int |du|^2$, since the energy is invariant under coordinate transformations.)

If we now add $c_{25} \int_{B(x_0,R)} d(\cdot,x_0)^{2-n} \, |du|^2$ to both sides of (4.5.10), i.e. we fill the hole (that explains why this device introduced by Widman is called the hole filling technique), we obtain with $\theta = \dfrac{c_{25}}{1 + c_{25}} < 1$

$$(4.5.11) \quad \int_{B(x_0,R)} d(\cdot,x_0)^{2-n} \, |du|^2 \le \theta \int_{B(x_0,2R)} d(\cdot,x_0)^{2-n} \, |du|^2 + c_{27} R^2$$

or, using the notation $\Phi(R) := \int_{B(x_0,R)} d(\cdot,x_0)^{2-n} \, |du|^2 + c_{27} R^2$

$$(4.5.12) \qquad \Phi(R) \le \theta_0 \, \Phi(2R) \qquad \text{with} \quad \theta_0 = \max(\theta, \tfrac{1}{2}) .$$

LEMMA 4.5.1 (de Giorgi) *For* $\alpha = \log_2(\theta_0^{-1})$ *and all* $r < R$

$$(4.5.13) \qquad \Phi(r) \le 2^\alpha \left(\frac{r}{R}\right)^\alpha \Phi(R) .$$

Proof of the lemma If $2^{-k-1} R \le r \le 2^{-k} R$,

$$\Phi(r) \le \Phi(2^{-k} R) \le \theta_0^k \phi(R) \qquad \text{by (4.5.12)} .$$

Writing $\theta_0^k = (2^{-k}) \log_2(\theta_0^{-1})$ and using $2^{-k} \le 2 \frac{r}{R}$,

$$\Phi(r) \le 2^{\alpha}\left(\frac{r}{R}\right)^{\alpha} \Phi(R)$$

which proves the lemma.

Since $0 < \theta_0 < 1$, $\alpha > 0$, and hence Thm. 4.5.1 will follow from (4.5.13) in conjunction with the following well-known Dirichlet growth theorem of Morrey, noting that the right hand side of (4.5.13) is finite by (4.2.1) or by (4.5.11)

THEOREM 4.5.2 (Morrey) *If* $f \in H_2^1(B(x_1,d))$ *satisfies*

$$\int_{B(x_1,d)\cap B(x_0,\rho)} |\nabla f|^2 \le M^2 \rho^{n-2+2\beta}$$

for all $x_0 \in B(x_1,d)$ *and all* $\rho > 0$ *for some positive constants* β *and* M , *then* $f \in C^{0,\beta}(B(x_1,d))$, *and*

$$|f(x) - f(y)| \le c_n M |x-y|^{\beta}$$

for all $x,y \in B(x_1,d)$, *where* c_n *depends only on* n .

For a proof, cf. e.g. [M3] .

The preceding proof of Thm. 4.5.1 was taken from [HJW]. It uses the method of [HW1]. Different proofs of Thm. 4.5.1 were obtained by Eliasson [Es], Sperner [Sp], and Tolksdorf [To].

4.6 APPLICATIONS TO THE BERNSTEIN PROBLEM

Actually, the dependence on the geometry of the domain in Thm. 4.5.1 can be considerably weakened. In [HJW], the following result is shown.

THEOREM 4.6.1 *Let again* $B(p,M) \subset Y$ *be a geodesic ball, disjoint to the cut locus of* p *, with* $M < \frac{\pi}{2\kappa}$ *, where* $-\omega^2 \le K \le \kappa^2$ *are curvature bounds on* $B(p,M)$.

Let $D(0,2d) = \{x \in \mathbb{R}^n : |x| < 2d\}$ *be a coordinate chart on the domain with metric tensor* $\gamma_{\alpha\beta}(x)$ *satisfying*

$$\lambda|\xi|^2 \le \gamma_{\alpha\beta}(x)\, \xi^\alpha \xi^\beta \le \mu|\xi|^2 , \qquad 0 < \lambda \le \mu \tag{4.6.1}$$

for all $x \in D(0,2d)$ *and all* $\xi \in \mathbb{R}^n$.

If $u : D(0,2d) \to B(p,M)$ *is harmonic, then for all* $x,y \in D(0,d)$

$$d(u(x),\, u(y)) \le \frac{c}{d^\beta}\, d(x,y)^\beta$$

for some $\beta \in (0,1)$ *and* $c > 0$ *, depending only on* n , $\dim Y$, ω , κ , M , λ *, and* μ *, but not on* d .

In the proof of Thm. 4.6.1, one has to use the Green function of the Laplace-Beltrami operator of the domain instead of the approximate fundamental solutions we use in the proof of Thms. 4.4.1 and 4.5.1. The truncated functions $g^\nu(x,y)$ of section 4.3 have to be replaced by mollifications of the Green function. The proof then yields the desired result because one can control the Green function only in terms of the ellipticity constants of the differential operator, i.e. by (4.6.1). The required estimates for the Green function depend on Moser's Harnack inequality and are carried out in [GW]. Also, Lemma 4.2.1 has to be proved in a different way to get the stronger estimate, again using Moser's Harnack inequality, cf. e.g. [GH].

Thm. 4.6.1 has the following

COROLLARY 4.6.1 *Let the manifold* X *be diffeomorphic to* $\mathbb{R}^n$ *, with a metric tensor* $\gamma_{\alpha\beta}(x)$ $(x \in \mathbb{R}^n)$ *satisfying*

$$\lambda|\xi|^2 \le \gamma_{\alpha\beta}(x)\ \xi^\alpha\xi^\beta \le \mu|\xi|^2 , \qquad 0 < \lambda \le \mu$$

for all $\xi \in \mathbb{R}^n$ *and* $x \in \mathbb{R}^n$,

Suppose $u : X \to Y$ *is harmonic and* $u(X) \subset B(p,M)$ *where* $B(p,M)$ *again satisfies the assumptions of Thm. 4.6.1.*

Then u *is constant.*

Cor. 4.6.1 in turn can be used to prove Bernstein type theorems for minimal submanifolds of Euclidean space, when combined with the following result of Ruh and Vilms [RV].

THEOREM 4.6.2 *Suppose* $F : M \to \mathbb{R}^{n+p}$ *is of class* C^3 *and immerses the* n*-dimensional manifold* M *into Euclidean* $(n+p)$*-space. Then its Gauss map* $G : F(M) \to G(n,p)$ *into the Grassmannian manifold of* n*-planes in* $(n+p)$*-space endowed with its standard Riemannian metric is harmonic if and only if* M *is immersed with parallel mean curvature field. This in particular is the case, if* $F(M)$ *is a minimal submanifold of* $\mathbb{R}^{n+p}$.

Cor. 4.6.1 and Thm. 4.6.2 yield the following Bernstein type theorem of [HJW].

THEOREM 4.6.3 *Suppose* $F : \mathbb{R}^n \to \mathbb{R}^{n+p}$ *is a* C^3*-immersion and* $X = F(\mathbb{R}^n)$ *is minimal or has parallel mean curvature field. Suppose there exists a fixed oriented* n*-plane* P_0 *, and a number* α_0

$$(4.6.2) \qquad \alpha_0 > \cos^m\left(\frac{\pi}{2\kappa\sqrt{m}}\right) , \qquad m = \min(n,p) , \qquad \kappa^2 = \begin{cases} 1 & \text{if } m = 1 \\ 2 & \text{if } m \ge 2 \end{cases}$$

and

$$(4.6.3) \qquad \langle P, P_0 \rangle \ge \alpha_0$$

holds for all oriented tangent planes P *of* X .

Suppose also that the metric

$$\gamma_{\alpha\beta}(x) = F_{x^\alpha}(x)\, F_{x^\beta}(x) \qquad (x \in \mathbb{R}^n)$$

of X *is uniformly equivalent to the Euclidean metric in the sense of (4.6.1).*

Then, X *is an affine linear subspace of* $\mathbb{R}^{n+p}$.

The conditions (4.6.2) and (4.6.3) guarantee that the image of the Gauss map of X is contained in a ball in $G(n,p)$ which satisfies the assumptions of Thm. 4.6.1, cf. [HJW].

If $p = 1$, then $m = \kappa = 1$ in (4.6.2), and hence Thm. 4.6.3 implies Moser's weak Bernstein theorem:

An entire solution of the minimal surface equation

$$\operatorname{div}\left\{\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right\} = 0$$

with $\sup|\nabla f| < \infty$ is linear.

Note that in the strong Bernstein theorem the assumption $\sup|\nabla f| < \infty$ is not necessary. On the other hand, this stronger version is only true for $n \le 7$, whereas Thm. 4.6.3 requires no restriction on the dimension.

The results of Thm. 4.6.3 seem to be also interesting, although probably not optimal, in codimension $p \ge 2$.

4.7 ESTIMATES AT THE BOUNDARY

In this section, we want to prove a-priori estimates at the boundary for weak solutions whose image is contained in a convex ball.

The following result can be found, e.g., in [GH].

THEOREM 4.7.1 *Suppose* $u : \Omega \to B(p,M)$ *is harmonic, where* $B(p,M)$ *again is a ball with* $M < \frac{\pi}{2\kappa}$ *and disjoint to the cut locus of* p . *Suppose* $\partial\Omega$ *is of class* C^2 , *and* $|K| \le \Lambda^2$ *for the sectional curvature of* Ω . *If* $g = u|\partial\Omega$ *is continuous, then for every* $\varepsilon > 0$ *we can find some* $\delta > 0$, *depending on* ω , κ , M , Λ , $i(\Omega)$, $\dim \Omega$, $\partial\Omega$, *the modulus of continuity of* g , *and on* ε , *for which*

$$d(u(y),\ u(x_0)) \le \varepsilon \qquad \text{for } y \in \Omega \cap B(x_0,\delta) \ . \tag{4.7.1}$$

If g *is Hölder continuous with some exponent* β , *then*

$$d(u(y),\ u(x_0)) \le c_\alpha |y - x_0|^\alpha \qquad \text{for } y \in \Omega \cap B(x_0,\delta) \tag{4.7.2}$$

where α *and* c_α *depend on* ω , κ , M , β , Λ , $i(\Omega)$, $\partial\Omega$, $\dim \Omega$, *and* $|g|_{C^\beta}$.

Proof W.l.o.g. $n \ge 3$. We need some definitions:

$$D(x_0,R) := \Omega \cap B(x_0,R) \ .$$

If $x_0 \in \partial\Omega$, let $c : [0,1] \to B(p,M)$ be the geodesic with $c(0) = p$, $c(1) = g(x_0)$, parametrized proportionally to arc length, and

$$p_t := c(t) \ ,$$

$$v_t := d^2(u(x),p_t) \ .$$

Furthermore, let $w_{t,R}$ be the solution of

$$\Delta w_{t,R} = 0 \qquad \text{on } D(x_0,R)$$

$$w_{t,R}|\partial D(x_0,R) = v_t|\partial D(x_0,R) \ .$$

As in the proof of Lemma 2.1.3 we derive for $y \in D(x_0,\frac{R}{2})$,

$\rho \le \min\left(\frac{R}{2}, \frac{\pi}{2\Lambda}, i(\Omega)\right)$

$$(4.7.3) \qquad \frac{1}{(n-2)\omega_n}\int_{D(y,\rho)} \Delta v_t\left(\frac{1}{d(y,z)^{n-2}} - \frac{1}{\rho^{n-2}}\right)dz \le w_{t,R}(y) - v_t(y) + \frac{\Lambda^2}{2\omega_n}\int_{D(y,\rho)} \frac{|w_{t,R} - v|}{d(y,z)^{n-2}}\,dz$$

using the fact that the boundary term on $\partial\Omega$ vanishes by definition of $w_{t,R}$. From the definition of v_t and $w_{t,R}$, we have

$$(4.7.4) \qquad v_t(y) = d^2(u(y),p_t) \le (1+t)^2M^2$$

and

$$(4.7.5) \qquad w_{t,R}(x_0) = v_t(x_0) = d^2(g(x_0),p_t) \le (1-t)^2M^2 .$$

We now want to exploit that the boundary values of $w_{t,R}$ on $\partial\Omega \cap D(x_0,R)$ are given by $d^2(g(x),p_t)$, i.e. controlled by assumption. Namely, given $\varepsilon' > 0$ and $R > 0$, $R \le R_0$, there exists some number $r = r(\varepsilon',R)$ (depending on ε', R, M, $\partial\Omega$, and the modulus of continuity of $d^2(g(x),p_t)$ and $\partial\Omega \cap D(x_0,R_0)$ with the property that

$$(4.7.6) \qquad w_{t,R}(y) \le w_{t,R}(x_0) + \frac{\varepsilon'}{2}$$

for all $y \in D(x_0,r)$. This is a result from potential theory (and can be found, e.g., in [GT], Thm. 8.27).

If $d^2(g(x),p_t)$ is Hölder continuous, we even have

$$(4.7.7) \qquad w_{t,R}(y) \le w_{t,R}(x_0) + \bar{c}|y - x_0|^{2\alpha} \qquad \text{for } y \in D(x_0,r) ,$$

where α, $\bar{c}$ depend ω, κ, M, β, $\partial\Omega$, and $|g|_{C^\beta}$.

We now want to apply an iteration procedure, and put

$$\bar{t} := \frac{\pi}{2M\kappa} - 1 ,$$

$$\varepsilon' := \min(M^2(1-(1-\bar{t})^2),\varepsilon)$$

$$t_i := i\bar{t} \qquad \text{for} \quad 1 \le i \le \mu-1$$

and $t_\mu := 1$, where μ is the smallest integer with $\mu\bar{t} \ge 1$. Furthermore, we start with some radius $R_0 < 1$ and define

$$R_i = \min\left(\frac{R_{i-1}}{2},\ r(\varepsilon', R_{i-1})\right) \qquad (i = 1,\dots,\mu) ,$$

where r is the same r as in (4.7.6).

Then, with

$$m_i := \max_{x\in D(x_0,R_{i-1})} (v_{t_i}(x))^{\frac{1}{2}} ,$$

by Lemma 2.1.1, (1.7.2), (4.7.3), (4.7.6) for $y \in D(x_0,R_i)$, $\rho_{i-1} = \frac{1}{2}R_{i-1}$

$$(4.7.8) \quad 2\kappa m_i \operatorname{ctg}(\kappa m_i) \int_{D(y,\rho_{i-1})} |du|^2 \left(\frac{1}{r(\cdot)^{n-2}} - \frac{1}{\rho_{i-1}^{n-2}}\right) + v_{t_i}(y)$$

$$\le w_{t_i,R_i}(y) + c_{26}\, R_0^2 \le (1-t_i)^2 M^2 + \varepsilon' \le M^2$$

choosing R_0 so small that $c_{26}\, R_0^2 \le \frac{\varepsilon'}{2}$. Furthermore,

$$m_1 \le \frac{\pi}{2\kappa} \qquad \text{by (4.7.4)} ,$$

and if $m_i \le \pi/2\kappa$, then by (4.7.8)

$$(v_{t_{i+1}})^{\frac{1}{2}} \le (v_{t_i})^{\frac{1}{2}} + \bar{t}M \le \frac{\pi}{2\kappa} ,$$

i.e. $m_{i+1} \le \pi/2\kappa$.

Therefore, by induction,

$$m_\mu \le \frac{\pi}{2\kappa} ,$$

and again from (4.7.8) and (4.7.6)

$$v_1(y) \le w_{1,R_\mu}(y) + c_{26}\, R_0^2 \le w_{1,R_\mu}(x_0) + \varepsilon = \varepsilon$$

for all $y \in D(x_0,R_\mu)$.

This gives the desired estimate of the modulus of continuity at the boundary, putting $\delta = R_\mu$.

In case the boundary data are Hölder continuous, we use (4.7.7) to get

$$d(u(y),u(x_0)) \leq (v_1(y))^{\frac{1}{2}} \leq c|y - x_0|^\alpha .$$

q.e.d.

4.8 C^1-ESTIMATES

Having established Hölder continuity of weakly harmonic maps in Thms. 4.5.1 and 4.7.1, it is well known that these maps are actually of class C^1 (and hence of class $C^{2,\alpha}$). Proofs of this assertion can be found in [LU] and [G], and more specifically for harmonic maps in [GH] and [Sp]. Instead of repeating those proofs, we contend ourselves to derive a-priori estimates for the gradient of harmonic maps (i.e. already assuming that the map is regular) which can be obtained in a very easy way following [JK1].

THEOREM 4.8.1 *Let* X *and* Y *be Riemannian manifolds,* $B(x_0,R_0) \subset X$, $R_0 < \min\left(i(x_0), \frac{\pi}{2\kappa_X}\right)$ *, where* $-\omega_X^2 \leq K_X \leq \kappa_X^2$ *are curvature bounds on* $B(x_0,R_0)$ *, and* $B(p,M) \subset Y$ *,* $M < \min\left(i(p), \frac{\pi}{2\kappa_Y}\right)$ *, where* $-\omega_Y^2 \leq K_Y \leq \kappa_Y^2$ *are curvature bounds on* $B(p,M)$ *. If* $u : X \to B(p,M)$ *is harmonic, then for all* $R \leq R_0$

$$|\nabla u(x_0)| \leq c_0 \cdot \max_{x \in B(x_0,R)} \frac{d(u(x),\ u(x_0))}{R} \tag{4.8.1}$$

where $c_0 = c_0(R_0,\omega_X,\kappa_X,\dim X, M, \omega_Y,\kappa_Y, \dim Y)$.

Proof The proof is based on an idea of E. Heinz [Hz1] and similar to the one of Lemma 2.8.3. Let $\dim X = n$, $\dim Y = N$. We define

$$\mu := \max_{x \in B(x_0,R_0)} (R_0 - d(x,x_0))\ |du(x)| .$$

Then there exists $x_1 \in B(x_0, R_0)$ with

$$\mu = (R_0 - d(x_1,x_0)) \cdot |du(x_1)|$$

and

(4.8.2) $$|du(x_0)| \le \frac{\mu}{R_0} .$$

We put

$$d := R_0 - d(x_1,x_0) .$$

We shall prove

(4.8.3) $$\mu \le \frac{\delta(\theta_0)}{2\theta} + \frac{b\theta}{2}\mu^2 \qquad \text{for all } \theta \le \theta_0$$

where θ_0 can be chosen so small (with the help of Thm. 4.5.1) that

$$\delta(\theta_0) \cdot b < 1 .$$

Then (4.8.1) follows as in the proof of Lemma 2.8.3.

We now use the functions k_i of Lemma 2.8.4 for $q = u(x_1)$. Then

(4.8.4) $$\frac{\mu}{d} = |du(x_1)| = |d(k \circ u)(x_1)| .$$

Moreover

(4.8.5) $|D^2 k| \le c_1 \cdot \frac{1}{t_0}$, where $c_1 = c_1(\omega_Y, M, N)$ (cf. (2.8.33)

and hence

(4.8.6) $$|\Delta(k \circ u)| \le \frac{c_1}{t_0} |du|^2 \qquad \text{(cf. (1.7.2)) .}$$

Furthermore, dk is an isometry at $u(x_1)$, and hence from (4.8.5)

(4.8.7) $|dk(y)| \le c_2$, $c_2 = c_2(\omega_Y, M, N, \kappa_Y)$ (cf. Lemma 2.8.4) .

We put

$$\delta = \delta(\theta) := \max_{x \in B(x_1, d\theta)} d(u(x), u(x_1)) .$$

By Thm. 4.5.1, δ can be made arbitrarily small by choosing θ sufficiently small. At the moment, we need only

$$\delta \le M .$$

By Lemma 2.7.5, putting $\Lambda_X = \max(\omega_X, \kappa_X)$

$$(4.8.8)\quad \frac{\mu}{d} = |dk{\circ}u(x_1)| \le \frac{c_3}{d^n\theta^n}\int_{d(x,x_1)=d\theta} |k(u(x)) - k(u(x_1))|$$
$$+ c_4 \int_{d(x,x_1)\le d\theta} \frac{|\Delta k{\circ}u|}{d(x,x_1)^{n-1}} + c_5\,\Lambda_X^2 \int_{d(x,x_1)\le d\theta} \frac{|k(u(x)) - k(u(x_1))|}{d(x,x_1)^{n-1}} .$$

By (4.8.7), $|k(u(x)) - k(u(x_1))| \le c_2\delta$, and by (4.8.6),

$$|\Delta(k{\circ}u)| \le \frac{c_1}{t_0}\cdot |du|^2 \le \frac{c_1}{t_0}\,\frac{\mu^2}{d^2(1-\theta)^2} .$$

Estimating the integrals, we also get volume factors

$$\left(\frac{\sinh(\Lambda_X d\theta)}{\Lambda_X d\theta}\right)^{n-1} ,$$

which will be included in the constants $(c_i \to c_i', \ i = 3,4,5)$. Hence

$$\frac{\mu}{d} \le \left(\frac{c_3' c_2 \delta}{d\theta} + \frac{c_4' c_1 \cdot \mu^2}{t_0 d(1-\theta)^2}\,\theta + c_5' \Lambda_X^2 c_2 \delta \cdot d\theta\right) \mathrm{vol}(S^{n-1}) ,$$

or, assuming $\theta \le \frac{1}{2}$ w.l.o.g.,

$$\mu \le \frac{\delta(\theta_0)}{2\theta} + \frac{b\theta}{2}\,\mu^2 \qquad \text{for all } \theta \le \theta_0 ,$$

i.e. (4.8.3). By definition of $\delta(\theta)$ and Thm. 4.5.1, $\delta(\theta_0)$ can be made arbitrarily small by choosing θ_0 sufficiently small, and the result follows as in the proof of Lemma 2.8.3.

q.e.d.

At the boundary, we have

THEOREM 4.8.2 *Let* Ω *be a bounded domain in some Riemannian manifold,* $\partial\Omega$ *of class* C^2, *and let* $u : \Omega \to B(p,M)$ *be harmonic, where* $B(p,m)$ *satisfies the same assumptions as in Thm. 4.8.1. Suppose* $u|\partial\Omega = \phi \in C^2$. *Then* $|u|_{C^1(\bar\Omega)}$ *can be bounded in terms of the geometric quantities of Thm. 4.8.1,* *bounds for the principal curvatures of* $\partial\Omega$ *,* $|\phi|_{C^2}$ *, and a lower bound for a number* τ *satisfying* $0 < \tau < \frac{\pi}{2\kappa_Y} - S$ *,* $\dot{B}(p,M+\tau)$ *disjoint to the cut locus of* p .

Proof The proof is again taken from [JK1] and refines an argument of [HKW1]. Let $d(x_0,\partial\Omega) = R_0$. By Thm. 4.8.1, it suffices to show

$$\max_{x\in B(x_0,R_0)} d(u(x),\, u(x_0)) \le cR_0 \;.$$

This in turn follows, if

$$d(u(x_2),\, u(x_1)) \le cR_0 \tag{4.8.9}$$

in case $x_1 \in \partial\Omega$, $d(x_0,x_1) = R_0$, $d(x_0,x_2) \le R_0$.

We choose some number $\tau > 0$ as described in the statement of the theorem; w.l.o.g.

$$\tau \le \frac{\pi}{4\kappa_Y} \;. \tag{4.8.10}$$

By Lemma 2.4.1, any two points in $B(p,M+\tau)$ can be joined by a unique geodesic arc inside $B(p,M+\tau)$.

By Thm. 4.7.1, we can calculate $R_1 > 0$ with the property that for all $R_0 \le R_1$ and $x \in \Omega \cap B(x_0,2R_0)$, x_1 as above

$$d(u(x),\, u(x_1)) \le \frac{\tau}{2} \;. \tag{4.8.11}$$

If $u(x) \ne u(x_1)$, we connect $u(x)$ to $u(x_1)$ by a geodesic arc and continue this arc beyond $u(x_1)$ until a distance τ . We thus reach some point $q(x) \in B(p,M+\tau)$.

By Thm. 4.7.1 again, we can find some subdomain $\Omega_0 \subset \Omega$ satisfying

$$B(x_0,R_0) \subset \Omega_0 \tag{4.8.12}$$

(4.8.13) $\Omega \cap B(x_1,\delta) \subset \Omega_0$ for some $\delta > 0$

(4.8.14) $u(\Omega_0) \subset B\left(q(x), \frac{\pi}{2\kappa_Y}\right)$ for all $x \in B(x_0,R_0)$ (cf. (4.8.10))

(4.8.15) $\partial\Omega_0 \in C^2$.

We then fix $x_2 \in B(x_0,R_0)$, assume $u(x_1) \neq u(x_2)$ w.l.o.g., and put $q = q(x_2)$.

By (4.8.14)

$$v(x) := d^2(u(x),q)$$

is subharmonic in Ω_0 .

Let h be the harmonic function on Ω_0 with the same boundary values, i.e.

$$(4.8.16) \quad \begin{aligned} \Delta h &= 0 \quad \text{in } \Omega_0 \\ h(x) &= d^2(u(x),q) \quad \text{for } x \in \partial\Omega_0 . \end{aligned}$$

By the maximum principle

(4.8.17) $v \le h$ in Ω_0 .

Now

$$\begin{aligned} d(u(x_1), u(x_2)) &= d(u(x_2),q) - d(u(x_1),q) \quad \text{by choice of } q \\ &\le \frac{1}{2\tau}\,(d^2(u(x_2),q) - d^2(u(x_1),q)) \\ &\le \frac{1}{2\tau}\,(h(x_2) - h(x_1)) \quad \text{by (4.8.16) and (4.8.17)} . \end{aligned}$$

Thus, (4.8.9) follows from a Lipschitz bound for the harmonic function h at the boundary, which in turn follows from standard barrier arguments, taking (4.8.12), (4.8.13), and (4.8.15) into account, cf. [GT], chapter 13.

q.e.d.

Different gradient estimates were provided by Giaquinta-Hildebrandt [GH], Sperner [Sp], and Choi [Ci] (only interior estimates). The latter two papers employ an auxiliary function introduced by Jäger-Kaul [JäK2], cf. 4.11.

4.9 HIGHER ESTIMATES

If we write the equations

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\gamma}\, \gamma^{\alpha\beta} \frac{\partial}{\partial x^\beta} u^i \right) + \gamma^{\alpha\beta}\, \Gamma^i_{jk} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} = 0$$

in terms of harmonic coordinates on domain and image, then the regularity properties of harmonic coordinates (cf. section 2.8) immediately imply $C^{2,\alpha}$-estimates for harmonic maps, again depending only on curvature bounds, injectivity radii, and dimensions, using standard results from potential theory. We have the following result of [JK1].

THEOREM 4.9.1 *Suppose that the assumptions of Thm. 4.8.1 hold and* τ *is chosen as in Thm. 4.8.2. Then the* $C^{2+\alpha}$*-norm of* u *on* $B\left(x_0, \frac{R_0}{2}\right)$ *is bounded in terms only of the quantities appearing in Thm. 4.8.1 and* τ *. A corresponding result holds at the boundary, provided* $\partial\Omega$ *and* $u|\partial\Omega$ *are of class* $C^{2+\alpha}$ *(for all* $\alpha \in (0,1)$ *). Similarly if* $u|\partial\Omega$ *is only of class* $C^{1+\alpha}$ *, then* $u \in C^{1+\alpha}(\bar{\Omega})$ *with appropriate estimates.*

Finally, Thm. 2.8.3 implies

THEOREM 4.9.2 *If under the assumptions of Thm. 4.9.1 the Riemann curvature tensors of domain and image are of class* C^k *or* $C^{k+\beta}$ *(* $k \in \mathbb{N}$*,* $\beta \in (0,1)$ *), then* u *is of class* C^{k+2} *or* $C^{k+3+\beta}$ *, resp., and the corresponding estimates depend in addition on the* C^k *or* $C^{k+\beta}$*-norm, resp., of the curvature tensors. A similar statement holds at the boundary, provided* $\partial\Omega$ *and* $u|\partial\Omega$ *are sufficiently regular.*

4.10 THE EXISTENCE THEOREM OF HILDEBRANDT-KAUL-WIDMAN

In this section, we shall establish the existence of a weakly harmonic map with given boundary data contained in a convex ball which admit an extension with finite energy. This map will be obtained as the minimum of energy among maps with image in this ball. The results of the preceding sections then imply regularity of this map, and hence we can solve the Dirichlet problem.

A useful tool will be the following maximum principle for energy minimizing maps which is taken from [J6] and based on the same idea as the one in [H1], Lemma 6.

LEMMA 4.10.1 *Suppose that* B_0 *and* B_1 , $B_0 \subset B_1$, *are closed subsets of a Riemannian manifold* N . *Suppose that there exists a projection map*

$$\pi : B_1 \to B_0 ,$$

which is the identity on B_0 *and which is of class* C^1 *and distance decreasing outside* B_0 , *i.e.*

$$|d\pi(v)| < |v| \quad \text{if} \quad v \in T_xN , \quad v \neq 0 , \quad x \in B_1 \backslash B_0 .$$

If $h : \Omega \to B_1$ *is an energy minimizing* W_2^1 *mapping with respect to fixed boundary values which are contained in* B_0 , *i.e.*

$$h(\partial\Omega) \subset B_0 , \tag{4.10.1}$$

then we also have

$$h(\Omega) \subset B_0 ,$$

if we choose a suitable representation of the Sobolev mapping h .

Proof Since $|d\pi(v)| < |v|$ for every nonzero $v \in T_xN$, $x \in B_1 \backslash B_0$, and since $\pi \circ h \in W_2^1(\Omega,N)$, we would have

$$E(\pi\circ h) < E(h) \ ,$$

contradicting the minimality of h, unless $dh = 0$ a.e. on $h^{-1}(B_1\backslash B_0)$. Thus $dh = d\pi\circ h$ a.e. on Ω, and since h and $\pi\circ h$ agree on $\partial\Omega$ by (4.10.1), we conclude from the Poincaré inequality that $\pi\circ h = h$ a.e. on Ω, which easily implies the claim.

LEMMA 4.10.2 *Suppose that* B_0 *and* B_1, $B_0 \subset B_1$, *are compact subsets of a Riemannian manifold* N, *and that every point in* $B_1\backslash B_0$ *can be joined to* ∂B_0 *by a unique geodesic normal to* ∂B_0, *and that the distance between every pair of such geodesics normal to* ∂B_0 *is in* $B_1\backslash B_0$ *always bigger than on* ∂B_0. *Then the same conclusion as in Lemma 4.10.1 holds.*

Proof We project $B_1\backslash B_0$ along normal geodesics onto ∂B_0 and apply Lemma 4.10.1.

q.e.d.

We shall see another useful consequence of Lemma 4.10.1 in chapter 5.

We are now ready to prove the existence of a weakly harmonic map.

LEMMA 4.10.3 *Suppose* $B(p,M)$ *is disjoint to the cut locus of* p, *and* $M < \frac{\pi}{2\kappa}$, *where, as usual,* κ^2 *is an upper curvature bound.*

If $g:\Omega \to B(p,M)$, Ω *being a bounded domain in some Riemannian manifold, has finite energy, then there exists a weakly harmonic map* $u:\Omega \to B(p,M)$ *with* $u-g \in \mathring{H}^1_2(\Omega,B(p,M))$. u *minimizes the energy among all such maps.*

Proof Since the cut locus of a point p is a closed set, we can find M^1, $M < M^1 < \frac{\pi}{2\kappa}$, for which $B(p,M^1)$ is still disjoint to the cut locus of p. We take a minimizing sequence for the energy in $V := \{v \in H^1_2(\Omega,B(p,M^1))\ ,\ v-g \in \mathring{H}^1_2\}$. Note that $g \in V$ and hence $V \neq \emptyset$. Such a sequence has a subsequence converging weakly in H^1_2, and the limit, denoted by u, minimizes energy in V because of the lower semicontinuity of the energy integral (cf. Lemma 1.3.1).

We then put $B_0 = B(p,M)$ and $B_1 = B(p,M^1)$. If $c(\cdot,t)$ is a smooth family of geodesics with $c(0,t) = p$, $c(1,t) \in \partial B(p,M^1)$, then (2.2.5) implies that the Jacobi fields $J_t(s) = \frac{\partial}{\partial t} c(s,t)$ are monotonically increasing for $s \in [0,1]$. Hence the assumptions of Lemma 4.10.2 are satisfied. Therefore, $u(\Omega) \subset B(p,M)$.

We identify $B(p,M^1)$ with its image in $\mathbb{R}^N$ under normal coordinates centred at p . If $\eta \in \overset{\circ}{H}{}^1_2 \cap L^\infty(\Omega,\mathbb{R}^N)$, we infer that for sufficiently small $|t| > 0$, $u + t\eta$ still maps Ω into $B(p,M^1)$. Hence $u + t\eta$ is a valid comparison map, and since u was minimizing, differentiating $E(u + t\eta)$ w.r.t. t at $t = 0$ implies (4.1.2), i.e. that u is weakly harmonic.

q.e.d.

Remark As easy examples show the map u constructed in Lemma 4.10.3 need not be minimizing among all maps $v : \Omega \to Y$ with $v-g \in \overset{\circ}{H}{}^1_2$ (Y is a target manifold containing $B(p,M)$) , not even among maps which are homotopic to u . Hence, u in general is only a local minimum of energy.

Lemma 4.10.3 together with the regularity results of the previous sections imply the existence theorem of Hildebrandt-Kaul-Widman [HKW3].

THEOREM 4.10.1 *Suppose again that* $B(p,M)$ *is disjoint to the cut locus of* p *and* $M < \frac{\pi}{2\kappa}$ *, where* κ^2 *is an upper bound for the sectional curvature of* $B(p,M)$ *. If* Ω *is a bounded domain in some Riemannian manifold and* $g : \Omega \to B(p,M)$ *has finite energy, then there exists a harmonic map* $u \in C^{2,\alpha}(\Omega,B(p,M))$ $(0 < \alpha < 1)$ *with* $u-g \in \overset{\circ}{H}{}^1_2(\Omega,B(p,M))$ *. At* $\partial\Omega$ *,* u *is as regular as* g *and* $\partial\Omega$ *permit.*

Actually, one can solve the Dirichlet problem for any continuous map

$w : \partial\Omega \to B(p,M)$, i.e. find a harmonic map $u \in C^{2,\alpha}(\Omega, B(p,M)) \cap C^0(\bar{\Omega}, B(p,M))$ with $u|\partial\Omega = w$, without assuming that w admits an extension of finite energy. In order to achieve this, one has to combine the a-priori estimates of the preceding sections with Leray-Schauder degree theory instead of using variational methods. For this, one first deforms w into constant boundary values, mapping $\partial\Omega$ onto p and then multiplies the nonlinearity in (1.3.1) by a parameter λ , $\lambda \in [0,1]$. Such a twofold deformation process was applied in [HKW2], for instance.

4.11 THE UNIQUENESS THEOREM OF JÄGER-KAUL

In this section, we want to prove the uniqueness and stability theorem of Jäger-Kaul [JäK2] for solutions of the Dirichlet problem with image contained in a convex ball.

THEOREM 4.11.1 *Suppose that* $u_i : \bar{\Omega} \to Y$, $i = 1,2$, *are harmonic maps of class* $C^0(\bar{\Omega}, Y) \cap C^2(\Omega, Y)$, Ω *is a bounded domain in some Riemannian manifold, and* $u_i(\bar{\Omega}) \subset B(p,M)$, *where* $B(p,M)$ *is a geodesic ball in* Y , *disjoint to the cut locus of* p *and with radius* $M < \frac{\pi}{2\kappa}$ (κ^2 *is an upper bound for the sectional curvature of* $B(p,M)$).

Then the function θ ,

$$\theta(x) := \frac{q_\kappa(d(u_1(x), u_2(x)))}{\cos(\kappa d(p, u_1(x))) \cdot \cos(\kappa d(p, u_2(x)))}$$

$$(q_\kappa(t) := \begin{cases} \frac{1}{\kappa^2}(1 - \cos \kappa t) , & \text{if } \kappa > 0 \\ \frac{t^2}{2} , & \text{if } \kappa = 0 \end{cases}) ,$$

satisfies the maximum principle

$$\sup_{\Omega} \theta \le \sup_{\partial\Omega} \theta . \tag{4.11.1}$$

In particular, if $u_1|\partial\Omega = u_2|\partial\Omega$ *, then*

$$u_1 \equiv u_2 .$$

The proof of Thm. 4.11.1 will actually show that we have strict inequality in (4.11.1) unless $\theta \equiv$ const. Furthermore, Thm. 4.11.1 also holds for weakly harmonic maps (cf. [JäKl]).

Proof We assume that θ has a positive maximum at some interior point $x_0 \in \Omega$. Then, θ is positive in a neighbourhood of x_0 , and $\log\theta > -\infty$ in this neighbourhood.

We define

$$\psi(x) := Q_\kappa(u_1(x),u_2(x)) = \begin{cases} \frac{1}{\kappa^2}(1-\cos\kappa d(u_1(x),u_2(x))) & \text{if } \kappa > 0 \\ \frac{1}{2} d^2(u_1(x),u_2(x)) & \text{if } \kappa = 0 \end{cases}$$

$$\phi_i(x) = \cos(\kappa d(p,u_1(x))) , \qquad i = 1,2 .$$

Then $\theta = \dfrac{\psi}{\phi_1 \cdot \phi_2}$, and consequently

$$\text{(4.11.2)} \qquad \operatorname{grad}\log\theta = \frac{\operatorname{grad}\psi}{\psi} - \frac{\operatorname{grad}\phi_1}{\phi_1} - \frac{\operatorname{grad}\phi_2}{\phi_2} ,$$

and

$$\text{(4.11.3)} \quad \Delta\log\theta = \frac{\Delta\psi}{\psi} - \frac{|\operatorname{grad}\psi|^2}{\psi^2} - \frac{\Delta\phi_1}{\phi_1} + \frac{|\operatorname{grad}\phi_1|^2}{\phi_1^2} - \frac{\Delta\phi_2}{\phi_2} + \frac{|\operatorname{grad}\phi_2|^2}{\phi_2^2} .$$

Since $x \to u(x) = (u_1(x),u_2(x)) \in B(p,M) \times B(p,M)$ is also harmonic, we can make use of the chain rule (1.7.2) in order to apply Lemma 2.5.1. This yields

$$\text{(4.11.4)} \qquad \Delta\psi \geq \frac{|\operatorname{grad}\psi|^2}{2\psi} - \kappa^2\psi(|du_1|^2 + |du_2|^2) ,$$

since

$$|\operatorname{grad}\psi|^2 = \sum_\alpha \langle(\operatorname{grad} Q_\kappa)\circ u,\ du(e_\alpha)\rangle^2 ,$$

where e_α is an orthonormal frame on Ω .

Similarly, from (2.5.2), since

$$\phi_i(x) = 1 - \kappa^2 Q_\kappa(p, u_i(x)) ,$$

we obtain

$$\Delta\phi_i(x) \leq -\kappa^2 \phi_i |du_i|^2 . \tag{4.11.5}$$

Finally, by (4.11.2),

$$-\tfrac{1}{2}\frac{|\text{grad } \psi|^2}{\psi^2} + \frac{|\text{grad } \phi_1|^2}{\phi_1^2} + \frac{|\text{grad } \phi_2|^2}{\phi_2^2} \geq -\left\langle \text{grad log } \theta, \tfrac{1}{2} \text{ grad log } \theta + \frac{\text{grad } \phi_1}{\phi_1} + \frac{\text{grad } \phi_2}{\phi_2} \right\rangle . \tag{4.11.6}$$

Putting

$$k(x) := \tfrac{1}{2} \text{ grad log } \theta + \frac{\text{grad } \phi_1}{\phi_1} + \frac{\text{grad } \phi_2}{\phi_2}$$

and plugging (4.11.4), (4.11.5), and (4.11.6) into (4.11.3), we obtain

$$\Delta \log \theta + \langle \text{grad log } \theta, k(x) \rangle \geq 0 .$$

Therefore, the assumption that θ has a positive maximum in the interior contradicts E. Hopf's maximum principle, and Thm. 4.11.1 is proved.