CHAPTER 3

THE HEAT FLOW METHOD

Existence, regularity, and uniqueness results for a nonpositively curved image

3.1 APPROACHES TO THE EXISTENCE AND REGULARITY QUESTION

There are four different approaches to the existence and regularity theory of harmonic maps available. The first one is the so-called heat flow method. In order to find a harmonic map homotopic to a given map $g : X \to Y$, one investigates the parabolic system

$$(3.1.1) \qquad \begin{aligned} \frac{\partial f(x,t)}{\partial t} &= \tau(f(x,t)) \qquad \text{for } x \in X \text{ and } t \geq 0 \\ f(x,0) &= g(x) \qquad \text{for } x \in X \end{aligned}$$

and one tries to prove that a solution of (3.1.1) exists for all $t \geq 0$ and that $f(\cdot,t)$ converges to a harmonic map f as $t \to \infty$. That means one tries to deform g into a homotopic harmonic map by an analogue of heat dispersion on manifolds. One should compare this method with the gradient flow descent method common in Morse theory. Whereas this method in our case would lead to an ordinary differential equation for a mapping from X into the Sobolev space $W_2^1(X,Y)$, i.e. an infinite dimensional target space, and follow the gradient lines of the energy functional, the heat flow method instead leads to a partial differential equation for a mapping from X into the finite dimensional manifold Y.

The second approach tries to establish regularity (and a-priori estimates) for weak solutions f of the elliptic system

$$(3.1.2) \qquad \int_X \{\gamma^{\alpha\beta}(x)\, g_{ij}(f(x))\, \frac{\partial f^i}{\partial x^\alpha}\frac{\partial \phi^j}{\partial x^\beta} - \gamma^{\alpha\beta}(x)\, \Gamma^i_{jk}(f(x))\, \frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta}\, \phi^i\}\, dX$$

$$\text{for all } \phi \in W_2^1 \cap L^\infty .$$

In case this approach works, it implies in particular the regularity of an energy minimizing map and hence establishes the existence by a variational method. Alternatively, it can be used in conjunction with Leray-Schauder degree theory to assert the existence of a solution.

The third approach uses perturbed energy functionals which satisfy the compactness condition (C) of Palais-Smale. It can reprove the results obtained by the first approach, requiring much deeper estimates, however.

The fourth approach is the so-called method of partial regularity. It tries to characterize the possible singularities of energy minimizing maps and then to show that under appropriate conditions those singularities cannot exist and that an energy minimizing map is hence regular. In contrast to the other approaches, here the techniques so far are restricted to energy minimizing maps. Nevertheless, a posteriori this method comprises the results obtained by the other ones, since in all cases, where those methods work, one can prove a uniqueness result with the implication that in those cases any harmonic map is energy minimizing.

The first method was initiated by Eells-Sampson [ES], the second one by Hildebrandt-Kaul-Widman [HKW3], the third one by Uhlenbeck [U] and the fourth one by Schoen-Uhlenbeck [SU1] and Giaquinta-Giusti [GG1], [GG2].

In the present notes, we shall only develop the first two methods. We believe that our presentations have some advantages compared to the ones existing in the literature, as either the estimates are more precise, the proofs are shorter, or the arguments are more elementary. In particular, all proofs are self-contained.

We start with the heat-flow method in the present chapter, and shall develop the second one in chapter 4.

During all of chapter 3, the manifold X will be assumed to be *compact*.

The results of this chapter are due to Eells-Sampson [ES] and Hartman [Ht]. We shall also use some ideas as presented by von Wahl [vW] and Jost [J4]. Similar arguments were also known to R. Schoen.

3.2 SHORT TIME EXISTENCE

The parameter t will be referred to as time parameter, while $x \in X$ is the space variable, according to the thermodynamic interpretation of the present method.

We shall start by proving the existence of a solution of (3.1.1) for small time.

LEMMA 3.2.1 *Suppose* $g \in C^{2+\alpha}(X,Y)$. *Then there is some* $\varepsilon > 0$ *depending only on the geometry of* X *and* Y *and on* g *with the property that (3.1.1) has a solution* $f(x,t)$ *for* $0 \le t < \varepsilon$.

Proof The linearization of the operator $\left(\frac{\partial}{\partial t} - \tau\right)$ at f is computed in local coordinates as

$$L_f^i(\phi) = \frac{\partial \phi^i}{\partial t} - \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}\,\gamma^{\alpha\beta} \frac{\partial \phi^i}{\partial x^\beta}\right) - \gamma^{\alpha\beta}\,\Gamma^i_{jk,\ell}\,\phi^\ell \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta}$$

$$- \gamma^{\alpha\beta}\,\Gamma^i_{jk}\left(\frac{\partial \phi^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta} + \frac{\partial f^j}{\partial x^\alpha}\frac{\partial \phi^k}{\partial x^\beta}\right) .$$

By the theory of linear parabolic equations, the system

$$L_f(\phi) = h(x,t)$$

$$\phi(x,0) = g(x)$$

for given h of class C^α in x and t and g of class $C^{2+\alpha}$ in x , has a unique solution $\phi(x,t)$ of class $C^{2+\alpha}$ in x and $C^{1+\alpha}$ in t .

Moreover, the corresponding a-priori estimates[1)] imply that L_f is a continuous bijective linear operator between the corresponding mapping spaces. The implicit function theorem then implies Lemma 3.2.1.

q.e.d.

COROLLARY 3.2.1 *The set of* $T \in (0,\infty)$ *for which the solution of (3.1.1) exists for* $t \in [0,T]$ *is open.*

This follows by taking $f(\cdot,T)$ as initial values in Lemma 3.2.1.

q.e.d.

Note that in contrast to the results in the following sections, for the small time existence of the solution of (3.1.1) we do not have to require any curvature assumptions for Y .

3.3 ESTIMATES FOR THE ENERGY DENSITY OF THE HEAT FLOW

We first show that the energy $E(f(\cdot,t))$ is a decreasing function of t . For,

$$(3.3.1)\quad \frac{d}{dt} E(f(\cdot,t)) = \frac{d}{dt}\,\tfrac{1}{2}\int \langle df,df\rangle = \int \left\langle \frac{\partial}{\partial t}\, df, df\right\rangle = \int \left\langle d\,\frac{\partial}{\partial t}\, f, df\right\rangle$$

$$= -\int \left\langle \frac{\partial}{\partial t}\, f, \tau(f)\right\rangle = -\int \left|\frac{\partial}{\partial t}\, f\right|^2$$

since f satisfies the equation (3.1.1), i.e. $\frac{\partial}{\partial t} f = \tau(f)$.

It is also interesting to compute the second time derivative of $E(f(\cdot,t))$, although this formula is not needed in the sequel. As in (1.6.5), we compute

$$\frac{\partial}{\partial t}\left|\frac{\partial}{\partial t}\, f\right|^2 = \Delta \left|\frac{\partial}{\partial t}\, f\right|^2 - \left|\nabla \frac{\partial f}{\partial t}\right|^2 + \left\langle R^Y\left(df\cdot e_\alpha, \frac{\partial f}{\partial t}\right) df\cdot e_\alpha, \frac{\partial f}{\partial t}\right\rangle ,$$

and hence, since X is compact

1) Note that since X is compact, $g(X)$ is bounded in Y .

$$(3.3.2) \qquad \frac{d^2}{dt^2} E(f(\cdot,t)) = \int_X \left| \nabla \frac{\partial f}{\partial t} \right|^2 - \int_X \left\langle R^Y\left(df\cdot e_\alpha, \frac{\partial f}{\partial t}\right) df\cdot e_\alpha, \frac{\partial f}{\partial t} \right\rangle .$$

We note that, in case Y is nonpositively curved,

$$\frac{d^2}{dt^2} E(f(\cdot,t)) \geq 0 .$$

From now on, we shall assume for the rest of this chapter, that Y has *nonpositive sectional curvature.*

As in 1.6, we look at the energy density of $f(x,t)$

$$e(f) = \tfrac{1}{2} \gamma^{\alpha\beta}(x)\, g_{ij}(f(x,t)) \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta} .$$

If $f(x,t)$ is a solution of (3.1.1), the calculations of 1.6 imply

$$(3.3.3) \qquad \Delta e(f) - \frac{\partial}{\partial t} e(f) = |\nabla df|^2 + {}^X R_{\alpha\beta}(x) \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta} g_{ij} - \gamma^{\alpha\beta}\gamma^{\delta\eta}\, {}^Y R_{ikj\ell} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\delta} \frac{\partial f^j}{\partial x^\beta} \frac{\partial f^\ell}{\partial x^\eta} .$$

Since X is a compact manifold of class C^3, its Ricci tensor is bounded. Since we assume that Y has nonpositive sectional curvature, (3.3.3) implies

$$(3.3.4) \qquad \Delta e(f) - \frac{\partial}{\partial t} e(f) \geq -c\, e(f) .$$

The constant c may still depend on t, since as $t \to \infty$, the image of $f(x,t)$ may become unbounded since we did not assume so far that Y is compact. This does not matter, however, since we shall see in 3.5 that for any $T < \infty$ and $t \in [0,T]$, $f(x,t)$ remains in a bounded subset of Y, possibly depending on T.

We now want to use (3.3.4) to derive estimates for $e(f)$.

For a given point $m \in X$, we choose a ball $B(m,\rho)$ satisfying the assumptions of Lemma 2.3.2. We note that $\rho > 0$ can be chosen uniformly for $m \in X$, since X is compact.

Plugging (3.3.4) into (2.7.6) and using (2.7.7), we obtain

$$(3.3.5) \quad e(f)(m,t) \le c_1 \int_{B(m,\rho,t_0,t)} e(f)(x,\tau)(t-\tau)^{-\frac{1}{2}} r(x)^{-n+1}\, dx\, d\tau$$

$$+ \frac{c_n}{\rho^{n+2}} \int_{B(m,\rho,t_0,t)} e(f) + \frac{c_n}{\rho^{n+1}} \int_{\substack{r(x)=\rho \\ t_0 \le \tau \le t}} e(f)$$

$$+ (t-t_0)^{-n/2} \int_{B(m,\rho)} e(f)(x,t_0)\, dx .$$

Here, c_1 depends on n and Λ^2 , a bound for the sectional curvature of X .

First of all, we observe that if $i(X) > 0$ is the injectivity radius of X , $\rho_0 = \min\left(i(X), \frac{\pi}{2\Lambda}\right)$, we can choose $\rho \in [\rho_0/2, \rho_0]$ with

$$(3.3.6) \qquad \int_{\substack{r(x)=\rho \\ t_0 \le \tau \le t}} e(f) \le \frac{2}{\rho} \int_{B(m,\rho,t_0,t)} e(f) .$$

We define

$$g_1(m,p,t) = t^{-\frac{1}{2}} \cdot d(m,p)^{-n+1}$$

$$g_k(m,p,t) = \int_{\substack{t_0 \le \tau \le t \\ d(x,p) \le \rho}} g_{k-1}(m,x,t-\tau)\, g_1(x,p,\tau)\, dx\, d\tau$$

and choose $\rho = \rho(p)$ in the definition of g_k in such a way that (3.3.6) is satisfied for p instead of m . We observe that

$$g_k(m,p,t) \le c_2 (t-t_0)^{\frac{1}{2}}\, d(m,p)^{-n+k}$$

and hence g_k is bounded for $k > n$.

Thus, if we iterate (3.3.5), using (3.3.5) again for $e(f)(x,\tau)$ in the first integral in (3.3.5), we obtain after a finite number of steps

$$(3.3.7) \qquad e(f)(m,t) \le c_3\, \rho^{-n-2} \int_{B(m,n\rho,t-n(t-t_0),t)} e(f)$$

$$+ c_4 (t-t_0)^{-n/2} \int_X e(f)(x,t-n(t-t_0))\, dx .$$

In order to locate the last integral at $t-n(t-t_0)$, we have used the fact that the energy decreases in time by (3.3.1).

Choosing $t_0 \geq 0$ in such a way that $t \geq n(t-t_0) \geq \varepsilon$ and using (3.3.1) again

$$e(f)(m,t) \leq c_5(t\rho^{-n-2} + \varepsilon^{-n/2}) \int_X e(f)(x,0)\,dx\,. \tag{3.3.8}$$

If we want to avoid the term with $\varepsilon^{-n/2}$, we can use (2.7.8) instead of (2.7.6) and obtain in a similar way

$$e(f)(m,t) \leq c_6\,\rho^{-2} \sup_{x\in X} e(f)(x,0)\,. \tag{3.3.9}$$

Namely, we then have the term

$$\int_{B(m,\rho)} e(f)(x,0)(t-t_0)^{-n/2} \exp\left(-\frac{r^2(x)}{4(t-t_0)}\right) dx$$

which is an approximate solution of the heat equation with initial values $e(f)(x,0)$, and we use that by the maximum principle the supremum over the space variables of a solution of the heat equation is nonincreasing in time.

We collect these estimates in

LEMMA 3.3.1 *Suppose* f *is a solution of (3.1.1) on* $[0,t]$. *If* $t \geq \varepsilon$ *and* $0 < R < \min\left(i(x), \frac{\pi}{2\Lambda}\right)$

$$e(f)(m,t) \leq c_5(tR^{-n-2} + \varepsilon^{-n/2}) \int_X e(f)(x,0)\,dx$$

Furthermore, for any $t_0 < t$, *in particular* $t_0 = 0$,

$$e(f)(m,t) \leq c_6\,R^{-2} \sup_{x\in X} e(f)(x,t_0)\,.$$

3.4 THE STABILITY LEMMA OF HARTMAN

We now let $f(x,t,s)$ be a smooth family of solutions of (3.1.1) depending on a parameter s and having initial values $f(x,0,s) = g(x,s)$, $0 \le s \le s_0$.

LEMMA 3.4.1 (Hartman [Ht]) *Suppose again, that* Y *has nonpositive sectional curvature.*

For every $s \in [0,s_0]$

$$\sup_{x\in X}\left(g_{ij}(f(x,t,s)) \cdot \frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}\right)$$

is nonincreasing in t . *Hence also*

$$\sup_{x\in X, s\in[0,s_0]}\left(g_{ij}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}\right)$$

is a nonincreasing function of t .

Proof As in 1.6, one calculates in normal coordinates

$$(3.4.1)\quad \left(\Delta - \frac{\partial}{\partial t}\right)\left(g_{ij}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}\right) = \frac{\partial^2 f^i}{\partial x^\alpha \partial s}\frac{\partial^2 f^j}{\partial x^\alpha \partial s} - R_{ikj\ell}\frac{\partial f^i}{\partial s}\frac{\partial f^k}{\partial x^\alpha}\frac{\partial f^j}{\partial s}\frac{\partial f^\ell}{\partial x^\alpha} ,$$

and since Y has nonpositive sectional curvature, hence

$$\left(\Delta - \frac{\partial}{\partial t}\right)\left(g_{ij}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}\right) \ge 0 .$$

The lemma then follows from the maximum principle for parabolic equations.

q.e.d.

We now assume that f_1 and f_2 are smooth homotopic maps from X to Y , and $h : X \times [0,1] \to Y$ is a smooth homotopy with $h(x,0) = f_1(x)$, $h(x,1) = f_2(x)$.

Since $h(x,s)$ is smooth in x and s , the curve $h(x,\cdot)$ connecting $f_1(x)$ and $f_2(x)$ depends smoothly on x . We let $g(x,\cdot)$ be the geodesic from $f_1(x)$ to $f_2(x)$ which is homotopic to $h(x,\cdot)$ and parametrized proportionally to arc length. Since Y is nonpositively curved, this geodesic arc is unique and hence depends smoothly on x . We define $\tilde{d}(f_1(x), f_2(x))$ to be the length of this geodesic arc.

We then put $f(x,0,s) = g(x,s)$.

COROLLARY 3.4.1 *Suppose, as before, that* Y *is nonpositively curved. Assume that the solution* $f(x,t,s)$ *of (3.1.1) exists for all* $s \in [0,1]$ *and* $t \in [0,T]$. *Then*

$$\sup_{x \in X} \tilde{d}(f(x,t,0),\ f(x,t,1))$$

is nonincreasing in t *for* $t \in [0,T]$.

Proof By construction, at $t = 0$

$$\sup_{x \in X, s \in [0,1]} \left\{ g_{ij} \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial s} \right\} = \sup_{x \in X} \tilde{d}^2(g(x,0),\ g(x,1)) \ .$$

On the other hand, for any $t \in [0,T]$

$$\tilde{d}^2(f(x,t,0),\ f(x,t,1)) \le \sup_{s \in [0,1]} g_{ij}(f(x,t,s)) \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial s}$$

since $f(x,t,\cdot)$ is a curve joining $f(x,t,0)$ and $f(x,t,1)$ in the homotopy class chosen for the definition of $\tilde{d}$. The claim then follows from Lemma 3.4.1.

q.e.d.

3.5 A BOUND FOR THE TIME DERIVATIVE

Our first application of Lemma 3.4.1 will be a bound for the time derivative of a solution of (3.1.1).

LEMMA 3.5.1 *Suppose* $f(x,t)$ *solves (3.1.1) for* $t \in [0,T)$ *and* Y *has non-positive sectional curvature. Then for all* $t \in [0,T)$ *and* $x \in X$

$$\left| \frac{\partial f(x,t)}{\partial t} \right| \le \sup_{x \in X} \left| \frac{\partial}{\partial t} f(x,0) \right| \ . \tag{3.5.1}$$

Proof This follows by putting

$$f(x,t,s) = f(x,t+s)$$

and applying Lemma 3.4.1 at $s = 0$.

q.e.d.

LEMMA 3.5.2 *Suppose* $f(x,t)$ *solves (3.1.1) for* $t \in [0,T)$ *and* Y *has nonpositive sectional curvature. Then for every* $\alpha \in (0,1)$

(3.5.2)
$$|f(\cdot,t)|_{C^{2+\alpha}(X,Y)} + \left|\frac{\partial f}{\partial t}(\cdot,t)\right|_{C^{\alpha}(X,Y)} \le c_7 .$$

c_7 *depends on* α , T *(only in case* $f(\cdot,t)$ *becomes unbounded, but anyway it will be finite for any finite* T *) , the initial values* $g(x) = f(x,0)$ *, and the geometry of* X *and* Y *, or more precisely on curvature bounds, injectivity radii and dimensions of* X *and* Y .

Proof We write (3.1.1) in the following way

(3.5.3)
$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}\,\gamma^{\alpha\beta}\frac{\partial f^i}{\partial x^\beta}\right) = -\gamma^{\alpha\beta}\,\Gamma^i_{jk}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta} + \frac{\partial f^i}{\partial t} .$$

If we centre our coordinate charts on X and Y at m and $f(m,t_0)$, then for a fixed neighbourhood $B(m,\rho) \times [t_0,t_1]$ of (m,t_0), $f(x,t)$ will stay inside this coordinate chart in Y by Lemmata 3.3.1 and 3.5.1. Furthermore, those lemmata also imply that the right hand side of (3.5.3) is bounded. This first implies a bound for $|f(\cdot,t)|_{C^{1+\alpha}(X,Y)}$ by elliptic regularity theory. But then the right hand side of (3.1.1) is bounded in $C^{\alpha}(X,Y)$, and (3.5.2) now follows from parabolic regularity theory.

The statements concerning the dependence of the estimates on the geometry follow from the results of section 2.8, where we constructed local coordinates for which the Hölder constants of the Christoffel symbols are bounded in terms of the quantities appearing in the statement of the lemma (cf. Thm. 2.8.2).

q.e.d.

LEMMA 3.5.3 *The solution of (3.1.1) exists for all* $t \in [0,\infty)$ *, if* Y *has nonpositive sectional curvature.*

Proof Lemma 3.2.1 shows that the set of $T \in [0,\infty)$ with the property that the solution exists for all $t \in [0,T]$ is open and nonempty, while Lemma 3.5.2 implies that it is also closed.

q.e.d.

3.6 GLOBAL EXISTENCE AND CONVERGENCE TO A HARMONIC MAP (THEOREM OF EELLS-SAMPSON)

We assume now, that $f(x,t)$ remains in a compact subset of Y for all t . This is trivially the case, if Y itself is compact.

If we use the energy decay formula (3.3.1), namely

$$\frac{\partial}{\partial t} E(f(\cdot,t)) = - \int_X \left|\frac{\partial f(x,t)}{\partial t}\right|^2 dX \ ,$$

observe that $E(f(\cdot,t))$ is by definition always nonnegative, and use the time independent C^{α}-bound for $\left|\frac{\partial f}{\partial t}\right|$, we obtain

LEMMA 3.6.1 *If* $f(x,t)$ *remains in a bounded subset of* Y *, then there exists a sequence* (t_n) *,* $t_n \to \infty$ *as* $n \to \infty$ *, for which* $\frac{\partial f}{\partial t}(x,t_n)$ *converges to zero uniformly in* $x \in X$ *as* $n \to \infty$ *.*

Now using the $C^{2+\alpha}$-bounds for $f(\cdot,t)$ of Lemma 3.5.2, we can assume, by possibly passing to a subsequence, that $f(x,t_n)$ converges uniformly to a harmonic map $f(x)$ as $t_n \to \infty$. In Cor. 3.4.1 which we may apply because of Lemma 3.5.3, we then put

$$g(x,0) = f(x,0,0) = f(x,t_n)$$
$$g(x,s_0) = f(x,0,s_0) = f(x)$$

By uniform convergence, some $f(\cdot,t_n)$ (and hence all, since $f(x,t)$ is continuous in t) are homotopic to f .

Since $f(x)$ as a harmonic map is a time independent solution of (3.1.1), $f(x,t,s_0) = f(x)$ for all t. Cor. 3.4.1 then implies

$$d(f(x,t_n+t),\ f(x)) \le d(f(x,t_n),\ f(x)) \qquad \text{for all} \quad t \ge 0 .$$

Hence it follows that the selection of the subsequence is not necessary and that $f(x,t)$ uniformly converges to $f(x)$ as $t \to \infty$.

We thus have proved the existence theorem of Eells-Sampson [ES] with the improvements by Hartman [Ht].

THEOREM 3.6.1 *Suppose* Y *is nonpositively curved. Then the solution of (3.1.1) exists for all* $t \in [0,\infty)$. *If the solution remains in a bounded subset of* Y, *in particular if* Y *is compact, then it converges uniformly to a harmonic map as* $t \to \infty$. *In particular, any map* $g \in C^{2+\alpha}(X,Y)$ *is homotopic to a harmonic map.*

Remarks 1) The result also holds, if we merely assume $g \in C^0$. A suitable modification of Lemma 3.2.1 pertains to this case, and we choose some $t_0 \in (0,\varepsilon)$, where ε is the time-range of Lemma 3.2.1. Then $f(x,t_0)$ is of class $C^{2+\alpha}$ in x and can be chosen as new initial values for the heat flow, and we apply the arguments of the preceding sections to these initial values.

2) If we take one branch of the curve $y = \frac{1}{x}$ and rotate it around the x-axis, we obtain a negatively curved surface of revolution. The image of a point on $y = \frac{1}{x}$ under this rotation yields a closed homotopically nontrivial curve which is not homotopic to any closed geodesic. It is not difficult to see that as $t \to \infty$ the solution of the heat equation with those initial values will disappear at infinity, not converging to anything. From this we see that the hypothesis in Thm. 3.6.1 that the solution remains in a bounded set is necessary for the existence of a harmonic map.

On the other hand, noncompactness of the target space does not inevitably prevent the solution of the heat flow from converging to a harmonic map as is seen by rotating the curve $y = x^2 + 1$ instead of $y = \frac{1}{x}$ around the x-axis.

3.7 ESTIMATES IN THE ELLIPTIC CASE

We now want to derive estimates for a harmonic map $f : X \to Y$. Since Y is nonpositively curved, (1.6.5) implies

$$\Delta e(f) \geq -c\, e(f) . \tag{3.7.1}$$

For simplicity, we assume $n = \dim X \geq 3$. We put $\rho_0 = \min\left(i(X), \frac{\pi}{2\Lambda}\right)$. By a suitable choice of $\rho \in (\frac{1}{2}\rho_0, \rho_0)$, (3.7.1) in conjunction with the representation formula (2.7.5) yields

$$e(f)(m) \leq \frac{c_8}{\rho^2} \int_{B(m,\rho)} \frac{e(f)(x)}{r(x)^{n-2}}\, dx . \tag{3.7.2}$$

Iteration of (3.7.2) yields as in 3.3

$$e(f)(m) \leq \frac{c_9}{\rho^2} \int_{B(m,\frac{n}{2}\rho)} e(f)(x)\, dx . \tag{3.7.3}$$

THEOREM 3.7.1 *If $f : X \to Y$ is harmonic, X compact and Y nonpositively curved,*

$$|f|_{C^{2+\alpha}(X,Y)} \leq c_9 ,$$

where c_9 depends on the energy $E(f)$ and on curvature bounds, injectivity radii and dimensions of X and Y.

Proof We again look at the equation

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\gamma}\, \gamma^{\alpha\beta} \frac{\partial}{\partial x^\beta} f^i\right) = -\gamma^{\alpha\beta}\, \Gamma^i_{jk} \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} .$$

(3.7.3) implies that the right hand side is bounded and that for every $m \in X$, a uniform neighbourhood $B(m,\rho)$ is mapped into the same coordinate chart on the image. Elliptic regularity theory implies $f \in C^{1+\alpha}$, which in turn implies that the right hand side is of class C^{α} and hence $f \in C^{2+\alpha}$.

The assertions about the dependence of c_9 on the geometry of X and Y follow, if we choose harmonic coordinates at m and $f(m)$. For those coordinates, the Christoffel symbols have the required regularity properties, as is shown in 2.8 (cf. Thm. 2.8.2).

q.e.d.

3.8 THE UNIQUENESS RESULTS OF HARTMAN

In this section, we shall be concerned with uniqueness properties of harmonic maps into nonpositively curved manifolds.

THEOREM 3.8.1 (Hartman [Ht]) *Let* $f_1(x)$, $f_2(x)$ *be two homotopic harmonic maps from* X *into the nonpositively curved manifold* Y. *For fixed* x, *let* $f(x,s)$ *be the unique geodesic from* $f_1(x)$ *to* $f_2(x)$ *in the homotopy class determined by the homotopy between* f_1 *and* f_2, *and let the parameter* $s \in [0,1]$ *be proportional to arc length.*

Then, for every $s \in [0,1]$, $f(\cdot,s)$ *is a harmonic map with* $E(f(\cdot,s)) = E(f_1) = E(f_2)$. *Furthermore, the length of the geodesic* $f(x,\cdot)$ *is independent of* x.

Hence any two harmonic maps can be joined by a parallel family of harmonic maps with equal energy.

Proof We let $f(x,t,s)$ be the solution of (3.1.1) with initial values $f(x,0,s) = f(x,s)$. $f(x,t,s)$ exists for all time by Lemma 3.5.3.

By Cor. 3.4.1, for any $s \in [0,1]$ and $t \in (0,\infty)$

$$\text{(3.8.1)} \qquad \sup_{x\in X} \tilde{d}(f(x,t,s),\ f_1(x)) \le \sup_{x\in X} \tilde{d}(f(x,s),\ f_1(x))$$

$$\le \sup_{x\in X} d(f_2(x),\ f_1(x)) \ .$$

Hence, $f(x,t,s)$ stays in a bounded subset of Y as $t \to \infty$.

Thm. 3.6.1 implies that $f(x,t,s)$ converges to a harmonic map $f_0(x,s)$ as $t \to \infty$.

We choose $x_0 \in X$ with

$$\tilde{d}(f_2(x_0),\ f_1(x_0)) = \sup_{x\in X} \tilde{d}(f_2(x),\ f_1(x))$$

and by construction therefore

$$\tilde{d}(f(x_0,s),\ f_1(x_0)) = \sup_{x\in X} \tilde{d}(f(x,s),\ f_1(x)) \quad \text{for all } s \ .$$

From (3.8.1)

$$\text{(3.8.2)} \qquad \tilde{d}(f(x_0,t,s),\ f_1(x_0)) \le \tilde{d}(f(x_0,s),\ f_1(x_0))$$

and similarly

$$\text{(3.8.3)} \qquad \tilde{d}(f(x_0,t,s), f_2(x_0)) \le \tilde{d}(f(x_0,s),\ f_2(x_0)) \ .$$

Note that all distances are measured by the length of that geodesic which is mentioned in the statement of the theorem.

Then (3.8.2) and (3.8.3) imply

$$\text{(3.8.4)} \qquad f(x_0,t,s) = f_0(x_o,s) = f(x_0,s) \quad \text{for all } s \ .$$

We now look at

$$e_s(f)(x,t,s) = g_{ij}(f(x,t,s)) \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial s} \ .$$

By Lemma 3.4.1

$$\text{(3.8.5)} \quad \sup_{x\in X} e_s(f)(x,t,s) \le \sup_{x\in X} e_s(f)(x,0,s) \quad \text{for every } s \in [0,1],\ t \in (0,\infty)$$

On the other hand, from (3.8.4)

(3.8.6) $$e_s(f)(x_0,t,s) = e_s(f)(x_0,0,s) = \sup_{x\in X} e_s(f)(x,0,s) .$$

Hence for all t, the supremum in (3.8.5) is attained at $x = x_0$ and is independent of t. Since by (3.4.1)

(3.8.7) $$\left(\Delta - \frac{\partial}{\partial t}\right) e_s(f) \geq 0 ,$$

the strong maximum principle implies that $e_s(f)(x,t,s)$ is independent of x and t, i.e.

$$e_s(f)(x,t,s) = e_s(f)(x_0,0,s) \qquad \text{for all } s .$$

Since s is the arc length parameter on the geodesic $f(x_0,\cdot)$, $e_s(f)(x_0,0,s)$ and hence $e_s(f)(x,t,s)$ is also independent of s. Thus for every x and t, $f(x,t,\cdot)$ is a curve of equal length from $f_1(x)$ to $f_2(x)$ parametrized proportionally to arc length. Since $f(x,0,\cdot)$ was a minimal geodesic, all $f(x,t,\cdot)$ are minimal geodesics and independent of t. In particular $f(x,t,s)$ is time independent for every s, and hence $f(x,0,s) = f(x,s)$ is harmonic, since $f(x,t,s)$ solves (3.1.1).

Returning to (3.4.1), since $g_{ij} \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial s}$ is constant and Y is nonpositively curved,

(3.8.8) $$\frac{\partial^2 f^i}{\partial x^\alpha \partial s} = 0$$

in normal coordinates, or in invariant notation

$$\nabla_{\frac{\partial}{\partial s}} \left(\frac{\partial f^i}{\partial x^\alpha} \frac{\partial}{\partial f^i}\right) = 0$$

where ∇ now is the covariant derivative in the bundle $f^{-1}(x,\cdot)TY$. This implies that the energy density

$$e(f)(x,s) = \gamma^{\alpha\beta}(x)\; g_{ij}(f,x,s))\; \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta}$$

is independent of s.

In particular, all the harmonic maps $f(\cdot,s)$ have the same energy.

q.e.d.

THEOREM 3.8.2 (Hartman [Ht]) *If* Y *has negative sectional curvature, then a harmonic map* $f : X \to Y$ *is unique in its homotopy class, unless it is constant or maps* X *onto a closed geodesic. In the latter case, nonuniqueness can only occur by rotations of this geodesic.*

Proof In this case, we see from (3.4.1), that since

$$R_{ikj\ell} \frac{\partial f^i}{\partial s} \frac{\partial f^k}{\partial x^\alpha} \frac{\partial f^i}{\partial s} \frac{\partial f^\ell}{\partial x^\alpha} = 0 \tag{3.8.9}$$

by the previous proof, either $\frac{\partial f}{\partial s} \equiv 0$ which means that the family $f(\cdot,x)$ is constant in s and hence consists of a single member, i.e. the harmonic map is unique, or the image of T_xS under df is a one-dimensional subspace of $T_{f(x)}Y$. Furthermore, if the harmonic map is not unique, then $f(x,s)$ for any $x \in X$ is a geodesic arc by the construction of the preceding proof. (3.8.9) implies again that df maps T_xX onto the tangent direction of this geodesic. This easily implies that X is mapped onto this geodesic.

We now have to show that this geodesic arc extends to a closed geodesic which is covered by $f(X)$.

Since X is compact, $f(X)$ is closed and hence covers some geodesic arc γ. Suppose this arc has an endpoint $p = f(x)$ for some $x \in X$. We choose $q \in \gamma$ within the injectivity radius of p. Then $d^2(q,f(y))$ is a subharmonic function on a suitable neighbourhood of $x \in X$ (by Lemmata 2.3.2 and 1.7.1) and has a local maximum at x which is a contradiction, unless $f(y) \equiv p$ for $y \in X$. Thus, if f is not constant, it has to cover a closed geodesic.

q.e.d.

Different proofs of Thms. 3.8.1 and 3.8.2 were obtained by Schoen-Yau

[SY3]. They show that under the hypotheses of those theorems, after lifting to suitable covers, the squared distance between two homotopic harmonic maps is a well defined smooth subharmonic function, in case Y is nonpositively curved, from which the argument proceeds in a similar way as above.

3.9 THE DIRICHLET PROBLEM

One can also solve the Dirichlet problem for harmonic mappings into nonpositively curved manifolds.

THEOREM 3.9.1 (Hamilton [Hm]) *Suppose* X *is a compact manifold with nonempty boundary* ∂X *,* Y *is complete (without boundary) and has nonpositive sectional curvature. If* $g : X \to Y$ *is a continuous map, then the parabolic system*

$$(3.9.1) \qquad \begin{aligned} \frac{\partial f}{\partial t}(x,t) &= \tau(f)(x,t) && \text{for} \quad (x,t) \in X \times (0,\infty) \\ f(x,0) &= g(x) && \text{for} \quad x \in X \\ f(y,t) &= g(y) && \text{for} \quad y \in \partial X \end{aligned}$$

has a smooth solution $f(x,t)$ *for all* $t \to (0,\infty)$ *. As* $t \in \infty$ *,* $f(x,t)$ *converges to the unique harmonic map homotopic to* g *with the same boundary values as* g *on* ∂X .

Instead of extending the Hölder estimates of the previous section to the boundary, Hamilton develops an L^p-regularity theory for harmonic maps for the proof of Thm. 3.9.1. Since the boundary values are fixed, $f(x,t)$ remains always in a bounded subset of Y as $t \to \infty$, even if Y is noncompact.

In case Y is simply connected, a simpler proof of Thm. 3.9.1 was obtained by Hildebrandt-Kaul-Widman [HKW1].

As an application of the maximum principle, Hamilton also showed that convex sets provide barriers for solutions of the heat equation.

THEOREM 3.9.2 *Suppose* $C \subset Y$ *is a convex set and* f *solves (3.9.1). If* $g(X) \subset C$ *, then* $f(x,t) \subset C$ *for all* $t \in [0,\infty)$.

3.10 AN OPEN QUESTION

A difficult open problem is to determine whether a solution of the heat equation (3.1.1) or (3.9.1) exists for all $t > 0$ without any curvature assumptions on Y .

Since there are manifolds X and Y and homotopy classes in $[X,Y]$ which do not contain harmonic representatives, as we shall see in chapter 5, even if the solution of the heat equation exists for all $t > 0$, in general it cannot converge uniformly to a harmonic map as $t \to \infty$.

There seems to be some indication that if one maps the unit ball D^n homotopically nontrivial onto the sphere S^n with constant boundary values, then the solution of (3.9.1) may cease to exist after a finite time, at least for large n .

Besides the results of this chapter and the case of "warped products" (cf. Lemaire [L3]), the existence of a solution of (3.9.1) for all time is only known in case $g(X)$ is contained in a ball $B(p,M) \subset Y$ which is disjoint to the cut locus of its centre p with $M < \frac{\pi}{2\kappa}$, where κ^2 is an upper bound for the sectional curvature on $B(p,M)$. This was carried out in [J4], combining some arguments of the present chapter with a result from elliptic regularity theory as shown in the next chapter and a stability inequality of (JäK2] analogous (but more difficult) to 3.4. A more general approach to long-time existence of solutions of nonlinear parabolic systems without divergence or variational structure by using stability inequalities was developed by von Wahl [vW]. For arbitrary Y , however, such stability inequalities do not hold, and von Wahl's approach is mainly aiming at applications different from harmonic maps.

Simon [Sm] showed that if f is a locally energy minimizing map between real analytic manifolds, then a solution of (3.1.1) exists for all time and converges to a harmonic map with the same energy as f, provided the initial values are already close to f in some high C^k-norm. It is not known whether the assumption that the manifolds involved are real analytic is necessary for Simon's theorem.