SQUARE ROOTS OF OPERATORS AND APPLICATIONS TO HYPERBOLIC P.D.E.'s

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INTRODUCTION

Throughout this paper $\mathcal{H}$ denotes a complex Hilbert space and $\mathcal{V}$ denotes a dense subspace, also with a Hilbert space structure, which is continuously embedded in $\mathcal{H}$. The two norms are denoted $\|\cdot\|$ and $\|\cdot\|_{\mathcal{V}}$.

For each $t \in [0,t_1]$, J_t denotes a sesquilinear form with domain $\mathcal{V} \times \mathcal{V}$ which satisfies

$$0 \le J_t[u,u] \quad , \text{ and}$$

$$\kappa \|u\|_{\mathcal{V}}^2 \le J_t[u,u] + \|u\|^2 \le M\|u\|_{\mathcal{V}}^2$$

for all $u \in \mathcal{V}$, where κ and M are positive numbers, independent of t and u.

The associated operators T_t are the operators with largest domains satisfying

$$J_t[u,v] = (T_t u,v)$$

for all $v \in \mathcal{V}$. They are non-negative self-adjoint operators and have non-negative square roots $T_t^{\frac{1}{2}}$ with domains $\mathcal{D}(T_t^{\frac{1}{2}}) = \mathcal{V}$. Indeed

$$J_t[u,v] = (T_t^{\frac{1}{2}}u, T_t^{\frac{1}{2}}v)$$

for all u and v in $\mathcal{V}$. See [7] for details.

If the forms J_t depend differentiably on t, we would like to know whether this same property is inherited by the operators $T_t^{\frac{1}{2}}$. In general it is not, as was shown in [8]. The forms constructed there are actually of the type $J_t = J_0 + tK$, with $J_0[u,u] \geq \|u\|^2$ and $|K[u,u]| \leq J_0[u,u]$ for all $u \in V$. The associated square roots $T_t^{\frac{1}{2}}$, as functions from $(-1,1)$ to $L(V,H)$, are not weakly differentiable at $t=0$.

The above question was raised by Kato in connection with the approach to second order evolution equations illustrated in the next section. In particular he asked whether positive results could be obtained for elliptic forms. The simplest such case occurs when $H = L_2(\mathbb{R})$, $V = H^1(\mathbb{R})$, and

$$J_t[u,v] = \int_{-\infty}^{\infty} a_t(x)u'(x)\,\overline{v'(x)}\,dx\,,$$

with $a_t \in L_\infty(\mathbb{R})$ and $\kappa \leq a_t(x) \leq M$ for each $t \in [0,t_1]$. It was suggested in [8] that this question is related to that of Calderón concerning the problem of showing the L_2-boundedness of the Cauchy integral on a Lipschitz curve. Indeed this turned out to be the case. It can be proved using the methods of [2] and [3] that if the functions $a_t \in C^m([0,t_1],L_\infty(\mathbb{R}))$ for some m, then $T_t^{\frac{1}{2}} \in C^m([0,t_1],L(V,H))$.

In the case when $H = L_2(\mathbb{R}^n)$, $V = H^1(\mathbb{R}^n)$, and

$$J_t[u,v] = \int_{\mathbb{R}^n} \sum_{j,k} a_{t,jk}(x)\,\frac{\partial u}{\partial x_k}(x)\,\overline{\frac{\partial v}{\partial x_j}(x)}\,dx\,,$$

with $a_{t,jk} \in L_\infty(\mathbb{R}^n)$ for each t, j, k, and

$$\kappa|\underset{\sim}{\zeta}|^2 \leq \sum a_{t,jk}\,\zeta_k\,\bar{\zeta}_j \leq M|\underset{\sim}{\zeta}|^2$$

for all $\underset{\sim}{\zeta} \in \mathbb{C}^n$, the corresponding result is known only when $|a_{t,jk} - \delta_{jk}| < \varepsilon$ for some ε depending on n (c.f. [1], [4]). It is not known whether or not it is true without this restriction.

This paper is concerned with forms associated with the Dirichlet problem on domains Ω in $\mathbb{R}^n$. It will be indicated how results of the above type can be obtained under mild regularity conditions on the coefficients and their derivatives with respect to t. The proofs are not as deep as those needed for L_∞-coefficients.

Full proofs of the theorems stated in this paper, and also of related results, will be published elsewhere. See [9].

AN APPLICATION

As an application involving the differentiability of $T_t^{\frac{1}{2}}$, the following method of treating second order evolution equations by reduction to first order evolution equations is presented. No attempt at maximum generality is made. Indeed the first order equations are treated using the pioneering work of Kato [6], published in 1953. Note however that the conclusion that $u(t) \in \mathcal{D}(T_t)$ for all t is quite strong, as $\mathcal{D}(T_t)$ may vary considerably with t.

<u>THEOREM</u> *Consider the initial value problem*

$$(*) \qquad \begin{cases} \dfrac{d^2u}{dt^2}(t) + T_t u(t) + F_t u(t) = f(t) , \quad 0 \le t \le t_1 , \\ u(0) = v , \\ \dfrac{du}{dt}(0) = v_1 . \end{cases}$$

Assume, in addition to the properties already specified, that $T_t \ge I$, $F_t \in L(V,H)$, $f(t) \in C([0,t_1], V)$, $v \in \mathcal{D}(T_0)$ *and* $v_1 \in V$. *Suppose, for each* $w \in V$, *that* $T_t^{\frac{1}{2}} w \in C^2([0,t_1], H)$ *and* $F_t w \in C^1([0,t_1], H)$.

Then there exists a unique solution $u(t)$ *of* $(*)$, *such that* $u(t) \in \mathcal{D}(T_t)$ *for all* t, $T_t u(t) \in C([0,t_1], H)$ *and*

$$u(t) \in C^2([0,t_1], H) \cap C^1([0,t_1], V) .$$

Proof Let

$$\underset{\sim}{v}(t) = \begin{bmatrix} \frac{du}{dt}(t) \\ T_t^{\frac{1}{2}} u(t) \end{bmatrix} , \quad \text{and}$$

$$A_t = \begin{bmatrix} 0 & T_t^{\frac{1}{2}} \\ -T_t^{\frac{1}{2}} & 0 \end{bmatrix} + \begin{bmatrix} 0 & F_t T_t^{-\frac{1}{2}} \\ 0 & -(\frac{d}{dt} T_t^{\frac{1}{2}}) T_t^{-\frac{1}{2}} \end{bmatrix} .$$

Then (*) becomes

$$(**) \qquad \begin{cases} \frac{d\underset{\sim}{v}}{dt}(t) + A_t \underset{\sim}{v}(t) = \underset{\sim}{f}(t) \\ \underset{\sim}{v}(0) = \underset{\sim}{v} \end{cases}$$

where

$$\underset{\sim}{f}(t) = \begin{bmatrix} f(t) \\ 0 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{v} = \begin{bmatrix} v_1 \\ T_0^{\frac{1}{2}} v \end{bmatrix} .$$

For suitable λ , $A_t + \lambda I$ are maximal accretive operators in $\underset{\sim}{H} = H \oplus H$ with domain $\underset{\sim}{V} = V \oplus V$. For $\underset{\sim}{w} \in \underset{\sim}{V}$, $A_t \underset{\sim}{w} \in C^1([0,t_1] , H)$ and $\underset{\sim}{f}(t) \in C([0,t_1] , \underset{\sim}{V})$. Also $\underset{\sim}{v} \in \underset{\sim}{V}$. It follows from standard results [6] that (**) has a unique solution

$$\underset{\sim}{v}(t) \in C^1([0,t_1] , \underset{\sim}{H}) \cap C([0,t_1] , \underset{\sim}{V}) .$$

On converting back to the original problem, we find that the theorem is proved. //

On letting T_t denote elliptic operators as specified in the next section, and F_t first order operators, results on hyperbolic partial differential equations are obtained.

Similar results under stronger conditions are presented in [11].

THE MAIN RESULT

Let $\mathcal{H} = L_2(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$, and let $\mathcal{V} = \mathring{H}^1(\Omega)$, the closure of $C_0^\infty(\Omega)$ in the Sobolev space $H^1(\Omega)$ with norm

$$\|u\|_{\mathcal{V}} = \|u\|_1 = \left\{\|u\|^2 + \sum_{j=1}^{n} \left\|\frac{\partial u}{\partial x_j}\right\|^2\right\}^{\frac{1}{2}} .$$

Consider the forms J_t defined on $\mathcal{V} \times \mathcal{V}$ by

$$J_t[u,v] = \int_\Omega \left\{\sum_{j,k} a_{t,jk}(x) \frac{\partial u}{\partial x_k}(x) \overline{\frac{\partial v}{\partial x_j}}(x) + a_t(x)\, u(x)\, \overline{v(x)}\right\} dx ,$$

where $a_{t,jk}$, $a_t \in L_\infty(\Omega)$ for each $t \in [0,t_1]$, and

$$\begin{cases} \kappa|\underset{\sim}{\zeta}|^2 \le \sum_{j,k} a_{t,jk}(x)\, \zeta_k \bar{\zeta}_j \le M|\underset{\sim}{\zeta}|^2 , & \text{and} \\ \kappa \le a_t(x) \le M , & \end{cases}$$

for all $x \in \Omega$, $\underset{\sim}{\zeta} \in \mathbb{C}^n$, and some M , $\kappa > 0$. Then these forms have the properties specified at the beginning of the paper. The corresponding operators T_t are defined by

$$T_t u = - \sum_{j,k} \frac{\partial}{\partial x_j} \left(a_{t,jk} \frac{\partial u}{\partial x_k}\right) + a_t\, u ,$$

where

$$u \in \mathcal{D}(T_t) = \{u \in \mathring{H}^1(\Omega) \,|\, T_t u \in L_2(\Omega)\} .$$

In order to proceed, we make some regularity assumptions on the region Ω and the coefficients $a_{t,jk}$. Let us first define the fractional order Sobolev spaces by quadratic interpolation.

$$H^s(\Omega) = \begin{cases} [H^1(\Omega) \, , \, H^2(\Omega)]_{s-1} \, , & 1 < s < 2 \, , \\ [L_2(\Omega) \, , \, H^1(\Omega)]_s \, , & 0 < s < 1 \, , \\ [(\mathring{H}^1(\Omega))^* \, , \, L_2(\Omega)]_{s+1} \, , & -1 < s < 0 \, . \end{cases}$$

For $0 < s < \frac{1}{2}$, we say that Ω has property (R_s) provided

$$\{f \in \mathring{H}^1(\Omega) \,|\, \nabla^2 f \in H^{-1+s}(\Omega)\} \subset H^{1+s}(\Omega) \, .$$

This property holds, for example, if Ω is a strongly Lipschitz bounded domain, i.e., if $\Omega \in N^{0,1}$, as can be shown on applying the results of [5].

The assumptions on $a_{jk,t}$ are made in terms of them being pointwise multipliers of $H^s(\Omega)$. We denote the space of such multipliers by $M^s(\Omega)$. That is,

$$M^s(\Omega) = \{b \in L_\infty(\Omega) \,|\, bu \in H^s(\Omega) \quad \text{for all} \quad u \in H^s(\Omega)\} \, ,$$

and

$$\|b\|_{M^s} = \|b\|_\infty + \sup\{\|bu\|_{H^s} \,\big|\, \|u\|_{H^s} = 1\} \, .$$

These spaces are well understood. For example, if $\Omega \in N^{0,1}$, then $C^{0,t}(\bar{\Omega}) \subset M^s(\Omega)$ when $0 < s < t \le 1$, and also $\chi_{\Omega_0} \in M^s(\Omega)$ if $0 < s < \frac{1}{2}$, where χ_{Ω_0} is the characteristic function of $\Omega_0 = \Omega_1 \cap \Omega$ with $\Omega_1 \in N^{0,1}$ [10].

Let m be a non-negative integer.

<u>THEOREM</u> *Let* J_t *denote the forms defined above, and suppose that* Ω *is sufficiently regular that property* (R_s) *is satisfied for some* $s \in (0,\frac{1}{2})$. *Suppose also that,*

$$a_{t,jk} \in C^m([0,t_1] \, , \, M^s(\Omega))$$

for $1 \le j \le n$ *and* $1 \le k \le n$. *Then*

$$T_t^{\frac{1}{2}} \in C^m([0,t_1] \, , \, L(V,H)) \, .$$

In order to prove this theorem, a more abstract result is first presented.

A HILBERT SPACE RESULT

Let B_t denote self-adjoint operators on a Hilbert space K such that $\|B_t\| \le \rho < 1$ for $t \in [0,t_1]$, and let A denote a one-one closed operator from H to K with domain V and closed range R. Then the forms J_t defined by

$$J_t[u,v] = ((I-B_t)Au,Av)$$

have the properties specified at the beginning of the paper, with $\|u\|_V = \|Au\|$, and $T_t = A^*(I-B_t)A$. Let E denote the orthogonal projection of K onto R, and, for each $s \ge 0$, let $R^s = R \cap D(|A^*|^s)$, which is a Hilbert space with norm $\||A^*|^s u\|$. In particular $R^0 = R$. Let m be a non-negative integer.

<u>THEOREM</u> *Suppose for some* $s \in (0,1)$, *that*

$$\underset{\sim}{B}_t = EB_t\big|_R \in C^m([0,t_1], L(R)) \cap L(R^s)) .$$

Then

$$T_t^{\frac{1}{2}} \in C^m([0,t_1], L(V,H)) .$$

Indeed, if

$$\begin{cases} \|\dfrac{d^j}{dt^j}\underset{\sim}{B}_t u\| \le \lambda^j\rho\|u\| & , u \in R, \text{ and} \\ \||A^*|^s \dfrac{d^j}{dt^j}\underset{\sim}{B}_t u\| \le \mu_s\lambda^j\rho\||A^*|^s u\| & , u \in R^s, \end{cases}$$

for some μ_s *and* λ, *and for* $0 \le j \le m$, *then there exists* $\kappa_s > 0$ *such that*

$$\left\| \frac{d^m}{dt^m} T_t^{\frac{1}{2}} u \right\| \le \{\kappa_s \mu_s \lambda^m \rho (1-\rho)^{-(m+1)} + \delta_{m0}\} \|u\|_V .$$

<u>OUTLINE OF PROOF</u> Let U denote the partial isometry from H to K such that $A^* = U^*|A^*|$ and that the kernel $N(U^*) = R$. Note that U is one-one. Then

$$T_t = A^*(I-B_t)A$$
$$= U^*|A^*|(I-\underset{\sim}{B}_t)|A^*|U .$$
$$\therefore \quad T_t^{\frac{1}{2}} = U^*\{|A^*|(I-\underset{\sim}{B}_t)|A^*|\}^{\frac{1}{2}} U .$$

So

$$T_t^{\frac{1}{2}} u = 2\pi^{-1} U^* \int_0^\infty \{I + \tau^2 |A^*|(I-\underset{\sim}{B}_t)|A^*|\}^{-1} |A^*|(I-\underset{\sim}{B}_t)|A^*| Uu \, d\tau$$

for $u \in D(T_t)$ [7]. On expanding as a power series in $\underset{\sim}{B}_t$ and ignoring problems about domains, we obtain

$$T_t^{\frac{1}{2}} = |A| - 2\pi^{-1} U^* \int_0^\infty \frac{\tau|A^*|}{1+\tau^2|A^*|^2} \underset{\sim}{B}_t \frac{1}{1+\tau^2|A^*|^2} \frac{d\tau}{\tau} A$$

$$- 2\pi^{-1} U^* \int_0^\infty \sum_{k=1}^\infty \frac{\tau|A^*|}{1+\tau^2|A^*|^2} \underset{\sim}{B}_t \left\{ \frac{\tau^2|A^*|^2}{1+\tau^2|A^*|^2} \underset{\sim}{B}_t \right\}^k \frac{1}{1+\tau^2|A^*|^2} \frac{d\tau}{\tau} A .$$

The (formal) expression for the derivative of $T_t^{\frac{1}{2}}$ can easily be derived. We will choose one term in its expansion and show how it can be estimated. Let

$$W_t = \int_0^\infty \frac{\tau|A^*|}{1+\tau^2|A^*|^2} \frac{d\underset{\sim}{B}_t}{dt} \frac{\tau^2|A^*|^2}{1+\tau^2|A^*|^2} \underset{\sim}{B}_t \frac{1}{1+\tau^2|A^*|^2} \frac{d\tau}{\tau} .$$

By assumption, there exist operators $B_{jst} \in L(R)$ such that

$$|A^*|^s \frac{d^j}{dt^j} \underset{\sim}{B}_t = B_{jst}|A^*|^s ,$$

and

$$\|B_{jst}\| \le \mu_s \lambda^j \rho .$$

Hence

$$W_t = \int_0^\infty \frac{\tau|A^*|}{1+\tau^2|A^*|^2} \frac{d\underset{\sim}{B}_t}{dt} \frac{\tau^{(2-s)}|A^*|^{(2-s)}}{1+\tau^2|A^*|^2} B_{0st} \frac{\tau^s|A^*|^s}{1+\tau^2|A^*|^2} \frac{d\tau}{\tau}$$

Now apply the following lemma [3].

<u>LEMMA</u>

$$\| \int_0^\infty S_\tau Z_\tau T_\tau \frac{d\tau}{\tau} \|$$

$$\le \| \int_0^\infty S_\tau S_\tau^* \frac{d\tau}{\tau} \|^{\frac{1}{2}} \| \int_0^\infty T_\tau^* T_\tau \frac{d\tau}{\tau} \|^{\frac{1}{2}} \sup_\tau \|Z_\tau\| ,$$

whenever S_τ , Z_τ *and* T_τ *are bounded operators which depend continuously on* τ , *and for which the operators on the right hand side exist in the strong topology.*

Note also that

$$\| \frac{\tau^\sigma|A^*|^\sigma}{1+\tau^2|A^*|^2} \| \le 1 \quad \text{if} \quad 0 \le \sigma \le 2 , \quad \text{and}$$

$$\| \int_0^\infty \left\{ \frac{\tau^\sigma|A^*|^\sigma}{1+\tau^2|A^*|^2} \right\}^2 \frac{d\tau}{\tau} \|^{\frac{1}{2}} = \left\{ \int_0^\infty \frac{\tau^{2\sigma}}{(1+\tau^2)^2} \frac{d\tau}{\tau} \right\}^{\frac{1}{2}} = c_\sigma < \infty$$

if $0 < \sigma < 2$. Therefore

$$\|W_t\| \le c_1 c_s (\lambda\rho)(\mu_s \rho) = c_1 c_s \mu_s \lambda \rho^2 .$$

On estimating the other terms and summing, we find that

$$\left\|\frac{d}{dt} T_t^{\frac{1}{2}} u\right\| \le 2\pi^{-1}\left\{c_{1-s}c_s\mu_s\lambda\rho + c_1 c_s \sum_{k=1}^{\infty} (k+1)\mu_s\lambda\rho^{k+1}\right\}\|Au\|$$

$$\le \kappa_s\mu_s\lambda\rho(1-\rho)^{-2}\|u\|_V .$$

The higher derivatives can be estimated in a similar way. Their continuous dependence on t poses no problem. //

Details concerning the above reasoning will be presented elsewhere. Note that the expansion of the square root as a power series is different from that used in [1], [2], [3] and [4], though the justification of the above reasoning is similar to that presented in [3]. Note also that the assumption that B_t be self-adjoint can be dropped.

PROOF OF THE MAIN RESULT

Let J_t denote the elliptic forms defined previously on $V = \mathring{H}^1(\Omega)$. Suppose, without loss of generality, that $M \le 1$. Let $\rho = 1-\kappa$, $K = \bigoplus_{n+1} L_2(\Omega)$, $A = \left\{I, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right\}^T$, and, for each $t \in [0,t_1]$, let B_t be the matrix of operators with components

$$\begin{cases} B_{t,00} = \text{multiplication by } b_t = 1 - a_t , \\ B_{t,jk} = \text{multiplication by } b_{t,jk} = \delta_{jk} - a_{t,jk} , \\ B_{t,0k} = B_{t,j0} = 0 , \end{cases}$$

where $1 \le j \le n$ and $1 \le k \le n$. Then $B_t \in L(K)$, $\|B_t\|_K \le \rho$, and the forms J_t can be expressed as

$$J_t[u,v] = ((I-B_t)Au, Av) .$$

The results of the preceding section apply to give a proof of the main theorem once it is verified that

$$EB_t|_{\mathcal{R}} \in C^m([0,t_1] \, , \, L(\mathcal{R}) \cap L(\mathcal{R}^s)) \ .$$

Now A is an isomorphism from $\mathcal{D}(|A|^{1+s})$ to $\mathcal{R}^s$, so this condition is a consequence of the following one:

$$(\#) \qquad EB_tA \in C^m([0,t_1] \, , \, L(V,\mathcal{R}) \cap L(\mathcal{D}(|A|^{1+s}) \, , \mathcal{R}^s)) \ .$$

Note that

$$\mathcal{D}(|A|^{1+s}) = \{f \in \mathring{H}^1(\Omega) \,|\, \nabla^2 f \in H^{-1+s}(\Omega)\} \ .$$

Let

$$K^s = L_2(\Omega) \oplus (\bigoplus_n H^s(\Omega)) \ .$$

The assumption that Ω satisfies property (R_s) implies that

$$A \in L(V,K) \cap L(\mathcal{D}(|A|^{1+s}),K^s) \ ,$$

while the assumption on B_t implies that

$$B_t \in C^m([0,t_1],L(K) \cap L(K^s)) \ .$$

Moreover

$$E \in L(K,\mathcal{R}) \cap L(K^s,\mathcal{R}^s) \ .$$

So (#) holds and the result follows. //

These sketchy details will be elaborated in a more comprehensive paper. It will be shown there that similar methods can be used for forms corresponding to Neumann and mixed boundary value problems. See [9].

To conclude, we remark that the results can readily be expanded to cover the case when the forms J_t have first order terms.

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