

THE INHERITANCE OF CONCAVITY BY
SOLUTIONS OF BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

A significant proportion of the literature dealing with boundary value problems is concerned with regularity inheritance results: under appropriate regularity assumptions for the data of the problem (the boundary, boundary function, differential operator and "force" function) one attempts to demonstrate the regularity of the solution. Very little, however, has been published concerning the inheritance of various concavity-like geometrical properties. In this paper, a very brief and incomplete history of the topic is outlined, after which a recent result of the author is sketched.

2. DEFINITIONS

The concept of α -concavity permits a unified presentation of concavity inheritance results. It is adapted from a definition of Brascamp and Lieb ([1]).

Let K be a convex subset of \mathbb{R}^n , and u be a positive real-valued function on K . Then for $\alpha > 0$, u is said to be α -concave when u^α is concave. Extending this definition in a natural way, one says that u is α -concave for $\alpha < 0$ when u^α is convex, and 0-concave (or *log concave*) when $\log(u)$ is concave. As a further extension, one may say that u is $(-\infty)$ -concave (or *pseudo-concave*) when

$u(x) \geq \min(u(y), u(z))$ for all x, y, z in K for which

$$x \in [y, z] = \{(1-\lambda)y + \lambda z; 0 \leq \lambda \leq 1\},$$

and that u is $(+\infty)$ -concave when "min" is replaced by "max" in the $(-\infty)$ -concave definition. Clearly, only constant functions are $(+\infty)$ -concave.

The naturalness of the definition of α -concavity is indicated by the fact that if K is open, α is real, and u is twice differentiable, positive and non-constant in K , then u is α -concave if and only if

$$\alpha \leq 1 - \sup\{u(x)u_{\theta\theta}(x)u_{\theta\theta}(x)^{-2}; x \in K, \theta \in S_n, \text{ and } u_{\theta}(x) \neq 0\},$$

where $S_n = \{t \in \mathbb{R}^n; |t| = 1\}$, and $u_{\theta}(x), u_{\theta\theta}(x)$ denote respectively the first and second derivatives of u in the direction θ . This is easily verified by calculating $(u^{\alpha})_{\theta\theta}$ and $(\log(u))_{\theta\theta}$.

A function u is pseudo-concave in a convex set K if and only if its upper level sets, $\{x \in K; u(x) > c\}$, are convex for all c . But all α -concave functions are β -concave for all $\beta \leq \alpha$. Hence all α -concave functions are pseudo-concave, whatever the value of α .

In this paper, *domain* will mean a non-empty open set.

3. HISTORY

In 1931, Gergen ([3]) showed that if Ω is a domain in \mathbb{R}^3 , starlike at $x \in \Omega$, and g_x denotes Green's function for the Laplacian on Ω with pole at x , then the upper level sets of g_x are starlike at x . In 1955, Gabriel ([2]) showed the same result with "starlike at x "

replaced by "convex". Thus Green's function is pseudo-concave when Ω is convex. Makar-Limanov ([4]) showed in 1971 that if u solves

$$\Delta u + 1 = 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded convex domain in \mathbb{R}^2 , then u is pseudo-concave. In fact, it is now known that a very minor addition to the proof indicates that u is $\frac{1}{2}$ -concave. This result is sharp, as is shown by the case where Ω is an equilateral triangle.

In 1976, Brascamp and Lieb ([1]) demonstrated the log-concavity of the first eigenfunction of the Laplacian on a bounded convex domain in \mathbb{R}^n , for $n \geq 1$. A few other results published in the last ten years show that solutions of various boundary value problems are either pseudo-concave or log-concave.

4. A RECENT RESULT

THEOREM *On a bounded convex domain Ω in \mathbb{R}^n , for $n \geq 2$, let u be the solution of*

$$\Delta u + f = 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega,$$

where f is a non-negative β -concave function on Ω for some $\beta \geq 1$. Then u is α -concave on $\bar{\Omega}$, if $\alpha = \beta/(1+2\beta)$. Thus if f is constant, so that f is $(+\infty)$ -concave, then u is $\frac{1}{2}$ -concave as in the Makar-Limanov result.

To prove this theorem it is necessary to have a means of testing whether a function is α -concave. A function \tilde{u} is constructed which is identical to u if and only if u is α -concave. For $\alpha > 0$ and x in $\bar{\Omega}$, define

$$\tilde{u}(x) = \sup ((1-\lambda)u(y)^\alpha + \lambda u(z)^\alpha)^{1/\alpha},$$

where the supremum is taken over all y and z in $\bar{\Omega}$, and λ in $[0,1]$, such that $x = (1-\lambda)y + \lambda z$. Clearly $\tilde{u}(x) \geq u(x)$ for all x in $\bar{\Omega}$.

Suppose u is not α -concave. Then $\tilde{u}(x) > u(x)$ for some x in Ω . So for a small enough $\varepsilon > 0$, $(1-\varepsilon)\tilde{u} - u$ achieves a positive maximum at some x in $\bar{\Omega}$ (as \tilde{u} is upper semicontinuous). $\Delta\tilde{u}$ does not necessarily exist, but by using a suitable generalisation, it is found from the classical maximum principle that

$$\begin{aligned} -\Delta\tilde{u}(x) &\geq (1-\varepsilon)^{-1}(-\Delta u(x)) \\ &> -\Delta u(x) = f(x) \end{aligned}$$

as $f(x) > 0$ unless f is identically zero. It can be shown that there exist y and z in $\bar{\Omega}$ with $\tilde{u}(x) = ((1-\lambda)u(y)^\alpha + \lambda u(z)^\alpha)^{1/\alpha}$, and a calculation shows that for such y and z , whenever $\frac{1}{3} \leq \alpha < \frac{1}{2}$,

$$\begin{aligned} -\Delta\tilde{u}(x) &\leq ((1-\lambda)(-\Delta u(y))^\beta + \lambda(-\Delta u(z))^\beta)^{1/\beta} \\ &= ((1-\lambda)f(y)^\beta + \lambda f(z)^\beta)^{1/\beta}, \end{aligned}$$

where $\beta = \alpha/(1-2\alpha)$. The resulting contradiction when f is β -concave completes the proof.

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