HARMONIC MORPHISMS ONTO RIEMANN SURFACES
— SOME CLASSIFICATION RESULTS

Paul Baird

1. INTRODUCTION

Let $\phi : M \to N$ be a mapping between smooth Riemannian manifolds. Then ϕ is called a *harmonic morphism* if $f \circ \phi$ is harmonic on $\phi^{-1}(V)$ for every function f harmonic on an open set $V \subset N$. Such mappings were first studied in detail by Fuglede [8] and Ishihara [10]. They established an alternative characterization as follows.

For a point $x \in M$ at which $d\phi(x) \neq 0$, let $V_x M$ denote the subspace of $T_x M$ given by $\ker d\phi(x)$, and let $H_x M$ denote the orthogonal complement of $V_x M$ in $T_x M$. Say that ϕ is *horizontally conformal* if the restriction mapping $d\phi(x)\big|_{H_x M} : H_x M \to T_{\phi(x)} N$ is conformal and surjective. Letting g,h denote the metrics of M,N respectively, this means that there exists a number $\lambda(x)$ such that $\lambda(x)^2\, g(X,Y) = h(d\phi(X), d\phi(Y))$ for each $x \in M$ with $d\phi(x) \neq 0$ and for all $X,Y \in H_x M$. Let $C_\phi = \{x \in M \mid d\phi(x) = 0\}$ denote the *critical set of* ϕ, and set $\lambda = 0$ on C_ϕ. Then we obtain a continuous function $\lambda : M \to \mathbb{R}$ called the *dilation of* ϕ. In general λ is not smooth, although clearly $\lambda^2 : M \to \mathbb{R}$ is a smooth function.

(1.1) *A map* $\phi : M \to N$ *is a harmonic morphism if and only if it is both harmonic and horizontally comformal* [8], [10].

It follows therefore that if ϕ is a harmonic morphism then $\dim M \geq \dim N$. If $\dim M = \dim N$, then in the case when $\dim M = 2$, the harmonic morphisms are precisely the weakly conformal mappings between surfaces. If $\dim M = \dim N \geq 3$, then any harmonic morphism must be a homothetic map. It is easy to check that the composition of two harmonic morphisms is also a harmonic morphism.

If $\phi : M \to N$ is a harmonic morphism, then the critical set C_ϕ forms a polar set in M [8]. At points where C_ϕ is a submanifold this means that $\operatorname{codim} C_\phi \geq m-2$, where $m = \dim M$. In certain circumstances, when M is the Euclidean space $\mathbb{R}^m$, the critical set forms a minimal cone in $\mathbb{R}^m$ [1]. Fuglede has shown that in the case when the vector field $\nabla\lambda^2$ is bounded away from the horizontal then C_ϕ is empty [9].

Independently, harmonic morphisms have been studied by Bernard, Cambell and Davie [4]. They show that a mapping between open subsets of Euclidean spaces is Brownian path preserving (in the context of stochastic processes) if and only if it is a harmonic morphism. That characterization of Brownian path preserving functions is due to P. Levy. They study in detail the case when M is an open subset of $\mathbb{R}^3$ and N a domain in the complex plane $\mathbb{C}$. One of the problems they pose is to classify all such harmonic morphisms. That classification is outlined below. One of the properties they observe of such mappings is that the fibres are straight lines. That ties in with results shown in [3], where the following is established.

(1.2) *If* $\phi : M \to N$ *is a harmonic morphism and* $n = \dim N$, *then*

(a) *if* $n = 2$ *the fibres over regular values of* ϕ *are minimal submanifolds of* M,

(b) *if* $n \geq 3$, *the fibres are minimal submanifolds if and only if* $\nabla\lambda^2$ *is vertical.*

From the above we see that the case when N is a Riemann surface is special. Indeed any conformal transformation of the range N will yield another harmonic morphism. One of the aims in the classification outlined below is to factor out such conformal transformations, restricting our attention to the structure of the fibres.

I would like to thank J. Eells, J. Jost and J.C. Wood for their helpful comments and correspondence.

I am especially indebted to the Centre for Mathematical Analysis, Canberra, for their support during the preparation of this work.

2. *The classification of harmonic morphisms from an open subset of 3-dimensional Euclidean space onto a Riemann surface.*

We obtain the classification in outline only, referring the reader to [2] for a detailed proof. Our aim is to establish the following theorem.

(2.1) THEOREM: *If* $\phi : M \to N$ *is a harmonic morphism from an open subset* M *of* $\mathbb{R}^3$ *onto a Riemann surface* N *. Then* ϕ *is the composition,* $\phi = \zeta \circ \gamma$ *, where* $\gamma : M \to P$ *is a harmonic morphism onto* $P \subset S^2$ *and* $\zeta : P \to N$ *is a weakly conformal map between Riemann surfaces. Furthermore the fibres of* γ *have the form*

$$s \mapsto sy + c(y)$$

for each $y \in P$ *, where* s *ranges over suitable values, and* c *is a conformal vector field over* P .

Conversely, any conformal vector field c *on an open subset* P *of* S^2 *yields a harmonic morphism* γ *as above.*

Outline of proof

Step 1. *Assume the critical set* C_ϕ *is empty.* This is a convenient assumption which we will be able to remove later (Step 6).

Step 2. *The map* ϕ *factors.* Thus $\phi = \zeta \circ \tilde{\phi}$, where $\tilde{\phi} : M \to \tilde{N}$, $\zeta : \tilde{N} \to N$ and N is the space of connected components of the fibres of ϕ . Furthermore $\tilde{N}$ can be given the structure of a Riemann surface with respect to which $\tilde{\phi}$ is a harmonic morphism with connected fibres and ζ is a conformal map between Riemann surfaces.

This follows since for each $x \in M$, $d\phi(x)\big|_{H_xM} : H_xM \to T_{\phi(x)}N$ is an isomorphism. Thus if W is a slice about x, that is a 2-dimensional submanifold of M which is everywhere bounded away from the vertical, then $\phi|_W : W \to N$ is locally a diffeomorphism by the inverse function theorem. Therefore W can be used to parametrize the points of $\tilde{N}$, and we can pull back the differentiable and conformal structure of N to $\tilde{N}$, thereby giving the above factorization.

Step 3. Suppose that $\phi : M \to N$ is a harmonic morphism from an open subset M of $\mathbb{R}^m$ onto a connected Riemann surface N. From Step 2 we can assume that the fibres are connected. From (1.2) the fibres are minimal in M.

In addition we will assume they are totally geodesic, so that the fibres are parts of $(m-2)$-planes in $\mathbb{R}^m$. We define the 'Gauss map' $\gamma : M \to G(m-2, \mathbb{R}^m)$, where $G(m-2, \mathbb{R}^m)$ denotes the Grassmannian of oriented $(m-2)$-planes in $\mathbb{R}^m$, by $\gamma(x) = V_xM$ for each $x \in M$. Then γ is constant along the fibres of ϕ and we obtain the commutative diagram

$$\begin{tikzcd} M \arrow[d, "\phi"'] \arrow[dr, "\gamma"] & \\ N \arrow[r, "\psi"'] & G(m-2, \mathbb{R}^m) \end{tikzcd}$$

for some map $\psi : N \to G(m-2, \mathbb{R}^m)$. In fact, writing $\psi(y) = e_3(y) \wedge \ldots \wedge e_m(y)$ for each $y \in N$, where $e_3(y), \ldots, e_m(y)$ is an orthonormal basis for the $(m-2)$-plane $\psi(y)$, the fibre of ϕ over has an expression

$$(s_3, \ldots, s_m) \mapsto \sum_{r=3}^{m} s_r e_r(y) + c(y)$$

for suitable $s_3, \ldots, s_m$ where $c(y)$ satisfies $\langle c(y), e_r(y) \rangle = 0$

$(r = 3, \ldots, m)$ with respect to the Euclidean inner product $\langle \ , \ \rangle$ on $\mathbb{R}^m$.

Step 4. *The map* ψ *defined above is holomorphic with respect to the natural complex structure on* $G(m-2, \mathbb{R}^m)$.

The Grassmannian $G(m-2, \mathbb{R}^m)$ is equivalent to the Grassmannian $G(2, \mathbb{R}^m)$ of oriented 2-planes in $\mathbb{R}^m$. This can be identified with the complex quadric hypersurface Q_{m-2} of $\mathbb{C}P^{m-1}$ from which it inherits a natural complex structure [5]. The proof that ψ is holomorphic is given in [2] and follows from the horizontal conformality of ϕ.

Step 5. *The case when* $m = 3$.

The Grassmannian $G(1, \mathbb{R}^3)$ is biholomorphic to the 2-sphere S^2. Writing $P = \psi(N) \subset S^2$, we gave the commutative diagram

$$\begin{tikzcd} M \subset \mathbb{R}^3 \arrow[d, "\phi"'] \arrow[dr, "\gamma"] & \\ N \arrow[r, "\psi"'] & P \subset S^2 \end{tikzcd}$$

Now for each $y \in N$, the vector $c(y)$ is perpendicular to the line $\psi(y)$. Thus $c(y)$ can be regarded as a vector in $T_{\psi(y)} S^2$. Otherwise said c is a section of the bundle $\psi^{-1}T\,S^2$. The two important results are

(i) ψ *is injective.*

(ii) c *can be regarded as a conformal vector field on* P.

See [2] for a detailed proof of these two statements. In fact, since ψ is holomorphic it is a branched covering map onto P. Removing branch points, the fibres over different sheets of the covering locally fill out open subsets of M. As we extend globally it is impossible for these open subsets to intersect in M. Thus M is the disjoint union, $M = \bigcup_i M_i$, of open sets corresponding to the number of sheets of the covering. Since any harmonic morphism is an open map, the image $\phi(M_i)$ is open.

Furthermore $\phi(M_i) \cap \phi(M_j)$ is empty since the fibres of ϕ are connected. Thus N is the disjoint union of open sets. The statement (i) now follows from the connectedness of N.

Statement (ii) is a result of the horizontal conformality of ϕ.

Step 6. *Reintroduce the critical set* C_ϕ.

The image of the critical set under ϕ consists of isolated points in N [8]. From the injectivity of ψ and the compactness of S^2, it follows that ψ has finite energy on deleted discs. By a Theorem of Sacks and Uhlenbeck [11], ψ extends over isolated points. Thus γ extends over C_ϕ also.

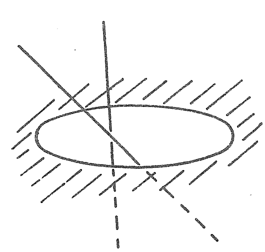
The above steps complete an outline of the proof of Theorem (2.1).

3. AN EXAMPLE

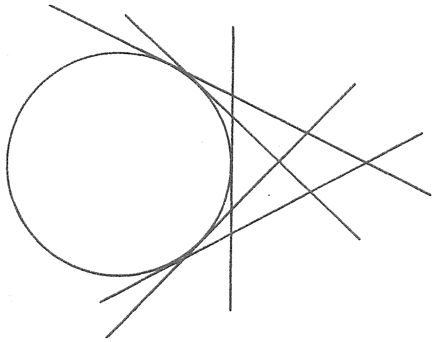
Every point $y \in S^2$ has an expression $y = (\cos t, \sin t\, e^{i\theta})$, where $t \in [0, \pi/2]$, $\theta \in [0, 2\pi)$. Let c be the vector field given by

$$c(y) = \sin t(0, i\, e^{i\theta})$$

at the point y (regarding S^2 as the unit sphere in $\mathbb{R}^3$). In fact c is a Killing vector field corresponding to rotations of S^2 about an axis. The corresponding harmonic morphism $\gamma : M \to P$ has domain M given by $M = \mathbb{R}^3 \backslash K$, where $K = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0,\ y^2 + z^2 \geq 1\}$, and range P given by the upper hemisphere (or lower one). The fibres twist through the hole $x = 0,\ y^2 + z^2 < 1$.

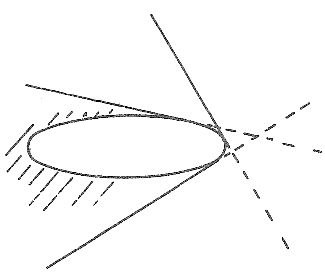


Fibres over the equatorial circle $C = \{(0, e^{i\theta}) \in S^2;\ \theta \in [0, 2\pi)\}$ intersect in K. The boundary of K is the envelope of these fibres. In fact the point of tangency of the fibres to the boundary of K is given by $s = 0$.



Each point of M lies on the fibre over a point of the upper hemisphere, and on a fibre over a point of the lower hemisphere. In this sense γ can be regarded as a multiple valued map. We now proceed by analogy to the construction of the Riemann surface of a multiple valued analytic function [12].

Take M and cut it along the set K. Points $(x,y,z) \in \mathbb{R}^3$, where $x = 0$, $y^2 + z^2 > 1$, can now be said to lie on one of two sheets. For a point p of the fibre over a point of C, say that p lies on the lower sheet if $s < 0$ and on the upper sheet if $s > 0$. Thus fibres over C pass from the lower sheet to the upper sheet.



Notice now that distinct fibres never intersect.

We now take two copies of the cut manifold and join along the edges created by the cuts, in such a way that fibres over corresponding points of C are identified. We thereby obtain a C^0-manifold $\tilde{M}$ homeomorphic to $S^2 \times \mathbb{R}$. The harmonic morphism γ now extends to a continuous single valued mapping $\tilde{\gamma} : \tilde{M} \to S^2$. The map $\tilde{\gamma}$ sends the interior of one of the cut manifolds to the upper hemisphere and the interior of the other to the lower hemisphere. The edges, which are identified, are mapped onto the equatorial circle.

4. *The classification of harmonic morphisms from an open subset of* S^3 *onto a Riemann surface.*

There is a one-to-one correspondence between harmonic morphisms $\nu : Q \to N$ where Q is an open subset of S^3 , and harmonic morphisms $\phi : M \to N$, where M is an open subset of $\mathbb{R}^4$ and ϕ has totally geodesic fibres which extend through the origin in $\mathbb{R}^4$. This is given by defining M to be the set $\mathbb{R}^+Q = \{\lambda x \in \mathbb{R}^4 \mid x \in Q,\ \lambda > 0\}$, and writing $\phi = \nu \circ \pi$, where $\pi : M \to Q$ is given by $\pi(x) = x/|x|$. As before we have the commutative diagram

$$\begin{tikzcd} M \arrow[d, "\phi"'] \arrow[dr, "\gamma"] & \\ N \arrow[r, "\psi"'] & G(2, \mathbb{R}^4) \end{tikzcd}$$

where $\psi : N \to G(2, \mathbb{R}^4)$ is holomorphic. Since the associated vector field c is identically zero, ψ must be injective and hence a biholomorphic map onto its image. Otherwise said, ψ is a holomorphic curve in the Grassmannian $G(2, \mathbb{R}^4)$. Conversely, given such a curve ψ , we can construct a corresponding harmonic morphism $\phi : M \to N$, with totally geodesic fibres which extend through the origin in $\mathbb{R}^4$. We therefore obtain the following classification.

(4.1) THEOREM: *If* $\nu : Q \to N$ *is a harmonic morphism from an open subset* Q *of* S^3 *onto a Riemann surface* N *, then* ν *is the composition,* $\nu = \zeta \circ \rho$ *, where* $\rho : Q \to P$ *is a harmonic morphism onto* $P \subset G(2, \mathbb{R}^4)$ *and*

$\zeta : P \to N$ *is a weakly conformal map between Riemann surfaces. Furthermore* P *is a holomorphic curve in the Grassmannian* $G(2, \mathbb{R}^4)$.

We consider the problem of which harmonic morphisms are defined globally on S^3 . A necessary condition is that the fibres of ν must not intersect in S^3 . This condition can be rephrased as follows.

The Grassmannian $G(2, \mathbb{R}^4)$ is biholomorphically equivalent to $S^2 \times S^2$. Write the corresponding holomorphic curve $\psi : N \to S^2 \times S^2$ as $\psi = (\psi_1, \psi_2)$. Then ψ_1, ψ_2 are harmonic maps. A detailed calculation verifies that the condition of non-intersecting fibres implies the strict inequality

$$e(\psi_1) > e(\psi_2) \qquad (\text{or } \; e(\psi_1) < e(\psi_2))$$

on the energy densities of ψ_1 and ψ_2 [2].

(4.2) THEOREM: *If* $\nu : S^3 \to S^2$ *is a harmonic morphism from the Euclidean 3-sphere onto* S^2 *, then up to a conformal transformation of* S^2 *,* ν *is the Hopf fibration.*

Proof: The Hopf fibration arises from the holomorphic curve $\tilde{\psi} : S^2 \to S^2 \times S^2$ given by $\tilde{\psi}(x) = (x, (1,0,0))$ for each $x \in S^2$ [2].

Now $\psi_i : S^2 \to S^2$ is harmonic. But any harmonic map from S^2 onto a Riemann surface is holomorphic [6].

Thus $\psi_i : S^2 \to S^2$ is a branched covering. Furthermore, since $e(\psi_i) \geq 0$, and $e(\psi_1) > e(\psi_2)$, we cannot have $e(\psi_2) = 0$ anywhere. Thus ψ_1 has no branch points and is a conformal diffeomorphism having degree 1 .

Any holomorphic map is an absolute minimum for the energy functional in its homotopy class. Since $e(\psi_2) < e(\psi_1)$, the energy of ψ_2 is strictly less than the energy of ψ_1 . But ψ_2 is holomorphic, from which we conclude that ψ_2 has degree 0 and hence is constant. Thus, up to an isometry of $S^2 \times S^2$, ψ is the holomorphic curve $\tilde{\psi}$.

REFERENCES

[1] P. Baird, 'Harmonic maps with symmetry, harmonic morphisms and deformations of metrics', *Research Notes in Math.* 87, Pitman (1983).

[2] P. Baird, 'On the classification of harmonic morphisms with totally geodesic fibres from an open subset of Euclidean space onto a Riemann surface', CMA report (1984).

[3] P. Baird and J. Eells, 'A conservation law for harmonic maps', Geometry Symp. Utrecht 1980. *Springer Notes* 894 (1981), 1-25.

[4] A. Bernard, E.A. Cambell and A.M. Davie, 'Brownian motion and generalized analytic and inner functions', *Ann. Inst. Fourier* 29 (1979), 207-228.

[5] J. Eells, *Gauss maps of surfaces*, (1983).

[6] J. Eells and L. Lemaire, 'A report on harmonic maps', *Bull. London Math. Soc.* 10 (1978), 1-68.

[7] J. Eells and L. Lemaire, 'On the construction of harmonic and holomorphic maps between surfaces', *Math. Ann.* 252 (1980), 27-52.

[8] B. Fuglede, 'Harmonic morphisms between Riemannian manifolds', *Ann. Inst. Fourier* 28 (b978), 107-144.

[9] B. Fuglede, 'A criterion of non-vanishing differential of a smooth map', *Bull. London Math. Soc.* 14 (1982), 98-102.

[10] T. Ishihara, 'A mapping of Riemannian manifolds which preserves harmonic functions', *J. Math. Kyoto Univ.* 19 (1979), 215-229.

[11] J. Sacks and K. Uhlenbeck, 'The existence of minimal immersions of two-spheres', *Ann. of Math.* 113 (1981), 1-24.

[12] C.L. Siegel, *Topics in complex function theory I*, Wiley (1969).