THE ITERATED GALERKIN METHOD FOR INTEGRAL EQUATIONS OF THE SECOND KIND

I.H. Sloan

1. INTRODUCTION

Consider the integral equation of the second kind

$$(1.1)\qquad y(t) = f(t) + \int_{\Omega} k(t,s)y(s)d\sigma(s) \ , \quad t \in \Omega \ ,$$

where Ω is either a bounded domain in $\mathbb{R}^d$ with a locally Lipschitz boundary or the smooth d -dimensional boundary of a bounded domain in $\mathbb{R}^{d+1}$, and $d\sigma(s)$ is the element of volume or surface area, as appropriate. Writing the equation as

$$(1.2)\qquad y = f + Ky \ ,$$

we shall assume that for each p in $1 \le p \le \infty$ K is a compact linear operator in L_p, $f \in L_p$, and the corresponding homogeneous equation has no non-trivial solution in L_p. It follows then from the Fredholm theorem that a (unique) solution $y \in L_p$ exists for each $f \in L_p$.

The Galerkin method, in which an approximate solution y_h is sought in a finite-dimensional space $S_h \subset L_\infty$ (see Section 2 for details), is a well understood numerical method for the solution of (1.1). Here we are more concerned with the iterated variant of the Galerkin method, i.e. with the approximation $y_h^{(1)}$ obtained by substituting the Galerkin approximation y_h into the right-hand side of the integral equation, giving

(1.3) $$y_h^{(1)} = f + Ky_h .$$

Higher iterates may be defined by

$$y_h^{(i+1)} = f + Ky_h^{(i)} , \qquad i = 1,2, \dots .$$

It is by now well known that the approximation $y_h^{(1)}$ often converges to the exact solution faster than y_h — that is, it exhibits 'superconvergence'. The earliest results of this kind [7, 8] did not give quantitative estimates for the improvement in the rate of convergence, but quantitative results are now available for many cases, including smooth kernels and Green's function kernels in one dimension [1, 2, 3], weakly singular kernels in one dimension [4, 5], and smooth kernels in higher dimensions [6]. In all of these cases the approximating spaces S_h were assumed to be of finite-element character, and that assumption will be continued here.

Recently, superconvergence results have been obtained by V. Thomée and myself [10] for some relatively formidable integral equations in higher dimensions. This paper gives a brief introduction to the methods and results of [10], with the emphasis on clarity and brevity rather than on generality or completeness.

2. SUPERCONVERGENCE THEORY

In this section p is a fixed number in $1 \le p \le \infty$, and q is the conjugate index defined by $1/p + 1/q = 1$, with $1/\infty = 0$.

The first step is to define the finite-dimensional space S_h . The details of the construction are not important here, but S_h is assumed to have an approximation property typical of piecewise-polynomial spaces of degree $\le r-1$, where r is a fixed positive integer. Specifically, letting h be the maximum diameter of a sub-region, S_h is assumed to be such that

$$\inf_{\chi \in S_h} \|g - \chi\|_{L_q} \le ch^s \|g\|_{W_q^s} , \qquad s = 0, \ldots, r , \tag{2.1}$$

for all g in the Sobolev space W_q^s. (In this paper c denotes a generic constant, which may take different values in different places, but which is always independent of h and of functions such as g.)

The Galerkin approximation y_h belongs to S_h, and satisfies

$$(y_h - f - Ky_h, \chi) = 0 \quad \text{for all} \quad \chi \in S_h , \tag{2.2}$$

where $(\cdot,\cdot)$ is the inner product

$$(u,v) = \int_\Omega u(s)\overline{v(s)}d\sigma(s) .$$

Letting P_h denote the L_2 projection onto S_h with respect to this inner product, we shall assume that $\|P_hK - K\|_{L_p} \to 0$ as $h \to 0+$. Then it is well known that the Galerkin approximation y_h exists and is unique for h sufficiently small. Thus y_h and the iterated Galerkin approximation $y_h^{(1)}$ defined by (1.3) are well defined.

In the following theorem K^* is the adjoint integral operator defined by

$$K^*v(t) = \int_\Omega \overline{k(s,t)}v(s)d\sigma(s) , \qquad t \in \Omega .$$

The theorem, which is a simplified version of a result stated in [10], links the superconvergence of $y_h^{(1)}$ to the smoothing properties of K^*.

THEOREM 1. *Assume that*

$$\|K^*v\|_{W_q^\ell} \le c\|v\|_{L_q} ,$$

for some ℓ *in* $0 \le \ell \le r$. *Then*

$$\|y_h^{(1)} - y\|_{L_p} \le ch^\ell \|y_h - y\|_{L_p} .$$

PROOF. From (1.2) and (1.3) we have

$$y_h^{(1)} - y = K(y_h - y) ,$$

and hence

$$\|y_h^{(1)} - y\|_{L_p} = \|K(y_h - y)\|_{L_p} = \sup_{v \in L_q} \frac{|(K(y_h - y), v)|}{\|v\|_{L_q}} \tag{2.3}$$

$$= \sup_{v \in L_q} \frac{|((y_h - y), K^*v)|}{\|v\|_{L_q}} \le \sup_{v \in L_q} \frac{\|y_h - y\|_{W_p^{-\ell}} \|K^*v\|_{W_q^\ell}}{\|v\|_{L_q}}$$

$$\le c\|y_h - y\|_{W_p^{-\ell}} ,$$

where

$$\|g\|_{W_p^{-\ell}} = \sup_{w \in W_q^\ell} \frac{|(g,w)|}{\|w\|_{W_q^\ell}} .$$

It remains to show that $\|y_h - y\|_{W_p^{-\ell}}$ has a suitable fast-convergence property. For $w \in W_q^\ell$ we have

$$(y_h-y,w) = ((I-K)(y_h-y),(I-K^*)^{-1}w)$$

$$= ((I-K)(y_h-y),(I-K^*)^{-1}w-\chi) ,$$

where χ is an arbitrary element of S_h, with the last step following from the defining property (2.2) for the Galerkin method. Then using the Hölder inequality we have

$$(2.4) \qquad |(y_h-y,w)| \le \|(I-K)(y_h-y)\|_{L_p} \inf_{\chi\in S_h} \|(I-K^*)^{-1}w-\chi\|_{L_q}$$

$$\le c\|y_h-y\|_{L_p} \; h^\ell \|(I-K^*)^{-1}w\|_{W_q^\ell}$$

$$\le ch^\ell \|y_h-y\|_{L_p} \|w\|_{W_q^\ell} ,$$

where in the second-last step we have used the approximation property (2.1), and in the last step the fact that $(I-K^*)^{-1}$ is a bounded operator in the space W_q^ℓ. To show the latter, first observe that the operator $I-K$ is one-to-one in L_p, from which it follows, using the Fredholm theorem, that $I-K^*$ is one-to-one in L_q, and hence in W_q^ℓ. By the assumption in the theorem, $I-K^*$ maps W_q^ℓ into itself. To show that this map is onto, let $v \in W_q^\ell$. Then $v \in L_q$, and from the Fredholm alternative in L_q there exists $z \in L_q$ such that $(I-K^*)z = v$. But $z = v+K^*z$, which belongs to W_q^ℓ because K^* maps L_q to W_q^ℓ. Thus $I-K^*$ has a bounded inverse in W_q^ℓ.

It follows from (2.4) that

$$\|y_h-y\|_{W_p^{-\ell}} \le ch^\ell \|y_h-y\|_{L_p} ,$$

and hence, from (2.3), that

$$\|y_h^{(1)}-y\|_{L_p} \le ch^{\ell}\|y_h-y\|_{L_p} . \qquad \square$$

Similar arguments are used in [10] to obtain a superconvergence result for $\|y_h^{(i)}-y\|_{W_p^m}$, $m \ge 0$, $i \ge 1$.

3. AN EXAMPLE

Consider the integral equation

$$(3.1) \qquad y(t) = f(t) + \lambda \int_0^1 \log|t-s|\, y(s)\,ds , \quad t \in [0,1] ,$$

with λ not a characteristic value of the equation, and $f \in W_\infty^1$. Here K is the integral operator

$$Kv(t) = \lambda \int_0^1 \log|t-s|\, v(s)\,ds ,$$

and $K^* = K$. In considering this example we shall assume that the approximation property (2.1) holds for all q in $1 \le q \le \infty$, with c independent of q .

Letting D denote the (weak) derivative, DK is the principal-value integral operator,

$$DKv(t) = \lambda \text{ p.v.} \int_0^1 \frac{1}{t-s}\, v(s)\,ds .$$

If q is any fixed number satisfying $1 < q < \infty$, it is known, from a celebrated theorem of M. Riesz, that the principal-value integral operator is a bounded operator from L_q to L_q . It follows that

$$(3.2) \qquad \|K^*v\|_{W_q^1} = \|Kv\|_{W_q^1} \le c\|v\|_{L_q} ,$$

so that Theorem 1 can be applied with $\ell = 1$. We conclude that for this example

$$\|y_h^{(1)} - y\|_{L_p} \leq ch\|y_h - y\|_{L_p} , \tag{3.3}$$

if p is any fixed number satisfying $1 < p < \infty$. A similar application of the more general theorem in [10] yields, for the W_p^1 norm of the error,

$$\|y_h^{(1)} - y\|_{W_p^1} \leq c\|y_h - y\|_{L_p} . \tag{3.4}$$

The above argument breaks down if $p = \infty$, because then $q = 1$, and the result of M. Riesz does not hold. However, in [10] it is shown that a similar L_∞ estimate can be obtained, at the expense of introducing a logarithmic factor: one obtains

$$\|y_h^{(1)} - y\|_{L_\infty} \leq ch \log \frac{1}{h} \|y_h - y\|_{L_\infty} . \tag{3.5}$$

Briefly, the argument is as follows. With Ω an interval, as in the present example, it is known that for large p

$$\|v\|_{L_\infty} \leq c\|v\|_{L_p}^{1-1/p}\|v\|_{W_p^1}^{1/p} ,$$

with c independent of p . Thus

$$\|y_h^{(1)} - y\|_{L_\infty} \leq c\|y_h^{(1)} - y\|_{L_p}^{1-1/p}\|y_h^{(1)} - y\|_{W_p^1}^{1/p} .$$

Now we can use (3.3) and (3.4), the only catch being that the constants in those expressions are not independent of p . An examination of the constant in the M. Riesz theorem shows that the constant in (3.2), and hence also the constants in (3.3) and (3.4), grow proportionally to p for large p, thus we obtain

$$\|y_h^{(1)}-y\|_{L_\infty} \le cph^{1-1/p}\|y_h-y\|_{L_p} \le cph^{1-1/p}\|y_h-y\|_{L_\infty}$$

for large p, with c independent of p. The result (3.5) now follows on setting $p = \log(1/h)$.

Similar arguments are used in [10] to obtain an L_∞ error estimate for the Galerkin approximation itself: it is shown that

$$\|y_h-y\|_{L_\infty} \le ch \log \frac{1}{h} \|f\|_{W^1_\infty},$$

provided P_h is bounded in L_∞ uniformly in h. Thus finally we conclude for this example that

$$\|y_h^{(1)}-y\|_{L_\infty} \le ch^2\log^2 \frac{1}{h} \|f\|_{W^1_\infty}. \qquad (3.6)$$

The examples discussed in [10] include the case of a logarithmic kernel in two dimensions, and the single-layer and double-layer operators over a smooth boundary which arise in the boundary-integral method for the solution of the Laplace equation in $\mathbb{R}^3$. A summary of the results for these examples, without any theory, has been given in another recent report [9].

REFERENCES

[1] G.A. Chandler, Global superconvergence of iterated Galerkin solutions for second kind integral equations, Technical Report, Australian National University, 1978.

[2] G.A. Chandler, Superconvergence for second kind integral equations,

in R.S. Anderssen, F.R. de Hoog and M.A. Lukas (Eds.), *The application and numerical solution of integral equations*, Sijthoff and Noordhoff, Alphen aan den Rijn, 1980, pp. 103-117.

[3] G.A. Chandler, Superconvergence of numerical solutions to second kind integral equations, Ph.D. Thesis, Australian National University, 1979.

[4] I.G. Graham, The numerical solution of Fredholm integral equations of the second kind, Ph.D. Thesis, University of New South Wales, 1980.

[5] I.G. Graham, Galerkin methods for second kind integral equations with singularities, Math. Comp. 39 (1982) 519-533.

[6] G.C. Hsiao and W.L. Wendland, The Aubin-Nitsche lemma for integral equations, J. Integral Eqns. 3 (1981) 299-315.

[7] I.H. Sloan, Error analysis for a class of degenerate-kernel methods, Numer. Math. 25 (1976) 231-238.

[8] I.H. Sloan, Improvement by iteration for compact operator equations, Math. Comp. 30 (1976) 758-764.

[9] I.H. Sloan, Fast convergence of the iterated Galerkin method for integral equations, in Proceedings of the CTAC-83 Conference, Sydney, August 1983, to appear.

[10] I.H. Sloan and V. Thomée, Superconvergence of the Galerkin iterates for integral equations of the second kind, Technical Report, University of New South Wales, 1983.

School of Mathematics
University of New South Wales
Kensington NSW 2033
AUSTRALIA