SPECTRAL PROPERTIES OF CERTAIN STIFF PROBLEMS
IN ELASTICITY AND ACOUSTICS, PART II

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ABSTRACT

We consider the vibration problem for an elastic bounded body with small compressibility, which is associated with a small parameter ε. As $\varepsilon \downarrow 0$ this is a stiff perturbation problem with non-analytic character, (in particular, the domain of the operator for $\varepsilon = 0$ is not dense in a standard space, whereas for $\varepsilon \neq 0$ it is). Nevertheless, analytic perturbation theory applies and we prove that the solution corresponding to each point of the resolvent set of the $\varepsilon = 0$ problem may be expanded as a series convergent for small $|\varepsilon|$; moreover, eigenvalues and eigenvectors have holomorphic expansions for small $|\varepsilon|$. Explicit computation of the first terms of the perturbation is given. The asymptotic behaviour of eigenvalues for large values of the spectral parameter is also given, and we show that it is not holomorphic in ε. The preceding techniques are applied to the problem of vibrations of a slightly viscous compressible fluid in a bounded vessel; an implicit function argument allows us to prove that infinitely many real eigenvalues converge as $\varepsilon \downarrow 0$ in an analytic way to the origin which is an eigenvalue of infinite multiplicity of the problem for $\varepsilon = 0$.

1. INTRODUCTION

The vibration problem for a) an elastic, slightly compressible body, b) a compressible slightly viscous fluid, were considered from the point of view of spectral perturbation in [3], where results in terms of continuity and convergence of the eigenvalues and eigenvectors were given. The present paper may be considered as an improved version of [3] by using analytic perturbation theory. In fact, the problem a) is a stiff problem for $\varepsilon \to 0$ (where ε is a parameter associated with the compressibility) as shown in Lions [7] and Pelissier [9]. If a standard $L^2(\Omega)$ framework is taken the corresponding operators are densely defined for $\varepsilon \neq 0$, but not for $\varepsilon = 0$ (where the domain is submitted to the constraint div U = 0 associated with the incompressibility condition). Nevertheless, the solution corresponding to any point of the resolvent set of the $\varepsilon = 0$ problem may be expanded as a series convergent for small $|\varepsilon|$; by constructing appropriate "transformation functions" [6] for the projections associated with the singularities, the perturbation problem is analogous to an analytic perturbation in a finite-dimensional space. Moreover, a standard argument [6] based on the self-adjoint character for real ε shows that branching of eigenvalues does not occur, and the eigenvalues are branches of holomorphic functions for small $|\varepsilon|$.

After the preceeding study (which is performed in section 3), we give in section 4 some explicit computations for the eigenvalues and eigenvectors for small $|\varepsilon|$. For fixed ε , the asymptotic distribution of eigenvalues (which may be seen in Grubb [5] for $\varepsilon \neq 0$ and in Metivier [8] for $\varepsilon = 0$) is given ; the non-analytic character of the perturbation $\varepsilon \to 0$ is shown in section 5 with some comments about this fact. Section 6 is devoted to the application of the preceeding techniques to the problem b) via a re-scaling $z = \varepsilon \zeta$ of the spectral

parameter and an application of the implicit function theorem in the framework of real analytic functions. It is shown that there are infinitely many real eigenvalues analytic in ε in the ζ variable, and consequently, in the standard spectral parameter z, converging to zero, which is an eigenvalue of infinite multiplicity of the $\varepsilon = 0$ problem.

As we already said, this paper contains sharper results than those of [3]; in order to be self-contained, we give in section 2 some results of [3] (with slight modifications); more explicit developments, as well as the case of mixed (Dirichlet and Neumann) boundary conditions may be seen in [3].

We now give some generalities about notation. The superscript $^-$ denotes the complex conjugate. Bold-face letters denote vectors in N-dimensional space:

$$u = (u_1, u_2, \ldots, u_N) \quad ; \quad L^2 = (L^2)^N .$$

$< , >$ denotes duality in the sense of distributions (or between H^{-1} and H^1_o, or even between $L^2(\Omega)$ and itself). The symbols $H^1(\Omega)$, $H^1_o(\Omega)$ denote the standard Sobolev spaces. Index L is often used for the limit problem. For instance, V, H (respectively V_L, H_L) are spaces used in the study of the Dirichlet problem for $\varepsilon \neq 0$ (respectively $\varepsilon = 0$).

2. SETTING OF THE PROBLEM AND EXPANSION OF SOLUTIONS

We consider a <u>slightly compressible</u> elastic body filling a bounded open region Ω of $R^N (N \geqq 2)$ with smooth boundary $\partial\Omega$. It is submitted to the action of the volume forces $f = (f_1, f_2 \ldots f_N)$

and is clamped at $\partial\Omega$. In the framework of linear elasticity, the displacement vector $u = (u_1, u_2 \ldots u_N)$ is the unique solution of the elasticity system

(2.1)
$$\left.\begin{array}{ll} -\mu\,\Delta\,u - \dfrac{1}{\varepsilon}\ \mathrm{grad}\ \mathrm{div}\,u = f & \text{in} \quad \Omega \\[2mm] u = 0 & \text{on} \quad \partial\Omega \end{array}\right\}$$

The Lamé's constants of the solid are $\mu > 0$ and $\lambda = (1/\varepsilon) - \mu$, with μ fixed and ε small (real positive in physical applications) parameter tending to zero (ε is associated with the small compressibility).

As proven by Lions [7], the solution u^ε of (2.1) converges as $\varepsilon \to 0$ to the solution of

(2.2)
$$\left.\begin{array}{ll} -\mu\,\Delta\,u + \mathrm{grad}\ p = f & \text{in} \quad \Omega \\[2mm] \mathrm{div}\ u = 0 & \text{in} \quad \Omega \\[2mm] u = 0 & \text{on} \quad \partial\Omega \end{array}\right\}$$

We consider the spaces $V = H_o^1(\Omega)$, $H = L^2(\Omega)$ and the sesquilinear hermitian and coercive form on $V \times V$:

$$a^\varepsilon(u,v) = \int_\Omega \left(\mu \sum_{i,r=1}^N \frac{\partial u_i}{\partial x_r} \frac{\partial \bar{v}_i}{\partial x_r} + \frac{1}{\varepsilon}\ \mathrm{div}\,u\ \mathrm{div}\,\bar{v}\right) dx$$

We denote A_ε the linear bounded operator from V to $V' = H^{-1}(\Omega)$ associated with the form a^ε , as well as the corresponding unbounded operator in H with domain $D(A_\varepsilon) \subset V$. It is a self-adjoint positive definite operator with compact resolvent, and its spectrum is formed by the sequence of eigenvalues with finite multiplicity

$$0 < \zeta_\varepsilon^1 \le \zeta_\varepsilon^2 \le \ldots\ldots \le \zeta_\varepsilon^n \le \ldots\ldots \rightarrow \infty$$

which are considered repeated according to their multiplicity.
As for the limit problem (2.2), let V_L be

$$V_L = \{u \; ; u \in H_0^1(\Omega), \; \mathrm{div}\, u = 0\}$$

and let H_L be its closure in $L^2(\Omega)$. We consider the sesquilinear
hermitian and coercive form on $V_L \times V_L$

$$a^L(u,v) = \mu \int_\Omega \sum_{i,k=1}^{N} \frac{\partial u_i}{\partial x_r} \frac{\partial \bar{v}_i}{\partial x_r} \, dx \; .$$

We denote A_L the bounded linear operator from V_L into V_L'
associated with the form a^L, as well as the unbounded self-adjoint
operator in H_L associated with the form a^L, and let

$$0 < \zeta_L^1 \le \zeta_L^2 \le \ldots\ldots \le \zeta_L^n \le \ldots\ldots \rightarrow +\infty$$

be the sequence of the corresponding eigenvalues.

We now consider the problem of finding $(u,p) \in H_0^1(\Omega) \times L^2(\Omega)/R$
satisfying

$$(2.3) \quad \left. \begin{array}{ll} -\mu \Delta u + \mathrm{grad}\, p = \zeta u + f & \text{in } \Omega \\[6pt] \mathrm{div}\, u = g & \text{in } \Omega \\[6pt] u = 0 & \text{on } \partial\Omega \end{array} \right\}$$

for given $(f,g) \in H^{-1}(\Omega) \times L^2(\Omega)/R$; here $L^2(\Omega)/R$ denotes the subspace of $L^2(\Omega)$ made of the functions whose mean value is zero.

If $\zeta \in \sigma(A_L)$, the spectrum of A_L, then (2.3) admits a solution if and only if the data (f,g) satisfy the compatibility condition

(2.4)
$$< f , v > = \int_{\Omega} g \bar{q} \, dx$$

for only solution (V,q) of the homogeneous problem.

We let us point out the following estimate of the solution of (2.3) :

PROPOSITION 2.1 - *Let* $\zeta \in \rho(A_L)$, *the resolvent set of* A_L ; *then, the solution of (2.3) satisfies:*

(2.5)
$$\| u \|_{H_0^1} + \| p \|_{L^2/R} \leq C(\zeta) \left\{ \| f \|_{H-1} + \| g \|_{L^2/R} \right\}$$

where:

(2.6)
$$C(\zeta) = (A + \frac{B + D|\zeta|}{\text{dist}(\zeta, \sigma(A_L))}) (E + F|\zeta|)$$

with A, B, D, E, F *depending only on* Ω *and* μ .

PROOF : The solution u,p can be taken under the form

$$u = V + W \quad , \quad p = p$$

where V is a continuous lift of $\text{div } V = g$ (which exists and is continuous from L^2/R into H_0^1 [12]); then the problem becomes

$$-\mu \Delta w + \text{grad } p - \zeta w = f + \mu \Delta v + \zeta v \left.\right\}$$

$$\text{div } W = 0$$

and we obtain (W,p) with

$$\| w \|_{V_L} \leq \frac{c_2}{\text{dist}(\zeta, \sigma(A_L))} \| f + \mu \Delta v + \zeta v \|_{V_L'}$$

and

$$\| p \|_{L^2/R} \leq c_1 \| \text{grad } p \|_{H^{-1}} \qquad \blacksquare$$

Now, for given $f \in H^{-1}$ and $\zeta \in \rho(A_L)$ we consider the problem:

(2.7)

$$u^\varepsilon \in H_0^1(\Omega) \left.\right\}$$

$$-\mu \Delta u^\varepsilon - \frac{1}{\varepsilon} \text{grad div } u^\varepsilon = f + \zeta u^\varepsilon \left.\right\}.$$

The following result completes those of Pelissier [9]:

PROPOSITION 2.2 - *For given* $\zeta \in \rho(A_L)$, $f \in H^{-1}$, *(2.7)* *has a unique solution* u^ε *for* $0 \neq \varepsilon \in \mathbb{C}$ *and* $|\varepsilon|$ *sufficiently small. It is a holomorphic function of* ε *with values in* $H_0^1(\Omega)$. *Moreover, when defined for* $\varepsilon = 0$ *by* u^0, *where* (u^0, p^0) *is the solution of the hereafter given problem (2.9), it is a holomorphic function for* $|\zeta|$ *sufficiently small.*

PROOF: As $A_\varepsilon - \zeta I$ has index $= 0$ it suffices to prove the surjectivity. We search for u^ε under the form:

(2.8)
$$u^\varepsilon = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots ; \ u^i \in H_0^1$$

where the series converges for $|\varepsilon|$ sufficiently small. In a standard way we insert (2.8) into (2.7) and we annihilate the coefficients of the powers of ε.

At order ε^{-1} we have:

$$\text{grad div } u^0 = 0$$

and because $u^0 \in H_0^1$ we have:

$$\text{div } u^0 = 0$$

so that $u^0 \in V_L$. For ε^0 we have

$$-\mu \Delta u^0 - \text{grad div } u^1 = f + \zeta u^0$$

and by writing $\text{div } u^1 = -p^0$, (2.7), (2.8) becomes

$$
(2.9) \qquad
\left.
\begin{aligned}
-\mu \Delta u^0 + \text{grad } p^0 &= f + \zeta u^0 && \text{in } \Omega \\
\text{div } u^0 &= 0 && \text{in } \Omega \\
u^0 &= 0 && \text{on } \partial\Omega
\end{aligned}
\right\}
$$

and by proposition 2.1 we have

$$\| u^0 \|_{H_0^1} + \| p^0 \|_{L^2/R} \leq c(\zeta) \| f \|_{H^{-1}} .$$

Then, for terms in ε^1 we have:

$$-\mu \Delta u^1 - \text{grad div } u^2 = \zeta u^1$$

and by writing

$$\text{div } u^2 = -p^1$$

we have:

$$- \mu \Delta u^1 + \text{grad } p^1 = \zeta u^1 \qquad \text{in} \quad \Omega$$
$$\text{div } u^1 = -p^0 \qquad \text{in} \quad \Omega$$
$$u^1 = 0 \qquad \text{on} \quad \partial \Omega$$

$$\|u^1\|_{H_0^1} + \|p^1\|_{L^2/R} \leq C(\zeta) \|p^0\|_{L^2/R} \leq c^2(\zeta) \|f\|_{H^{-1}} .$$

It is then a straightforward matter to go on to all terms in (2.6) and proving that it is a convergent series for $|\varepsilon| < C(\zeta)^{-1}$ and u^ε satisfies the estimate

(2.10)
$$\|u^\varepsilon\|_{H_0^1} \leq \frac{C(\zeta)}{1 - |\varepsilon| C(\zeta)} \|f\|_{H^{-1}}$$

and the conclusions follow. ■

If we denote by Π the projection of $L^2(\Omega)$ on H_L (or of $H^{-1}(\Omega)$ on V_L'), u^0 in (2.9) may be obviously written

(2.11)
$$(A_L - \zeta) u^0 = \Pi f$$

and the obtained results may be described by writing

(2.12)
$$u^\varepsilon(\zeta) \equiv (A_\varepsilon - \zeta)^{-1} f \rightarrow u^0(\zeta) \equiv (A_L - \zeta)^{-1} \Pi f$$

in H_0^1 strongly for any $f \in H^{-1}$. Moreover the convergence is uniform for $\zeta \in K$ compact of $\rho(A_L)$ (see (2.6) and the proof of proposition 2.2) and for f in a bounded set of H^{-1}.

3. HOLOMORPHIC BEHAVIOUR OF EIGENVALUES AND EIGENVECTORS FOR SMALL ε.

Let us consider a simple closed path γ contained in a compact set K of $\rho(A_L)$ enclosing an eigenvalue ζ_L^* of A_L (i.e. one of the eigenvalues ζ_L^i). The corresponding (orthogonal) eigenprojection

$$\frac{-1}{2\pi i} \int_\gamma (A_L - \zeta)^{-1} d\zeta$$

of H_L may also be considered, when acting after the projection Π, as an eigenprojection in $L^2(\Omega)$; we shall denote it by P_L^*:

$$(3.1) \qquad P_L^* = \frac{-1}{2\pi i} \int_\gamma (A_L - \zeta)^{-1} \Pi \, d\zeta .$$

We also consider (if $\gamma \in \rho(A_\varepsilon)$) the projection

$$(3.2) \qquad P_\varepsilon = \frac{-1}{2\pi i} \int_\gamma (A_\varepsilon - \zeta)^{-1} d\zeta .$$

and we have :

<u>PROPOSITION 3.1</u> - *If $\gamma \subset \rho(A_L)$ is chosen as above, then, for sufficiently small $|\varepsilon|$, $\gamma \subset \rho(A_\varepsilon)$ and the projection P_ε of (3.2) is a holomorphic function of ε with values in $\mathcal{L}(L^2(\Omega))$ taking for $\varepsilon = 0$ the value P_L^* of (3.1). In consequence, the dimension of the range of P_ε is independent of ε for small $|\varepsilon|$ (and then equal to the dimension of the range of P_L^*).*

<u>PROOF</u> : From proposition 2.2 and the remark at the end of section 2 about uniform convergence for $\zeta \subset K$ compact set of $\rho(A_L)$ the projection P_ε is defined for small $|\varepsilon|$. The convergence of the integrals is

obvious from the uniform convergence in (2.12) when applied to a fixed $f \in L^2$ and is a holomorphic function of ε with values in L^2. According to a classical property (cf. Kato [6] section VII.1.1), P_ε is holomorphic with values in $\mathcal{L}(L^2)$. Finally, the property about the dimension of the range is classical. ∎

At the present state, it is to be noticed that the operators A_ε are self-adjoint only for real ε, and the range of P_ε has not necessarily a basis formed by eigenvalues. We have :

THEOREM 3.1 - Let ζ^* be an eigenvalue of A_L (i.e. one of the ζ_L^i) of multiplicity $m \geq 1$. Then, for the sufficiently small $|\varepsilon|$ (complex in general) the operator A_ε has in the vicinity of ζ^* m eigenvalues (not necessarily distinct) which are holomorphic functions of ε, real for real ε :

$$
(3.3) \quad \left.
\begin{aligned}
\zeta_\varepsilon^{(1)} &= \zeta^* + \varepsilon \zeta_1^{(1)} + \varepsilon^2 \zeta_2^{(1)} + \cdots \\[2mm]
\zeta_\varepsilon^{(m)} &= \zeta^* + \varepsilon \zeta_1^{(m)} + \varepsilon^2 \zeta_2^{(m)} + \cdots
\end{aligned}
\right\}
$$

and for real ε there are m associated linearly independent eigenvectors (which are analytic functions of ε such that for $\varepsilon = 0$ they span $\mathrm{Ker}\,(A_L - \zeta^*)$).

PROOF : Let us construct the "transformation functions" $U(\varepsilon)$ and $U(\varepsilon)^{-1}$ associated with the projections P_ε in the vicinity of $\varepsilon = 0$ (cf. Kato [6] p. 33, 56, 99, 102, 369) :

$$
(3.4) \quad U(\varepsilon) = [I - (P_\varepsilon - P_L^*)^2]^{-\frac{1}{2}} [P_\varepsilon P_L^* + (I - P_\varepsilon)(I - P_L^*)]
$$

which are holomorphic functions of ε for small $|\varepsilon|$ with values in $\mathsf{L}(H)$ satisfying:

(3.5)
$$U(\varepsilon)P_L^* U(\varepsilon)^{-1} = P_\varepsilon \; ; \; U(\varepsilon)^{-1}P_\varepsilon U(\varepsilon) = P_L^*.$$

Moreover, for ε real, P_ε is orthogonal and $U(\varepsilon)$, $U(\varepsilon)^{-1}$ are unitary. We then consider the operators

(3.6)
$$\tilde{A}_\varepsilon = U(\varepsilon)^{-1}A_\varepsilon U(\varepsilon)$$

which are image of A_ε under the transformation $U(\varepsilon)$. They are self-adjoint for real ε. Of course, the eigenvalue problem for A_ε is the same as for $\tilde{A}_\varepsilon$. Moreover, the projection P_ε commutes with A_ε and P_L^* commutes with $\tilde{A}_\varepsilon$: consequently the spectral problem for A_ε in $P_\varepsilon H$ is analogous to the spectral problem for $\tilde{A}_\varepsilon$ in $P_L^* H$, i.e. the $m \times m$ matrix $P_L^* \tilde{A}_\varepsilon P_L^*$.
Now we have :

LEMMA 3.1 - *The* $m \times m$ *matrix* $P_L^* \tilde{A}_\varepsilon P_L^*$ *is holomorphic for sufficiently small* $|\varepsilon|$ *and takes for* $\varepsilon = 0$ *the value* $P_L^* \tilde{A}_L P_L^*$.

In order to prove this lemma, we see from proposition 2.2 that A_ε^{-1} is a holomorphic function of ε for small $|\varepsilon|$ with values in $\mathcal{L}(H)$, taking the value $A_L^{-1}\Pi$ for $\varepsilon = 0$. Thus

$$\tilde{A}_\varepsilon^{-1} = U(\varepsilon)^{-1}A_\varepsilon^{-1}U(\varepsilon)$$

is a holomorphic function of ε for small $|\varepsilon|$ with values in $\mathcal{L}(H)$ and taking the value $A_L^{-1}\Pi$ for $\varepsilon = 0$. Let us consider :

$$\tilde{A}_\varepsilon u^\varepsilon = \phi$$

if we take $\phi \in P_L^* H$ we see that $u^\varepsilon \in P_L^* H$ and consequently the operator

$$P_L^* \tilde{A}_\varepsilon^{-1} P_L^* \equiv (P_L^* \tilde{A}_\varepsilon P_L^*)^{-1}$$

is an operator (in fact a $m \times m$ matrix) which is holomorphic and take the value $P_L^* A_L^{-1} P_L^*$ for $\varepsilon = 0$. As a consequence, the matrix $P_L \tilde{A}_\varepsilon P_L$ is holomorphic for small ε and take the value $P_L^* A_L P_L^*$ for $\varepsilon = 0$. Lemma 3.1 is proven. At the present state the eigenvalue problem is analogous to the classical one for a holomorphic $m \times m$ matrix which is self-adjoint for real ε (cf. Kato [6] chap.2) and the conclusions follow. Let us recall in this connection that without the self-adjointness hypothesis, the eigenvalue ζ^* splits into m branches as an algebraic singularity; the fact that the branches are holomorphic is a consequence of the fact that the partial eigenprojections for the branches are orthogonal (and then bounded) for real ε (see Kato [6] section 2.1.6). ■

4. EXPLICIT COMPUTATION OF EXPANSION (3.3)

Let us consider as in the preceeding section that ζ^* is an eigenvalue of A_L with multiplicity $m \geq 1$. We consider the expansion of the m analytic eigenvectors associated with the expansion (3.3) for the eigenvalues.

(4.1)
$$u_\varepsilon^{(\ell)} = u_0^{(\ell)} + \varepsilon u_1^{(\ell)} + \varepsilon^2 u_2^{(\ell)} + \ldots \quad \ell = 1, 2, \ldots m,$$

where $u_i^{(\ell)} \in H_0^1(\Omega)$ and the $u_0^{(\ell)}$ span $\mathrm{Ker}\,(A_L - \zeta^*)$. The eigenvectors and eigenvalues of (3.3), (4.1) satisfy

(4.2)
$$-\mu\Delta u_\varepsilon^{(\ell)} - \frac{1}{\varepsilon} \text{ grad div } u_\varepsilon^{(\ell)} = \zeta_\varepsilon^{(\ell)} u_\varepsilon^{(\ell)}.$$

After inserting (3.3), (4.1) into (4.2) we proceed as in the proof
of proposition 2.2. At order ε^{-1} and ε^{0} we have respectively:

$$\text{div } u_0^{(\ell)} = 0 \qquad \text{in } \Omega$$

and by denoting :

$$\text{div } u_1^{(\ell)} = - p_0^{(\ell)}$$

we see that $(u_0^{(\ell)}, p_0^{(\ell)})$ are solutions of the eigenvalue problem for A_L :

(4.3)
$$\left. \begin{array}{rll} -\mu \Delta v + \text{grad } q = \zeta^{*} v & \text{in } \Omega \\ \text{div } v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{array} \right\}.$$

We shall see that the $(u_0^{(\ell)}, p_0^{(\ell)})$ are defined later (partially at
least) as eigenvectors of some eigenvalue problem in $\text{Ker}(A_L - \zeta^{*})$. For
the time being, we denote $(v^{(j)}, q^{(j)})$ a basis satisfying the
orthogonality conditions

(4.4)
$$\delta_{ij} = (v^{(j)}, v^{(i)})_{H_L} = (q^{(j)}, q^{(i)})_{L^2(\Omega)} , \quad i,j = 1,2,\ldots,m ,$$

and determining $u_0^{(\ell)}, p_0^{(\ell)}$ amounts to determining $\alpha_j^{(\ell)}$ $(\ell,j = 1,2,\ldots m)$
with

(4.5)
$$u_0^{(\ell)} = \sum_{j=1}^{m} \alpha_j^{(\ell)} v^{(j)} ; \quad p_0^{(\ell)} = \sum_{j=1}^{m} \alpha_j^{(\ell)} q^{(j)} ; \quad \ell = 1,2,\ldots m.$$

Now the expansion of (4.2) at order ε^1 gives :

$$-\mu\Delta u_1^{(\ell)} + \text{grad} p_1^{(\ell)} - \zeta^* u_1^{(\ell)} = \zeta_1^{(\ell)} u_0^{(\ell)} \quad \text{in} \quad \Omega$$

(4.6)
$$\text{div } u_1^{(\ell)} = -p_0^{(\ell)} \quad \text{in} \quad \Omega$$

$$\underline{u}_1^{(\ell)} = 0 \quad \text{on} \quad \partial\Omega$$

where

$$p_1^{(\ell)} \equiv -\text{div } u_2^{(\ell)}.$$

We consider (4.6) as a system with unknowns $u_1^{(\ell)}$, $p_1^{(\ell)}$; according to (2.4), the compatibility condition is :

$$\zeta_1^{(\ell)} (u_0^{(\ell)}, v^{(j)})_{\mathbb{H}_L} = -(p_0^{(\ell)}, q^{(j)})_{L^2}$$

which, with (4.4), (4.5) becomes

(4.7)
$$-\zeta_1^{(\ell)} \alpha_j^{(\ell)} = \sum_{i=1}^{m} \alpha_i^{(\ell)} \int_\Omega q^{(i)} \bar{q}^{(j)} dx, \quad j, \ell = 1, 2, \ldots m$$

and consequently $-\zeta_1^{(\ell)}$ and $(\alpha_1^{(\ell)}, \ldots \alpha_m^{(\ell)})$ are the eigenvalues and the corresponding eigenvectors of the $m \times m$ self-adjoint non-negative matrix M with coefficients

$$M_{ij} = \int_\Omega q^{(i)} \bar{q}^{(j)} dx.$$

Consequently the coefficients $\zeta_1^{(\ell)}$ are well determined real ≤ 0 numbers. If they are distinct we are in the case of maximum splitting of the eigenvalue (3.3) and the $u_0^{(\ell)}$ are also well determined

(up to a constant factor for each one). If there are multiple eigenvalues, the corresponding $u_0^{(\ell)}$ are only subject to be a basis of the corresponding eigenspace.

At this stage, (4.6) are compatible and $u_1^{(\ell)}$ are defined up to an additive eigenvector which is determined at the same time that $\zeta_2^{(\ell)}$ in order to satisfy the compatibility condition for the terms of order ε^2 in (4.2) and so on.

REMARK 4.1 : *We saw that* $\zeta_1^{(\ell)} \leq 0$ *and consequently, for real* ε, *the eigenvalues decrease as* ε *increase. This result is natural from a physical point of view : as* $\varepsilon \downarrow 0$ *the rigidity of the system increases and the eigen-frequencies must increase.* ■

5. ASYMPTOTIC DISTRIBUTION OF THE n-TH EIGENVALUE AS n→∞.

It is worthwhile comparing formula (3.3), associated with a fixed eigenvalue ζ^* of A_L with the formula giving the asymptotic distribution of eigenvalues. System (2.1) is elliptic in the sense of Agmon, Douglis, Nirenberg [1] for complex ε with

(5.1) $\varepsilon \notin \{-\frac{1}{\mu}, \frac{-1}{2\mu}, 0\}$

and consequently it has index 0 (either as a bounded operator from V into V' or as an unbounded operator in H) for all ε satisfying (5.1). According to general results on asymptotic distribution of eigenvalues of elliptic systems (see for instance Grubb [5] for a presentation of these results), we define:

$$\nu(t, A_\varepsilon) = \sum_{\zeta_\varepsilon^j \leq t} 1$$

and we have (with arbitrary $\delta > 0$) :

$$\nu(t,A_\varepsilon) = C_{A_\varepsilon} t^{N/2} + 0(t^{(N-\frac{1}{2}+\delta)/2}) \text{ as } t \to +\infty$$

(5.2)

$$C_{A_\varepsilon} = \frac{\omega_N}{(2\pi)^N} \frac{(N-1)}{N} \mu^{-N/2} \text{meas}(\Omega) \left[1 + \frac{(\varepsilon\mu)^{N/2}}{N-1}(1+\varepsilon\mu)^{-N/2}\right]$$

(ω_N is the measure of the unit sphere of R^N).

On the other hand, for $\varepsilon = 0$, the asymptotic distribution of eigenvalues for a class of systems containing (2.3) is given in Metivier [8]. The result for our system is :

$$\nu(t,A_L) \sim C_{A_L} t^{N/2}$$

(5.3)

$$C_{A_L} = \frac{\omega_N}{(2\pi)^N} \frac{(N-1)}{N} \mu^{-N/2} \text{meas}(\Omega)$$

and we see that :

(5.4)
$$C_{A_L} = \lim_{\varepsilon \to 0} C_{A_\varepsilon}$$

From (5.2) and (5.3) we deduce an asymptotic behaviour of the n-th eigenvalue :

$$\zeta_\varepsilon^n \sim C_{A_\varepsilon}^{-2/N} n^{2/N} \text{ as } n \to \infty$$

(5.5)

$$\zeta_L^n \sim C_{A_L}^{-2/N} n^{2/N} \text{ as } n \to \infty$$

where from (5.2), (5.3)

(5.6)
$$C_{A_\varepsilon}^{-2/N} = C_{A_L}^{-2/N} \left[1 - \frac{2}{N(N-1)} (\varepsilon\mu)^{N/2} + \ldots \right]$$

This formula is consistent with the physical considerations of remark 4.1 and shows that (for $N \neq 2$) in general, the analytic dependence of ζ_ε^n on ε does not hold for the asymptotic behaviour $n \to \infty$.

This phenomenon may be explained by the following considerations. According to theorem 3.1, each eigenvalue ζ_L^n of A_L is approached analytically by eigenvalues ζ_ε^n of A_ε, but there is no reason for obtaining all eigenvalues of A_ε in this way. There may perhaps exist other eigenvalues of A_ε not depending analytically on ε. We show this with an example inspired by the considerations of Metivier [8], section 6:

EXAMPLE : *Let* $\Omega =]0, 2\pi[^N$. *We consider problems analogous to* (2.1), (2.2) *with periodicity conditions. A Fourier expansion easily shows that eigenvalues of* A_L *and* A_ε *are :*

$$\zeta_L = \mu |\nu|^2 \quad \text{with multiplicity } N - 1$$
$$\zeta_\varepsilon = \mu |\nu|^2 \quad \text{with multiplicity } N - 1$$
$$\Lambda_\varepsilon = (\mu + \tfrac{1}{\varepsilon}) |\nu|^2 \quad \text{simple}$$

for $\nu \varepsilon z^N$.

The eigenvalues Λ_ε *are not analytic in* ε. ▦

6. APPLICATION TO THE VIBRATIONS OF A FLUID WITH SMALL VISCOSITY

We consider a bounded domain Ω of R^N as in the preceeding sections. If u and p denote the velocity vector and pressure

perturbation of a slightly viscous barotropic fluid, the linearized

vibration in a vessel with rigid boundary satisfies the system:

$$
\left.
\begin{array}{ll}
\dfrac{\partial u_i}{\partial t} = -\dfrac{\partial P}{\partial x_i} + \varepsilon\,\dfrac{\partial \sigma_{ij}^{v}(u)}{\partial x_j} & \text{in } \Omega \\[4mm]
\dfrac{\partial p}{\partial t} = -\,\mathrm{div}\; u & \text{in } \Omega \\[4mm]
\left.
\begin{array}{l}
u = 0 \ \text{if} \ \ \varepsilon > 0 \\[2mm]
u\cdot n = 0 \ \text{if} \ \ \varepsilon = 0
\end{array}
\right\} & \text{on } \partial\Omega \\[4mm]
\sigma_{ij}^{v}(u) \equiv \lambda(\mathrm{div}\,u)\delta_{ij} + \mu\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) &
\end{array}
\right\}
$$

(6.1)

where $\varepsilon\sigma$ is the viscosity tensor and $\varepsilon\lambda, \varepsilon\mu$ are the viscosity

coefficients, which satisfy

$$
\mu > 0, \quad 3\lambda + 2\mu > 0.
$$

The spectral properties of system (6.1) were considered in [4].

For $\varepsilon > 0$ the spectrum consists of isolated eigenvalues with finite

multiplicity and of the essential spectrum formed by the points

$$
[\varepsilon(\lambda+2\mu)]^{-1}, \quad [\varepsilon(\lambda+3\mu)]^{-1}.
$$

For $\varepsilon = 0$, the spectrum consists of the sequences of eigenvalues $\pm\, i\omega_k$

(where $-\omega_k^2$ are the eigenvalues of the laplacian with Neumann boundary

condition in Ω) and the eigenvalue 0 with infinite multiplicity

(essential spectrum).

The spectral problem associated with (6.1) is

$$(6.2) \qquad \left. \begin{array}{c} -\varepsilon\dfrac{\partial\sigma^{v}_{ij}}{\partial x_{j}} + \dfrac{\partial p}{\partial x_{i}} = zu_{i} \\[4mm] \text{div } u = zp \end{array} \right\}.$$

We now study the eigenvalues of (6.2) in the vicinity of $z = 0$ as $\varepsilon \to 0$. After eliminating p between the two equations of (6.2) and under the re-scaling $z = \varepsilon\zeta$, the problem becomes:

$$(6.3) \qquad -\mu\Delta u - (\lambda+\mu-1/(\zeta\varepsilon^{2})) \text{ grad div } u = \zeta u$$

with of course the Dirichlet boundary condition for u . Under this form we have an implicit eigenvalue problem (the spectral parameter ζ appears in the left hand side) for a perturbation of the form of that of section 2. We introduce the new small parameter η defined by:

$$\frac{1}{\eta} = (\lambda + \mu) - \frac{1}{\zeta\varepsilon^{2}}$$

or equivalently

$$(6.4) \qquad F(\eta,\varepsilon) \equiv \eta - \frac{\zeta\varepsilon^{2}}{(\lambda+\mu)\zeta\varepsilon^{2}-1} = 0.$$

The problem of finding the eigenvalue ζ as a function of the parameter η is exactly the problem of section 3, (where the parameter is denoted η instead of ε). Consequently, if we replace ζ in (6.4) by the functions $\zeta_{\eta}^{(\ell)}$ of theorem 3.1, equation (6.1) becomes an implicit equation to find $\eta = \eta(\varepsilon)$ such that as replaced into (6.3) with the corresponding $\zeta_{\eta}^{(\ell)}$ (and the corresponding eigenvector) gives a solution of the eigenvalue problem.

In order to perform this program, let ζ^* be an eigenvector with multiplicity m of the limit operator A_L. According to theorem 3.1 let us take :

$$(6.5) \qquad \zeta = \zeta(\eta) = \zeta_\eta^{(\ell)}$$

for small real η in (6.4), which becomes an implicit equation to be studied in the vicinity of $\varepsilon = 0$, $\eta = 0$ (and consequently $\zeta = \zeta^*$). One sees that

$$F_\eta'(0,0) = 1$$

and consequently there exists $\eta(\varepsilon)$, $\zeta(\varepsilon)$ real-analytic for small real ε associated with each of the analytic branches $(\ell = 1,\ldots m)$ of (3.3). We have proved the following theorem, which improves theorem 3.1 of [3].

THEOREM 6.1 *Let ζ be an eigenvalue with multiplicity m of A_L. Then, for real ε with sufficiently small $|\varepsilon|$ the eigenvalue problem (6.2) has the m real analytic eigenvalues (not necessarily distinct)*

$$
(6.6) \qquad
\left.
\begin{aligned}
z_\varepsilon^{(1)} &= \varepsilon\zeta^* + \varepsilon^2 z_2^{(1)} + \varepsilon^3 z_3^{(1)} + \ldots \\
&\ldots\ldots \\
z_\varepsilon^{(m)} &= \varepsilon\zeta^* + \varepsilon^2 z_2^{(m)} + \varepsilon^3 z_3^{(m)} + \ldots
\end{aligned}
\right\}
$$

and m associated linearly independent eigenvectors which are real analytic functions of ε such that for $\varepsilon = 0$ they span $\mathrm{Ker}(A_L - \zeta^*)$.

7. A COMPLEMENTARY REMARK

Barry Simon has kindly pointed out to us that the results of section 3 can be viewed in the following more general setting. The form $a^L(u,v)$, that we shall now denote $a^o(u,v)$, with domain $V_L = \{u; u \in H^1_0 (\Omega), \operatorname{div} u = 0\}$ can be considered as a non-densely defined closed form in the space $H = L^2(\Omega)$. Now, as is remarked in [11, section 4] << the extension of the usual theory of densely defined closed forms to the general case is quite elementary >>. The notions of closed, closability and closure are unchanged; there is a one to one correspondence between closed positive quadratic forms t on H and operators T which are self-adjoint on $\overline{D(T)}$=closure of $D(T)$ in H. If the form t with eventually non dense domain $D(t) \subseteq H$ is closed its resolvent is defined as the operator which is $(T-\zeta)^{-1}$ on $\overline{D(t)}$ and zero on $D(t)^\perp$; i.e. if π denotes the orthogonal projection of H onto $\overline{D(t)}$ we define the resolvent as $(T-\zeta)^{-1}\pi$. We can then say that the forms $t_n \to t$ as in the strong resolvent sense (s.r.s.) if $(T_n+1)^{-1}$ converges to $(T_\infty+1)^{-1}$ strongly. The convergence of the u^ε solution of (2.1) to u^o solution of (2.2) proved by Lions [7], can also be obtained as a consequence of a theorem of Kato and B. Simon [10], [11].

With the convention that $(A_0-\zeta)^{-1} = (A_L-\zeta)^{-1}\pi$, proposition 2.2 means that $(A_\varepsilon-\zeta)^{-1}$ is for $|\varepsilon|$ small a bounded holomorphic family of operators for ζ in a compact set of $\rho(A_L)$. But the previous considerations suggest that one can use the defintion of Kato [6], Chap.VII,§4.2 saying that the family $a_\varepsilon(u,v)$ is for $|\varepsilon|$ small an holomorphic family of type (a) of eventually non-densely defined closed forms on H. Then theorem 3.1 means that the results on the perturbations of eigenvalues and eigenvectors can be extended to this more general situation.

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