CORNER SINGULARITIES AND BOUNDARY INTEGRAL EQUATIONS

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INTRODUCTION

It is well known that if U is the solution to an elliptic boundary value problem (BVP) posed on a region Ω, then, in general, U will not be smooth at any corner point of Ω. Because of this, special care is needed when designing numerical methods for solving the BVP, especially if high accuracy is needed near the corners, see e.g. [8]. As might be expected, similar difficulties arise when the BVP is reformulated as a boundary integral equation (BIE), that is, the solution to the BIE fails, in general, to be smooth at the corner points of the boundary.

In this paper we discuss the double layer potential equation for a polygon, a BIE which occurs as a reformulation of the Dirichlet problem for Laplace's equation. With this simple example, we illustrate some of the difficulties created by the presence of corners, and how one might deal with them. A more detailed technical discussion of this and similar problems can be found in papers such as [2], [3], [4] and [5].

1. THE INTEGRAL EQUATION

Let Ω be a polygonal domain. That is, Ω is a bounded, connected, open subset of $\mathbb{R}^2$ whose boundary $\Gamma = \partial\Omega$ is made up of a finite number of straight line segments. Let $x_1, x_2, \ldots, x_N$ be the vertices of Ω, numbered so that x_{j+1} follows x_j as one proceeds anticlockwise around Γ. We adopt the convention that subscripts are evaluated modulo N

wherever necessary, so, for example, $x_0 = x_N$. We also rule out the possibility that Ω has "cracks", that is, we require Ω to be a Lipschitz domain (cf. (2.5) below).

The classical Dirichlet problem for the Laplace operator

$$\Delta = \left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2$$

is to find, for a given $g \in C^0(\Gamma)$, a function $U \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ satisfying

(1.1a) $\Delta U = 0$ on Ω

(1.1b) $U = g$ on Ω .

The traditional way to reformulate this BVP as a BIE is as follows.

Let $n(y)$ denote the unit inner normal at $y \in \Gamma, y \neq x_j$, let σ denote the usual surface measure on Γ (i.e. $d\sigma$ is the "element of arc length"), and let

$$E(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$$

be the logarithmic potential. The double layer potential with density function u on Γ is defined by

$$Wu(x) = \int_\Gamma u(y) \frac{\partial}{\partial n(y)} E(x-y)\, d\sigma(y) .$$

Because E is a fundamental solution to Laplace's equation in two dimensions, i.e.

$$-\Delta E = \delta \qquad \text{on} \quad \mathbb{R}^2 \ ,$$

where, as usual, δ denotes the Dirac measure or unit mass at the origin, one can easily show $Wu \in C^{\infty}(\Omega)$ and

$$\Delta Wu = 0 \qquad \text{on} \quad \Omega \ , \tag{1.2}$$

assuming only that $u \in L^1(\Gamma)$. If one also assumes u is continuous at $x \in \Gamma$, then

$$\lim_{z \to x, z \in \Omega} Wu(z) = \frac{1}{2}\,(I+T)u(x) \ , \qquad x \in \Gamma \ , \tag{1.3}$$

where

$$Tu(x) = \int_{\Gamma} k(x,y)u(y)\,d\sigma(y) + \frac{\theta(x)}{\pi}\,u(x)$$

$$k(x,y) = 2\,\frac{\partial}{\partial n(y)}\,E(x-y) = \frac{1}{\pi}\,\frac{(x-y)\cdot n(y)}{|x-y|^2}$$

$$\theta(x) = \begin{cases} 0 & \text{for} \quad x \in \Gamma \ , \ x \neq x_j \\ \theta_j & \text{for} \quad x = x_j \end{cases}$$

and θ_j is the angle shown in figure 1. Notice $\pi - \theta_j$ is the interior angle at x_j.

If $u \in C^0(\Gamma)$ satisfies the second kind Fredholm integral equation

$$(1.4) \qquad (I+T)u = 2g \quad \text{on} \quad \Gamma$$

then we see from (1.2) and (1.3) that

$$U(x) = \begin{cases} Wu(x) & \text{for} \quad x \in \Omega \\ g(x) & \text{for} \quad x \in \Gamma \end{cases}$$

is a solution to (2.1), and the usual maximum principle argument shows that this is the only solution. Thus, the problem of solving the BVP (1.1) reduces to that of solving the BIE (1.4), and it is known, see eg. [6], that the integral equation (1.4) is well posed on $C^0(\Gamma)$, that is

$$I + T : C^0(\Gamma) \to C^0(\Gamma)$$

is an isomorphism. Note that throughout the paper the term "isomorphism" will mean "isomorphism of normed spaces", in other words an isomorphism is a bounded, linear, onto mapping which has a bounded inverse.

2. AN EQUIVALENT SYSTEM OF INTEGRAL EQUATIONS

In this section we construct a system of integral equations on the unit interval which is equivalent to the BIE (1.4).

Denote the part of Γ lying strictly between x_{j-1} and x_j by Γ_j, i.e.

$$\Gamma_j = \{(1-t)x_{j-1} + tx_j : 0 < t < 1\} \quad ,$$

and choose a sufficiently small $d_j > 0$ such that $d_{j-1} + d_j <$ length of Γ_j. Let $x_{j-\frac{1}{3}}$ and $x_{j+\frac{1}{3}}$ be the points lying on Γ_j and Γ_{j+1} respectively which are at distance d_j from x_j, and define parametrizations

$$\left.\begin{aligned} \gamma_j^1(t) &= (1-t)x_{j-1} + tx_{j-\frac{2}{3}} \\ \gamma_j^2(t) &= (1-t)x_{j-\frac{2}{3}} + tx_{j-\frac{1}{3}} \\ \gamma_j^3(t) &= (1-t)x_j + tx_{j-\frac{1}{3}} \end{aligned}\right\} \qquad 0 \leqq t \leqq 1 \quad .$$

Put

$$\Gamma_j^m = \{\gamma_j^m(t) : 0 < t < 1\} \quad , \qquad m = 1(1)3 \ ,$$

then

$$\Gamma_j = \Gamma_j^1 \cup \left\{x_{j-\frac{2}{3}}\right\} \cup \Gamma_j^2 \cup \left\{x_{j-\frac{1}{3}}\right\} \cup \Gamma_j^3 \ , \quad j = 1(1)N \ ,$$

as illustrated in Figure 1. Notice Γ_j^3 is oriented "backwards" so

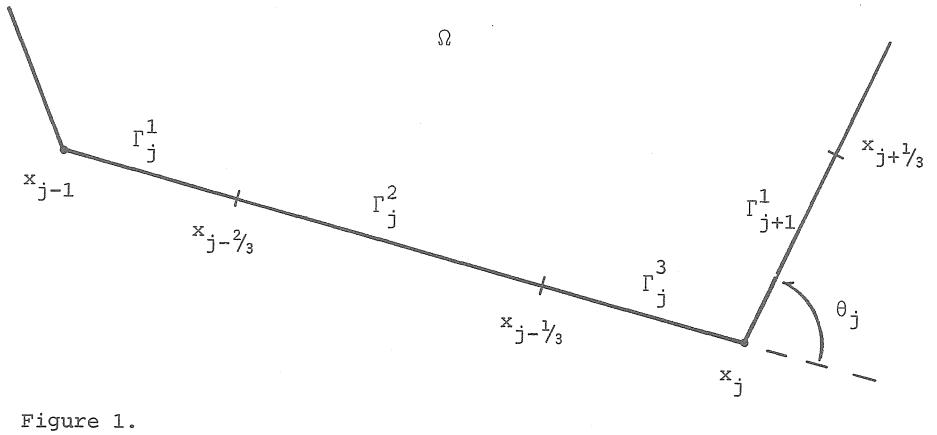

Figure 1.

$$\gamma_j^3(0) = x_j = \gamma_{j+1}^1(0) \quad .$$

To each function u on Γ, associate the vector of functions

$$\underline{u} = (u_1^1, u_1^2, u_1^3, \ldots, u_j^1, u_j^2, u_j^2, \ldots, u_N^1, u_N^2, u_N^3)$$

defined on the unit interval by

$$u_j^m(t) = u[\gamma_j^m(t)] \quad , \quad 0 \leqq t \leqq 1 \quad . \tag{2.1}$$

Introduce the $3N \times 3N$ operator $\underline{T}$ by putting

$$\underline{T}\,\underline{u}(s) = Tu(x) \quad , \qquad x = \gamma_i^\ell(s)$$

then

$$\underline{T} = (T_{ij}^{\ell m}) \qquad i,j = 1(1)N \qquad \ell,m = 1(1)3$$

where

$$T_{ij}^{\ell m}\, u_j^m(s) = \int_0^1 k_{ij}^{\ell m}(s,t)\, u_j^m(t)\, dt \quad , \quad 0 < s \leqq 1,$$

$$k_{ij}^{\ell m}(s,t) = k(x,y)\, \frac{d\gamma_j^m}{dt} \quad , \; x = \gamma_i^\ell(s), \; y = \gamma_j^m(t).$$

Corresponding to our original integral equation (1.4), we have a $3N \times 3N$ system of integral equations

$$(\underline{I} + \underline{T})\underline{u} = 2\underline{g} \tag{2.2}$$

on the unit interval, the correspondence between the solutions u and $\underline{u}$ being given by (2.1) .

If $x,y \in \Gamma_j$, then $(x-y)\cdot n(y) = 0$ so $k(x,y) = 0$ and

$$T_{ij}^{\ell m} = 0 \qquad j = 1(1)N \qquad \ell,m = 1(1)3 \ .$$

With this in mind it is clear that unless

(2.3) $$j = i-1 \quad \text{and} \quad \ell = 1 \quad \text{and} \quad m = 3$$

or

(2.4) $$j = i+1 \quad \text{and} \quad \ell = 3 \quad \text{and} \quad m = 1 \ ,$$

i.e. unless Γ_i^{ℓ} and Γ_j^{m} meet at a corner, the kernel $k_{ij}^{\ell m}$ is C^{∞} on $[0,1]\times[0,1]$. On the other hand, one finds that

$$T_{j+1,j}^{13} = T_{j,j+1}^{31} = R_{\theta_j}$$

where

$$R_\theta v(s) = \begin{cases} \displaystyle\int_0^1 r_\theta(s,t)v(t)dt & \text{for } 0 < s \leq 1 \\ \dfrac{\theta}{\pi} v(0) & \text{for } s = 0 \end{cases}$$

with the singular kernel

$$r_\theta(s,t) = \frac{1}{\pi} \frac{s \cdot \sin\theta}{s^2+t^2+2st\cos\theta} \ .$$

Note that our assumptions on the domain Ω imply

$$(2.5) \qquad 0 < |\theta_j| < \pi \quad , \qquad j = 1(1)N \quad ,$$

and that, owing to a scale invariance property d_j does not appear in R_{θ_j} .

Write

$$\theta = (\theta_1, \ldots, \theta_N)$$

and gather together the $T_{ij}^{\ell m}$ satisfying (2.3) or (2.4), i.e. those with singular kernels, to form the $3N \times 3N$ operator $\underline{R}_\theta$. Then gather together the remaining $R_{ij}^{\ell m}$, i.e. those with smooth kernels, to form a $3N \times 3N$ operator $\underline{K}$. Thus we have the decomposition

$$\underline{T} = \underline{R}_\theta + \underline{K}.$$

The general form of $\underline{R}_\theta$ can be seen by looking at the case $N = 3$:

$$\underline{R}_\theta = \begin{bmatrix} & & R_{\theta_3} \\ & R_{\theta_1} & \\ R_{\theta_1} & & R_{\theta_2} \\ & R_{\theta_2} & \\ R_{\theta_3} & & \end{bmatrix}$$

(Each block of the matrix is 3×3 and only the nonzero entries are shown.)

If Γ were a smooth, closed curve (i.e. no corners), then the kernel k of T would be smooth and T would be a regularising operator so the principal part of $I + T$ would be just the identity operator I. However, in our case

(2.6) $$\underline{I} + \underline{T} = \underline{A}_\theta + \underline{K} \quad ,$$

where

$$\underline{A}_\theta = \underline{I} + \underline{R}_\theta$$

is now the principal part, with $\underline{K}$ regularising.

3. THE BASIC SCALAR INTEGRAL EQUATIONS

We begin by diagonalising $\underline{A}_\theta$. Once again, the general case is illustrated by taking $N = 3$. If

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & & & & & 1 \\ & \sqrt{2} & & & & \\ & & 1 & 1 & & \\ & & 1 & -1 & & \\ & & & & \sqrt{2} & \\ & & & & 1 & 1 \\ & & & & 1 & -1 \\ & & & & & & \sqrt{2} \\ 1 & & & & & & & 1 \end{bmatrix}$$

then

(3.1) $$Q^{-1}\underline{A}_\theta Q$$

$$= \operatorname{diag}(I-R_{\theta_3}, I, I+R_{\theta_1}, I-R_{\theta 1}, I, I+R_{\theta_2}, I-R_{\theta_2}, I, I+R_{\theta_3}) \quad ,$$

so we turn to consider the scalar integral equations

(3.2) $$(I \pm R_\theta)v = f \quad \text{on} \quad [0,1] \quad .$$

Let $1 \leq p \leq \infty$ and $\alpha \in \mathbb{R}$. For functions $v \in L^1_{loc}(0,1]$, put

$$\|v\|_{p,\alpha} = \|t \mapsto t^{\alpha} v(t)\|_p ,$$

where $\|\cdot\|_p$ is the usual norm on $L^p(0,1)$, and for functions $w \in L^1_{loc}[0,\infty)$ put

$$\|w\|_{p,\alpha} = \|y \mapsto e^{-\alpha y} w(y)\|_p ,$$

where $\|\cdot\|_p$ is the usual norm on $L^p(0,\infty)$. Denote the corresponding Banach spaces by $L^p_\alpha(0,1)$ and $L^p_\alpha(0,\infty)$ respectively, then

$$J_p v(y) = \begin{cases} e^{-y/p} v(e^{-y}) & \text{if} \quad 1 \leq p < \infty \\ v(e^{-y}) & \text{if} \quad p = \infty \end{cases}$$

defines a linear isometry

$$J_p : L^p_\alpha(0,1) \to L^p_\alpha(0,\infty) \quad , \quad 1 \leq p \leq \infty .$$

Notice that $L^p_0 = L^p$. A simple calculation reveals that

$$\mathcal{R}_{\theta,1/p} = J_p R_\theta J_p^{-1} \tag{3.3}$$

is the Wiener-Hopf operator defined by

$$\mathcal{R}_{\theta,\mu} w(x) = \int_0^\infty \kappa_{\theta,\mu}(x-y) w(y)\, dy \quad , \quad 0 < x < \infty$$

$$\kappa_{\theta,\mu}(x) = \frac{1}{\pi} \frac{e^{-\mu x} \sin\theta}{e^x + e^{-x} + 2\cos\theta} \quad , \quad x \in \mathbb{R} .$$

Using the results in [7], many of the essential properties on $\mathcal{R}_{\theta,\mu}$ can be deduced from a knowledge of the Fourier transform of its kernel:

$$\hat{\kappa}_{\theta,\mu}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} \kappa_{\theta,\mu}(x)\, dx = \frac{\sinh[\theta(\xi+i\mu)]}{\sinh[\pi(\xi+i\mu)]} \quad , \quad -1 < \mu < 1 .$$

For our purposes, it suffices to know that the norm of $\mathcal{R}_{\theta,\mu}$ as an operator on $L^p_\alpha(0,\infty)$ satisfies

$$\|\mathcal{R}_{\theta,\mu}\|_{p,\alpha} \leq \int_{-\infty}^{\infty} |e^{-\alpha x} \mathcal{r}_{\theta,\mu}(x)|\,dx = \int_{-\infty}^{\infty} \mathcal{r}_{|\theta|,\mu+\alpha}(x)\,dx = \hat{\mathcal{r}}_{|\theta|,\mu+\alpha}(0) ,$$

that is,

$$\|\mathcal{R}_{\theta,\mu}\|_{p,\alpha} \leq \begin{cases} \dfrac{\sin[|\theta|(\mu+\alpha)]}{\sin[\pi(\mu+\alpha)]} & \text{if } 0 < |\mu+\alpha| < 1 \\ \dfrac{\theta}{\pi} & \text{if } \mu+\alpha = 0 . \end{cases} \tag{3.4}$$

Since J_p is an isometry, we see from (3.3) that the norm of R_θ as an operator on $L^p_\alpha(0,1)$ satisfies

$$\|R_\theta\|_{p,\alpha} = \|\mathcal{R}_{\theta,1/p}\|_{p,\alpha} \tag{3.5}$$

and therefore:

THEOREM: *If* $|\alpha + 1/p| < (1 + |\theta|/\pi)^{-1}$, *then*

$$\|R_\theta\|_{p,\alpha} \leq \frac{\sin[|\theta|(\alpha + 1/p)]}{\sin[\pi(\alpha + 1/p)]} < 1$$

and so

$$(I \pm R_\theta) : L^p_\alpha(0,1) \to L^p_\alpha(0,1)$$

is an isomorphism with

$$\|(I \pm R_\theta)^{-1}\|_{p,\alpha} \leq (1 - \|R_\theta\|_{p,\alpha})^{-1} .$$

Proof: This is just a matter of checking

$$|\mu| < (1 + |\theta|/\pi)^{-1} \Rightarrow \frac{\sin[|\theta|\mu]}{\sin[\pi\mu]} < 1 ,$$

then using (3.4) and (3.5), and writing $(I \pm R_\theta)^{-1} = \sum_{n=0}^{\infty} (\mp R_\theta)^n$. ∎

By developing a suitable regularity theory for the corresponding Wiener-Hopf equation, one can show that if f in (3.2) is smooth on $[0,1]$, then the solution v satisfies

$$D^m v(s) = O(s^{\alpha-m}) \quad \text{as} \quad s \downarrow 0 \tag{3.6}$$

for $m \in \{1,2,3,\ldots\}$ and

$$\alpha < (1+|\theta|/\pi)^{-1} ,$$

cf. [3].

If we define the product space

$$\underline{L}^p = \prod_{}^{3J} L^p(0,1)$$

then it is clear from (3.1) and the theorem above (with $\alpha = 0$) that

$$\underline{A}_\theta : \underline{L}^p \to \underline{L}^p$$

is an isomorphism for $1/p < \min_j (1+|\theta_j|/\pi)^{-1}$. Since $\underline{K} : \underline{L}^p \to \underline{L}^p$ is compact $(1 \leq p \leq \infty)$, it is easy to show from (2.6) that

$$\underline{I} + \underline{T} : \underline{L}^p \to \underline{L}^p$$

is an isomorphism for

$$\max_j (1+|\theta_j|/\pi) < p \leq \infty .$$

By considering $\underline{I} + \underline{T}$ as an operator on a suitable product of weighted Sobolev spaces, one can show that the solution of (2.2) also satisfies (3.6) near the corner x_j (with $\theta=\theta_j$), provided g is smooth on $\bar{\Gamma}_j$ and $\bar{\Gamma}_{j+1}$.

In section 1, when discussing the Dirichlet problem, we assumed the data g to be continuous. If, for example, g is only assumed to be in

$L^2(\Gamma)$, then (1.3) holds only σ-almost everywhere and the limit must be taken nontangentially. A solution to the Dirichlet problem, in this generalized sense, is still furnished by (1.5). (Some recent results in this direction for Lipschitz domains may be found in [9].) In practice, one is mostly interested in the case of piecewise smooth data. By considering the obvious corresponding system of integral equations, one easily sees that jumps in g cause no real problems, it is only the presence of corners in the region Ω that gives rise to singularities in the derivatives of the density u .

4. REMARKS ON NUMERICAL METHODS

The decomposition (2.6) of $\underline{I}+\underline{T}$ into a principal part $\underline{A}_\theta$ and a compact perturbation $\underline{K}$ is especially useful for the analysis of numerical methods of solving (2.2). Indeed, the usual collective compactness arguments, see e.g. [1], effectively reduce the problem to that of solving the simplified system of equations

$$\underline{A}_\theta \underline{u} = 2\underline{g} ,$$

and of course the diagonalization (3.1) reduces this to the problem of solving the scalar equations

$$(I \pm R_\theta) v = f \quad \text{on} \quad [0,1] . \tag{4.1}$$

If (4.1) is considered as an operator equation on a Banach space X , then a projection method for obtaining an approximate solution consists in choosing a (finite dimensional) subspace X_n of X and a projection operator

$$P_n : X \to X_n ,$$

i.e. $P_n^2 = P_n$, then looking for $v_n \in X_n$ such that

$$(I \pm P_n R_\theta) v_n = P_n f \ . \tag{4.2}$$

For example, if one chooses mesh points

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

then one might take

$$X_n = \{v : v \text{ is a function on } [0,1] \text{ which coincides on each subinterval } (t_{i-1}, t_i) \text{ with a polynomial of degree } \leq k-1 \} \ .$$

In the Galerkin method, $X = L^2(0,1)$ and $P_n = P_n^G$ is the Hilbert space projection onto X_n . In the collocation method, $X = C[0,1]$ and $P_n = P_n^C$ is the interpolation operator for the chosen collocation points (k on each subinterval). After expressing v_n in terms of a basis for X_n , (4.2) reduces to an $nk \times nk$ system of algebraic equations; see [1]. The coefficients of this system involve integrals which will nearly always be approximated by numerical quadratures, and evaluating these coefficients may require more computing time than solving the linear system.

One can think of the standard error analysis for v_n in terms of stability and consistency. The stability part is a matter of showing that $(I \pm P_n R_\theta)^{-1}$ exists and of estimating its norm as $n \to \infty$. The consistency part is a matter of estimating the norm of

$$(I \pm P_n R_\theta)(v_n - v) = (P_n - I) v \ ,$$

and it is clear that the two results imply an error estimate for v_n , viz.

$$\|v_n - v\|_X \leq \|(I \pm P_n R)^{-1}\|_X \, \|(I - P_n) v\|_X \ . \tag{4.3}$$

The consistency part is reasonably simple once one knows the qualitative behaviour (3.6) of the derivatives of v. A suitable type of mesh is given by

$$t_i = (i/n)^q , \quad i = 0(1)n , \; q \geq 1 .$$

If $q = 1$, then the points t_i are equally spaced, but as q is increased the mesh points are concentrated near zero where the solution is changing rapidly. One finds, see [3], that for the Galerkin projection

$$\| (I-P_n^G)v \|_2 \leq c\, n^{-k} \sup_{0<s\leq 1} | s^{k-\alpha} D^k v(s) | = O(\frac{1}{n^k})$$

provided $q > r/(\alpha+\frac{1}{2})$, where the constant c is independent of u and n. Similarly, if one chooses points ξ_j satisfying

$$-1 < \xi_1 < \xi_2 < \ldots < \xi_k < 1$$

and takes the collocation points on (t_{i-1}, t_i) to be

$$x_{ij} = \frac{1}{2}\{ (1-\xi_j)t_{i-1} + (1+\xi_j)t_i \} , \quad i = 1(1)n , \quad j = 1(1)k ,$$

i.e. the x_{ij} are the ξ_j shifted to the i^{th} subinterval, then

$$\| (I-P_n^C)v \|_\infty \leq c\, n^{-k} \sup_{0<s\leq 1} | s^{r-\alpha} D^r v(s) | = O(\frac{1}{n^k})$$

provided $q > r/\alpha$.

The question of stability, on the other hand, is much deeper, owing to the fact that since R_θ is not compact we cannot appeal to collective compactness. If one can show that there is a $\delta > 0$ for which

$$\| P_n R_\theta \|_X \leq 1-\delta \tag{4.4}$$

unformly in n sufficiently large, then the simple geometric series bound

$$\| (I \pm P_n R_\theta)^{-1} \|_X \leq \delta^{-1}$$

follows immediately. In the case of the Galerkin method, (4.4) is easy because $\|P_n^G\|_2 = 1$, however for the collocation method $\|P_n^C\| > 1$ in general. In [2], (4.4) was established by ad hoc techniques for some particular collocation schemes, as well as certain stability results based on the matrix analogues of (4.3) and (4.4). A disadvantage of stability analyses based on (4.4) is that this condition is sufficient but not necessary, and indeed there are probably stable schemes for which (4.4) does not hold. A further complication is that the use of numerical quadratures effectively modifies the projection operator P_n, so the easy stability proof for the Galerkin method is not as useful as it may appear.

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