PROBLEMS WITH DIFFERENT TIME SCALES

Heinz-Otto Kreiss

1. INTRODUCTION

Perhaps the simplest problem with different time scales is given by the initial value problem for the ordinary differential equation

$$(1.1) \qquad \varepsilon dy/dt = ay + e^{it}, \quad t \geq 0, \quad y(0) = y_0 .$$

Here ε, a are constants with $0 < \varepsilon << 1$, $|a| = O(1)$ and Real $a \leq 0$. The solution of (1.1) is given by

$$(1.2) \qquad y(t) = y_S(t) + y_R(t),$$

where

$$y_S(t) = e^{it}(-a+i\varepsilon)^{-1}, \quad y_R(t) = e^{(a/\varepsilon)t}(y_0 - y_S(0)) .$$

Thus it consists of the slowly varying part $y_S(t)$ and the rapidly changing part $y_R(t)$. There are two fundamentally different situations

1) $\underline{a = -1}$. In this case $y_R(t)$ decays rapidly and outside a boundary layer the solution of (1.1) varies slowly. Many people have developed numerical methods to solve problems of this kind (see for example [15]) and we shall not consider this case.

2) $\underline{a = i}$ is purely imaginary. Now $y_R(t)$ does not decay and $y(t)$ is highly oscillatory everywhere. In many applications one is not interested in the fast time scale. Therefore it is of interest to

develop methods to prepare the initial data such that the fast time scale is suppressed. We shall describe one such method.

Initialisation. It prepares the initial data in such a way that the fast time scale is not activated. In the above example we need only to choose

$$y_0 = y_S(0) = (-a+i\varepsilon)^{-1} = -a^{-1}(1+i\varepsilon/a-(\varepsilon/a)^2+\ldots) \ . \tag{1.3}$$

Then $y_R(t) \equiv 0$ and the solution of our problem consists only of the slowly varying part $y_S(t)$. For more complicated problems one can determine $y_S(0)$ only approximately. The rapidly changing part will always be present but we can reduce its amplitude to the size $O(\varepsilon^p)$, $p = 1,2,\ldots$. An effective way to do this is to use the "bounded derivative principle" which is based on the following observation:

If $y(t)$ varies on the slow time scale then $d^\nu y/dt^\nu \sim O(1)$ for $\nu = 1,2,\ldots,p$ where $p > 1$ is some suitable number. Therefore our principle is

Choose the initial value $y(0) = y_0$ *such that for* $t = 0$

$$d^\nu y/dt^\nu\Big|_{t=0} \sim O(1) \ , \quad \nu = 1,2,\ldots,p \ . \tag{1.4}$$

Using the differential equation we can express the derivatives at $t = 0$ in terms of $y(0)$. Therefore we can determine $y(0)$ such that (1.4) is satisfied without solving the differential equations.

Let us apply this principle to our example. $dy/dt\Big|_{t=0} = O(1)$ if and only if

$$ay(0) = -1 + O(\varepsilon)$$

i.e.

(1.5) $$y(0) = -1/a + \varepsilon y_1(0)\ ,\quad dy/dt\Big|_{t=0} = ay_1\ ,\quad y_1 = O(1)\ .$$

If we choose $y(0)$ according to (1.5) then

$$y_R(0) = y(0) - y_S(0) = -1/a + \varepsilon y_1 - 1/(-a+i\varepsilon) = O(\varepsilon)\ ,$$

i.e. the amplitude of $y_R(t)$ is $O(\varepsilon)$ for all times. We consider now the second derivative. The differential equation gives us

$$\varepsilon d^2y/dt^2 = ady/dt + ie^{it}$$

Thus $d^2y/dt^2\Big|_{t=0} = O(1)$ if and only if

$$ady/dt\Big|_{t=0} = a^2y_1 = -i + O(\varepsilon)\ ,$$

i.e.

$$y_1 = -i/a^2 + \varepsilon y_2\ ,\quad d^2y/dt^2\Big|_{t=0} = a^2y_2\ ,$$

and by (1.5)

(1.6) $$y(0) = -1/a(1+i\varepsilon/a) + \varepsilon^2 y_2\ .$$

In this case we obtain for the amplitude

$$y_R(0) = y(0) - y_S(0) = O(\varepsilon^2)\ .$$

The above procedure can be continued. If we choose the initial data such that the first p time derivatives are $O(1)$ then the amplitude of the fast part of the solution is $O(\varepsilon^p)$. We are going to indicate that results of this kind are valid for very general systems of linear and nonlinear ordinary and partial differential equations.

2. SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

In this section we consider systems of ordinary differential equations

$$(2.1) \qquad dy/dt = \frac{1}{\varepsilon} A(t)y + f(y,t) \ , \quad y(0) = y_0 \ , \quad t \geq 0 \ .$$

Here $\varepsilon > 0$ is a small constant, $y = (y^{(1)}, y^{(2)}, \ldots, y^{(n)})^T$ is a vector function with n components, $A(t) = - A^*(t)$ is a skew-hermitien matrix and $f(y,t)$ is a vector function which depends nonlinearly on y and t. We assume also that $y_0 = 0(1)$ and that $A(t)$ and $f(y,t)$ have p derivatives with respect to all variables and that these derivatives can be estimated by expressions $K_1 |y|^m + K_2$, $K_j = 0(1)$, $m > 0$ a natural number. Thus there are two time scales present, a slow $0(1)$ and a fast $0(1/\varepsilon)$.

We assume also that the eigenvalues $\mathcal{H}$ of $A(t)$ are either identically zero or different from zero for all times, i.e. the rank of $A(t)$ is independent of t. One can prove

<u>THEOREM 2.1</u> *There is an interval* $0 \leq t \leq T$ *and a constant* K_p *such that*

$$\sup_{0 \leq t \leq T} \sum_{\nu=0}^{p} |d^\nu y(t)/dt^\nu| \leq K_p \sum_{\nu=0}^{p} |d^\beta y/dt^\nu|_{t=0}| \ .$$

Here $K_p = 0(1)$ *and* T *does not depend on* ε.

We shall now discuss how to choose the initial data. Without restriction we can assume that (2.1) has the form

$$(2.2)\qquad \begin{aligned} du/dt &= \frac{1}{\varepsilon} A_1(t)u + g(y,t) \\ dv/dt &= h(y,t) \ , \quad y = (u,v)^T \ , \end{aligned}$$

where A_1 has full rank and h is independent of ε

$dy/dt\big|_{t=0} = 0(1)$ if and only if

$$A_1 u(0) = 0(\varepsilon) \ .$$

Thus

$$(2.3)\qquad u(0) = \varepsilon q_1 \ , \quad q_1 = 0(1)$$

and

$$(2.4)\qquad du/dt\big|_{t=0} = A_1(0)q_1 + g(0,v(0),0) + 0(\varepsilon) \ .$$

(2.3) guarantees that $d^2v/dt^2 = 0(1)$. For d^2u/dt^2 we have

$$\begin{aligned} d^2u/dt^2\big|_{t=0} &= \frac{1}{\varepsilon} (A_1(0)du/dt + (dA_1/dt)u)\big|_{t=0} + 0(1) \\ &= \frac{1}{\varepsilon} A_1(0)(A_1(0)q_1 + g(0,v(0),0)) + 0(1) \end{aligned}$$

Therefore $d^2u/dt^2\big|_{t=0} = 0(1)$ if and only if

$$q_1(0) = - A_1^{-1}(0)g(0,v(0),0) + \varepsilon q_2 \ , \quad q_2 = 0(1)$$

or

$$(2.5)\qquad u(0) = - \varepsilon A_1^{-1}(0)g(0,v(0),0) + 0(\varepsilon^2) \ .$$

(2.5) tells us that $u(0)$ is determined by $v(0)$ up to terms of order $O(\varepsilon^2)$. This process can be continued. If we demand that p time derivatives are bounded independently of ε then $u(0)$ is determined by $v(0)$ up to terms of order $O(\varepsilon^p)$.

We can now use our results to derive reduced systems. We know, that if we choose the initial data such that p time derivatives are bounded independently of ε, then u is determined by v up to terms of order $O(\varepsilon^p)$. This relation can be derived for every fixed t with $0 \leq t \leq T$. Thus we can replace the differential equation for u by the above relation between u and v. We obtain reduced systems which become more and more refined depending on the number of time derivatives which stay bounded. The crudest system is

$$(2.6) \qquad u = 0 \, , \quad dv/dt = h(0,v,t)$$

and an improved system is given by

$$(2.7) \qquad A_1(t)u(t) + \varepsilon g(0,v(t),t) = 0 \, , \quad dv/dt = h(u,v,t) \, .$$

3. PARTIAL DIFFERENTIAL EQUATIONS

The earlier results can be generalized to partial differential equations. In this section we consider equations of the form

$$(3.1) \qquad u_t = \frac{1}{\varepsilon} P_0(x,t,\partial/\partial x)u + P_1(u,x,t,\varepsilon\partial/\partial x)u + F(x,t)$$

is s space dimensions. Here P_0, P_1 are first order differential operators with symmetric matrices as coefficients. The main assumption is that the number of eigenvalues $\varkappa \neq 0$ of the symbol

$$\hat{P}_0(x,t,i\omega) = i \sum_{j=1}^{s} A_j(x,t)\omega_j$$

does not depend on the frequency ω and x , t . Then the bounded derivative principle is valid, i.e. the following theorem holds

<u>THEOREM 3.1</u> *Assume that all derivatives*

$$\partial^{|\nu|+\mu} u(x,t)/\partial x_1^{\nu_1} \ldots \partial x_s^{\nu_s} \partial t^{\mu}\Big|_{t=0} , \quad |\nu| + \mu := \Sigma\nu_i + \mu \le p$$

are bounded independently of ε . *Then we can estimate these derivatives independently of* ε *in a time interval* $0 \le t \le T$, T *independent of* ε , *provided* $p \ge [\frac{1}{2}s] + 2$. *Here* $[\frac{1}{2}s]$ *is the largest integer with* $[\frac{1}{2}s] \le s/2$.

As an application we consider the shallow water equations which play a central role in geophysics. In meteorology they govern two classes of motions with different time scales, consisting of low frequency Rossby waves and high frequency inertial gravity waves. Often one is not interested in the inertial gravity waves which then have to be filtered out. Initialisation procedures have been considered for a long time and we refer to [3] for a more detailed account of the development. In Cartesian coordinates x and y , directed eastward and northward respectively, the shallow water equations including the effect of gravity (see for example [11]) are expressed by

$$\begin{aligned} & u_t + uu_x + vu_y + gh_x - fv = 0 , \\ (3.2) \qquad & v_t + uv_x + vv_y + gh_y + fu = 0 , \\ & h_t + (uh)_x + (vh)_y + h_0(u_x+v_y) = 0 . \end{aligned}$$

Here t is time, u and v are the velocity components in the x and y directions, h_0 is the mean height of the homogeneous fluid, h denotes the deviation from the mean height $g \cong 10\ ms^{-2}$ is the constant gravity acceleration and f denotes the coriolis force. For simplicity only, we assume that f is constant. Using typical scale parameters (see for example [3]), we can write (3.2) in nondimensional form

$$\frac{du}{dt} + \varepsilon^{-1}(h_x - fv) = 0 \ ,$$

$$(3.3) \qquad \frac{dv}{dt} + \varepsilon^{-1}(h_y + fu) = 0 \ ,$$

$$\frac{dh}{dt} + \varepsilon^{-2}(1+\varepsilon^2 h)(u_x + v_y) = 0 \ , \qquad \varepsilon << 1 \ .$$

Here

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \ .$$

The system (3.3) is not exactly of the form as discussed earlier. However the proofs in [5], [14] can be modified to cover this case because we can symmetrize the equations by introducing new variables

$$u(1+\varepsilon\Phi) = \varepsilon^{\frac{1}{2}}\tilde{u} \ , \quad v(1+\varepsilon\Phi) = \varepsilon^{\frac{1}{2}}\tilde{v} \ .$$

We shall now use the bounded derivative principle to determine the relations the initial data have to satisfy such that only the Rossby waves are present. The first time derivatives are bounded independently of ε at $t = 0$ if

$$(3.4) \qquad h_x - fv = \varepsilon a \ , \quad h_y + fu = \varepsilon b \ , \quad u_x + v_y = \varepsilon^2\delta \ .$$

Thus the data have to be (approximatively) in geostrophic balance and the divergence has to be small. If we choose the initial data according to (3.4) then these relations hold also at later times. We can therefore assume that a, b, δ are smooth functions and can rewrite the equations (3.3) in the form

$$du/dt + a = 0 \ , \quad dv/dt + b = 0 \ , \quad dh/dt + (1+\varepsilon^2 h)\delta = 0 \tag{3.5}$$

Also, we can write (3.4) as

$$\varepsilon(a_x+b_y) = h_{xx} + h_{yy} + f(u_y-v_x) \ , \quad a_y - b_x = \varepsilon f\delta \ ,$$
$$u_x + v_y = \varepsilon^2\delta \ . \tag{3.6}$$

The second time derivatives are bounded independently of ε if da/dt , db/dt , $d\delta/dt$ are $O(1)$. This leads to

$$\varepsilon^2 d\delta/dt = du_x/dt + dv_y/dt = (du/dt)_x + (dv/dt)_y - J(u,v) =$$
$$= -(a_x+b_y) - J(u,v) = \tag{3.7}$$
$$= \varepsilon^{-1}(h_{xx}+h_{yy}+f\xi) - J(u,v) \ ,$$

where $\xi = u_y - v_x$ is the vorticity and $J(u,v) = (u_x)^2 + 2u_yv_x + (v_y)^2$ the Jacobian. Thus we have to choose the initial data such that the balance equation

$$h_{xx} + h_{yy} + f\xi + \varepsilon J(u,v) = O(\varepsilon^3) \tag{3.8}$$

is satisfied. Using (3.7) we can write (3.6) in the form

$$(3.9) \qquad a_x + b_y + J(u,v) = O(\varepsilon^2) , \quad a_y - b_x = \varepsilon f\delta , \quad u_x + v_y = \varepsilon^2\delta .$$

$da/dt = O(1)$, $db/dt = O(1)$ give us correspondingly

$$\delta_x + u_x h_x + v_x h_y - fb = O(\varepsilon) ,$$

$$\delta_y + u_y h_x + v_y h_y + fa = O(\varepsilon) ,$$

or, using the geostrophic relations

$$(3.10) \qquad \begin{aligned} &\delta_x + f(u_x v - v_x u) - fb = O(\varepsilon) , \\ &\delta_y + f(u_y v - v_y u) + fa = O(\varepsilon) . \end{aligned}$$

The last two relations are compatible because if we crossdifferentiate them we obtain (3.7). Thus δ is determined by

$$(3.11) \qquad \delta_{xx} + \delta_{yy} + f((u_x v - v_x u)_x + (u_y v - v_y u)_y) = O(\varepsilon) .$$

We shall now discuss how one can find initial data such that the relations (3.9) and (3.11) are satisfied. Let the vorticity $u_y - v_x = \xi$ be given. Then we determine preliminary velocities u^* , v^* from

$$u_x^* + v_y^* = 0 , \quad u_y^* - v_x^* = \xi$$

and determine δ as the solution of

$$\delta_{xx} + \delta_{yy} + f((u_x^* v^* - v_x^* u^*)_x + (u_y^* v^* - v_y^* u^*)_y) = 0 .$$

The final values u , v , h , a and b are the solutions of

(3.12a) $$u_x + v_y = \varepsilon^2\delta \ , \quad u_y - v_x = \xi \ , \quad h_{xx} + h_{yy} + f\xi - J(u,v) = 0 \ ,$$

(3.12b) $$a_x + b_y + J(u,v) = 0 \ , \quad a_y - b_x = \varepsilon f\delta \quad .$$

Instead of solving the original system with proper initial data we can also replace (3.3) by a reduced system. In [4] we have made a detailed study of the system

$$du/dt + a = 0 \ , \quad dv/dt + b = 0 \ ,$$

$$a_x + b_y + J(u,v) = 0 \ , \quad a_y - b_x = \varepsilon f\delta \quad .$$

REFERENCES

1. Baer, F., 1977, Adjustment of initial conditions required to suppress gravity oscillations in nonlinear flows. *Beitr. Phys. Atmos.*, 50, 350-366.

2. Barker, J., 1982, Interactions of fast and slow waves in problems with two time scales, Ph.D. thesis, California Inst. of Technology, Pasadena, will partially appear in Math. Meth. Appl. Sci. and SIAM Journal of Analysis.

3. Browning, G., A. Kasahara, and H.-O. Kreiss, 1980, Initialization of the primitive equations by the bounded derivative method. *J.Atmos. Sci.*, 37, 1424-1436.

4. Browning, G. and H.-O. Kreiss, 1982, Initialization of the shallow water equations with open boundaries by the bounded derivative method. *Tellus*, 34, 334-351.

5. Browning, G. and H.-O. Kreiss, 1982, Problems with different time scales for nonlinear parital differential equations. SIAM J. Appl. Math., Vol. 42.

6. Browning, G. and H.-O. Kreiss, 1983, Scaling and computation of atmopsheric motions, to appear.

7. Gustafsson, G. and H.-O. Kreiss, Difference approximations of hyperbolic problems with different time scales. I. The reduced problem. Report No.86, Uppsala University, Dept. of Computer Sciences. To appear in SIAM J. Num. Anal.

8. Gustafsson, B., 1980, Asymptotic expansions for hyperbolic systems with different time scales. SIAM J. Num. Anal., Vol. 17, No.5, pp.623-634.

9. Gustafsson, B., 1980, Numerical solution of hyperbolic systems with different time scales using asymptotic expansions. J. Comp. Phys. 36, pp.209-235.

10. Guerra, J. and B. Gustafsson, 1982, A semi-implicit method for hyperbolic problems with different time-scales, Uppsala University, Dept. of Computer Sciences, Report No.90.

11. Kasahara, A., 1974, Various vertical coordinate systems used for numerical weather prediction. *Mon. Wea. Rev.*, 102, 509-522.

12. Kasahara, A., 1982, Nonlinear normal mode initialization and the bounded derivative method. Review of Geophysics and Space Physics, Vol. 20, No.3, 385-397.

13. Kreiss, H.-O., 1979, Problems with different time scales for ODE. SIAM J. Num. Anal., 16, 980-998.

14. Kreiss, H.-O., 1980, Problems with different time scales for PDE. Comm. Pure Appl. Math., 33, 399-439.

15. Kreiss, H.-O., 1978, Difference methods for stiff ordinary differential equations, SIAM J. Num. Anal., Vol. 15, 21-58.

16. Kreth, H. 1977, Time-discretisations for nonlinear evolution equations. Lecture Notes in Mathematics Nr. 679, Srpinger-Verlag.

17. Leith, C.E., 1980, Nonlinear normal mode initialization and quasi-geostrophic theory. J. Atmos. Sci., 37, 954-964.

18. Machenhauer, B., 1977, On the dynamics of gravity oscillations in a shallow water model, with applications to normal mode initialization. Beitr. Phys. Atmos., 50, 253-271.

19. Oliger, J. and A. Sundstrom, 1978, Theoretical and practical aspects of some initial-boundary value problems in fluid dynamics, SIAM J. Appl. Math., 35, 419-446.

20. Tadmor, E., 1982, Hyperbolic systems with different time scales. Comm. Pure Appl. Math., 35, 839-866.

Department of Applied Mathematics,
California Institute of Technology
PASADENA, California 91125 U.S.A.