A LINEARIZED ELLIPTIC FREE BOUNDARY VALUE PROBLEM

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This is a report on joint work with John van der Hoek. We consider the flow of an irrotational inviscid and incompressible fluid under a thin body of convex plan form at a non-uniform small clearance from a plane ground surface. The problem is relevant to vehicle aero-dynamics, especially for racing cars. It was brought to our attention by E.O. Tuck who considered certain aspects of the problem in [3].

Following Tuck we take the body to be fixed and the flow to have a uniform velocity at infinity of U in the positive x-direction. The plan form of the body is assumed to be a bounded convex domain Ω in $\mathbb{R}^2$ which is symmetric with respect to the x-axis and has a smooth boundary $\partial\Omega$.

For each point $q \in \partial\Omega$ let $\theta = \theta(q)$ denote the angle measured in the anticlockwise direction between the positive x-axis and the outward unit normal $\nu = \nu(q)$ at q, with $-\pi \leq \theta(q) \leq \pi$. See the diagram.

The leading and trailing edges of Ω determined by the transition points $p = (a, b)$ and $\bar{p} = (a, -b)$ in $\partial\Omega$ are the sets

$$\Gamma_L(p) = \{q \in \partial\Omega : |\theta(q)| \geq |\theta(p)|,\ q \neq p \text{ or } \bar{p}\} \qquad \text{and}$$

$$\Gamma_T(p) = \{q \in \partial\Omega : |\theta(q)| \leq |\theta(p)|,\ q \neq p \text{ or } \bar{p}\} .$$

The distance between the body and the ground surface at the point $(x, y) \in \bar{\Omega}$ is $h(x, y)$. We assume that h is a positive smooth function on $\overline{\Omega}$, symmetric about the x-axis. Let ϕ be the velocity potential of the flow at the ground surface.

The problem reduces to the study of the following mixed free-boundary value problem:

Find $p \in \partial\Omega$, $\phi \in C^1(\overline{\Omega})$ such that

(1) $$\text{div } (h \text{ grad } \phi) = 0 \quad \text{in} \quad \Omega ,$$

(2) $$\phi = Ux \quad \text{on} \quad \Gamma_L(p) ,$$

(3) $$|\text{grad } \phi| = U \quad \text{on} \quad \Gamma_T(p) ,$$

with the supplementary condition

(4) $$\frac{\partial}{\partial\nu} (Ux + \phi) = 0 \quad \text{at} \quad p \quad \text{and} \quad \overline{p} .$$

Under the assumption that this problem has a solution, Tuck investigated the position of the transition points p, $\overline{p}$ relative to the lateral extremities of Ω. In the special case of an exponentially increasing clearance and a circular plan form, he found numerically that $|\theta(p)| < \frac{\pi}{2}$.

In this paper we consider a corresponding linearized mixed boundary value problem. Indeed we set $\phi = Ux + \Psi$ and assume $|\text{grad } \Psi| << U$. Under this approximation, we must dispense with the supplementary condition (4) and treat $p \in \partial\Omega$ as a parameter. Equations (1), (2), (3) become

(5) $$\text{div } (h \text{ grad } \Psi) = -U \frac{\partial h}{\partial x} \quad \text{in} \quad \Omega ,$$

(6) $\Psi = 0$ on $\Gamma_L(p)$,

(7) $\dfrac{\partial \Psi}{\partial x} = 0$ on $\Gamma_T(p)$.

We consider questions of existence, uniqueness and regularity of solutions of (5), (6), (7) in Sobolev spaces $H^s(\Omega)$. For this and other notation, see for example Lions and Magenes [1] .

Define a properly elliptic operator Au by $Au = \text{div}\ (h\ \text{grad}\ u)$. For real s let $H^s_A(\Omega)$ denote the space of $u \in H^s(\Omega)$ for which $Au \in L^2(\Omega)$, together with the graph norm. The trace maps $\left(\dfrac{\partial}{\partial \nu}\right)^j$ on smooth functions on $\overline{\Omega}$, $j = 0, 1, 2 \ldots$, extend by continuity to bounded operators

$$\gamma_j : H^s_A(\Omega) \to H^{s-j-\frac{1}{2}}(\partial\Omega) .$$

For $u \in H^s_A(\Omega)$ let $\gamma_L u$ denote the restriction to $\Gamma_L = \Gamma_L(p)$ of $\gamma_0 u$, $\gamma_T u$ the restriction to $\Gamma_T = \Gamma_T(p)$ of $\gamma_1 u$, and $B_T u$ the restriction to Γ_T of $\gamma_0 \dfrac{\partial u}{\partial x}$.

Related to the problem (5), (6), (7) is the operator

$$(A, \gamma_L, B_T) : H^s_A(\Omega) \to L^2(\Omega) \times H^{s-\frac{1}{2}}(\Gamma_L) \times H^{s-1\frac{1}{2}}(\Gamma_T)$$

for which we have the following result.

THEOREM 1 If $\frac{1}{2} + \frac{1}{\pi}|\theta(p)| < s < 1\frac{1}{2}$ then (A, γ_L, B_T) is Fredholm of index 0 . If in addition $|\theta(p)| \le \frac{\pi}{2}$ then (A, γ_L, B_T) is an isomorphism.

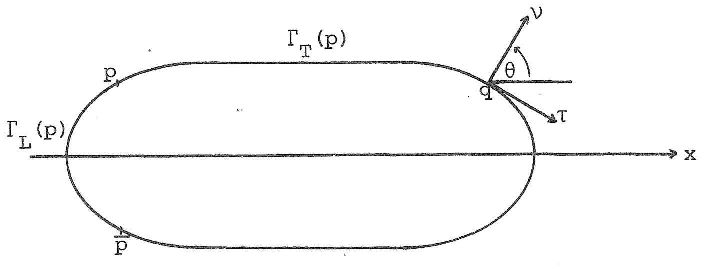
COROLLARY If $\frac{1}{2} + \frac{1}{\pi}|\theta(p)| < s < 1\frac{1}{2}$ and $|\theta(p)| \le \frac{\pi}{2}$ then problem (5), (6), (7) has a unique solution in $H^s(\Omega)$.

The theorem is proved by constructing a homotopy from (A, γ_L, B_T) to (A, γ_L, γ_T) in the space of semi-Fredholm operators from $H^s_A(\Omega)$ to $L^2(\Omega) \times H^{s-\frac{1}{2}}(\Gamma_L) \times H^{s-1\frac{1}{2}}(\Gamma_T)$. The index of (A, γ_L, B_T) is then equal to the index of (A, γ_L, γ_T) which can be calculated using the Lax-Milgram theorem and results of Shamir [2]. Uniqueness when $|\theta(p)| \leq \frac{\pi}{2}$ is a consequence of the Hopf maximum principle. So is the following result, which relates to a conjecture of Tuck [3] that for the linearized problem (5), (6), (7) to have a $C^1(\overline{\Omega})$ solution, it is necessary that $|\theta(p)| = \frac{\pi}{2}$, that is the transition points $p, \overline{p}$ must lie at the lateral extremities of Ω .

<u>THEOREM 2</u> If (5), (6), (7) has a solution $\Psi \in C^1(\overline{\Omega})$, $\frac{\partial h}{\partial x} \geq 0$ on Ω (or $\frac{\partial h}{\partial x} \leq 0$ on Ω) and $h \not\equiv 0$ then $|\theta(p)| > \frac{\pi}{2}$.

Finally we have the following regularity theorem, proved by localizing the problem. Recall also that, by the Sobolev imbedding theorem, if $1 < s < 1\frac{1}{2}$ then $H^{s+1}(\Omega) \subset C^{1,s-1}(\overline{\Omega})$.

<u>THEOREM 3</u> If $1 < s < 1\frac{1}{2}$ and $|\theta(p)| = \frac{\pi}{2}$ then the solution $\Psi \in H^s(\Omega)$ of (5), (6), (7) belongs to $H^{s+1}(\Omega)$.



Diagram

REFERENCES

[1] Lions, J.L. and Magenes, E., *"Non-homogeneous boundary value problems and applications"*, Vol I, Springer Verlag, Berlin, Heidelberg, New York, 1972.

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