

UNCERTAINTY PRINCIPLES AND INTERFERENCE PATTERNS

John F. Price

The classical Heisenberg uncertainty principle

$$(1) \quad \Delta q \Delta p \geq \hbar/2$$

has been one of the key relationships in quantum mechanics for over fifty years. It does have a number of weaknesses, however, particularly related to the fact that the standard deviations Δq and Δp only give very general information about the spreads of the probability density functions of position and momentum respectively. This paper surveys a number of recent inequalities which describe more subtle relationships between position and momentum or, in mathematical terms, between a function and its Fourier transform. For example, local uncertainty principle inequalities assert that if the uncertainty of momentum Δp is small, then not only is the uncertainty of position Δq large, but the probability of the system being localized at any point is also small.

So as to add a little more interest, I have applied in turn each of the inequalities, starting with (1), to the proposition by Niels Bohr that in the double-slit experiment you can have an interference pattern or know the paths of the particles, but not both. In some ways I could not have chosen a worse test-case since it turns out that for this example they are all out-performed by Poisson summation. Nevertheless it does provide an opportunity to display and contrast some of their features. Also in the end we arrive at a rigorous justification of Bohr's original argument which apparently is new.

I am grateful to J.B.M. Uffink and J. Hilgevoord of the University of Amsterdam for sending me a copy of their preprint [3]. It aroused my interest in the Einstein-Bohr debate, particularly in the area of the two-slit experiment. Also they first pointed out the significance of the notions of W - and w -widths discussed later.

QUANTUM MECHANICAL BACKGROUND

In the Schrödinger interpretation the states of a one-dimensional quantum mechanical system are given by complex-valued functions $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$. The position and momentum operators are

$$q : f \mapsto xf \quad \text{and} \quad p : f \mapsto -i\hbar f'$$

respectively, where \hbar denotes Planck's constant and $\hbar = h/2\pi$. The probability density functions (pdf's) of position and momentum are

$$|f|^2 \quad \text{and} \quad \hbar^{-1} |F(\cdot/\hbar)|^2$$

respectively, where

$$F(y) = \int f(x) \exp(-2\pi ixy) \, dx$$

is the Fourier transform of f . (The Fourier transform is extended from $L^1(\mathbb{R})$ to $L^2(\mathbb{R})$ in the usual manner. Also, unless stated otherwise, all integrals are over \mathbb{R} .)

The expected values of position and momentum are

$$\langle q \rangle = \int x |f(x)|^2 \, dx \quad \text{and} \quad \langle p \rangle = \int (y/\hbar) |F(y/\hbar)|^2 \, dy$$

and their standard deviations are

$$\Delta q = \left(\int (x - \langle q \rangle)^2 |f(x)|^2 \, dx \right)^{\frac{1}{2}},$$

$$\Delta p = \left(\int (y - \langle p \rangle)^2 \hbar^{-1} |F(y/\hbar)|^2 \, dy \right)^{\frac{1}{2}}$$

whenever $\langle p \rangle$ and $\langle q \rangle$ exist. (If need be, $\langle q \rangle$ will be defined as

$$\langle q \rangle = \lim_{a \rightarrow \infty} \int_{|x| \leq a} x |f(x)|^2 \, dx,$$

and similarly for $\langle p \rangle$.)

It is now evident that the classical uncertainty principle inequality (1) is a simple consequence of the inequality

$$(2) \quad \int x^2 |f(x)|^2 dx \int y^2 |F(y)|^2 dy \geq (16\pi^2)^{-1} \left(\int |f(x)|^2 dx \right)^2$$

for all $f \in L^2(\mathbb{R})$. (For a proof see [5, p.117].)

SINGLE SLIT EXPERIMENT

The first weakness of (1) is that for quite reasonable states f , Δq and Δp can be infinite. For example, consider the classical experiment in which there is a parallel stream of particles passing through a single slit as in Figure 1. If the slit has width $2a$, the state function of the system (in the vertical direction) is

$$f = (2a)^{-\frac{1}{2}} 1_{[-a,a]} ,$$

where 1_E denotes the indicator function of E . Hence

$$F(y) = (2a)^{\frac{1}{2}} (\sin 2\pi ay) / 2\pi ay .$$

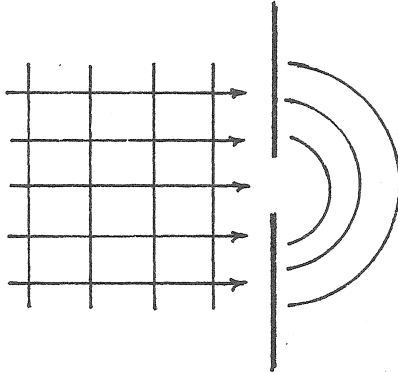


Figure 1

This means that the probability function of momentum is

$$(3) \quad \frac{2a}{h} \left(\frac{\sin 2\pi ay/h}{2\pi ay/h} \right)^2$$

so that $\Delta p = \infty$. Hence, given Δp , (1) tells us nothing about the uncertainty of position. (Of course, in this case it is trivial to calculate directly that $\Delta q = 3^{-\frac{1}{2}} a$.)

In this and related cases, ad hoc measures are often employed to quantify the "widths" or "spread" of the relevant distributions. For example, the width of the momentum pdf is frequently defined as $\lambda = h/2a$, this being the wavelength of the function $\sin^2(2\pi ay/h)$. Hence

$$\Delta q \cdot \lambda = h/(2 \cdot 3^{\frac{1}{2}}),$$

an impoverished analogue of (1).

The crux of the problem is that the weight y^2 grows too rapidly in the definition of Δp forcing it to be infinite. With this in mind, and in an attempt to provide a more uniform approach, in 1982 Michael Cowling and I obtained the following generalization of (1) [4, Theorem 5.1]. (Let $t^\# = 2t/(t-2)$.)

THEOREM 1. *Suppose $p, q \in [1, \infty]$ and $\theta, \phi \geq 0$. There exists a constant K such that*

$$\|f\|_2 \leq K \| |x|^\theta f \|_p^\alpha \| |y|^\phi f \|_q^{1-\alpha}$$

for all tempered distributions f with the property that f and F are locally integrable functions if and only if

(i) $\theta > 1/p^\#$, $\phi > 1/q^\#$ and α satisfies

$$\alpha(\theta - 1/p^\#) = (1-\alpha)(\phi - 1/q^\#),$$

OR (ii) $(p, \theta) = (2, 0)$ and $\alpha = 1$,

OR (iii) $(q, \phi) = (2, 0)$ and $\alpha = 0$.

(If both the last cases occur, α is arbitrary.)

DOUBLE SLIT EXPERIMENT

Suppose we have the situation described in Figure 1 but with two slits each of width $2a$ with centres distance $2A$ apart where $A > a$. Suppose also that there is a screen situated at distance d from the diaphragm which detects the arrivals as in Figure 2.

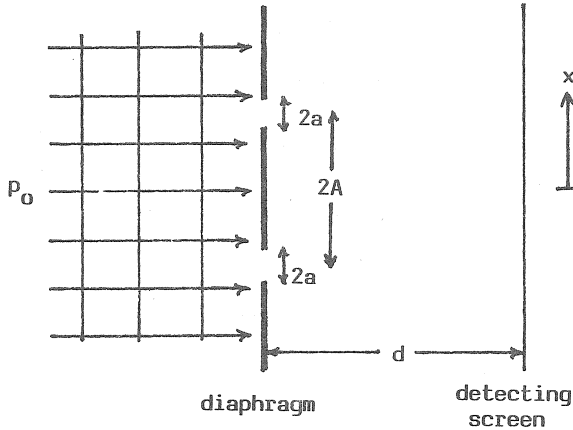


Figure 2

The state function for the vertical component of the incoming particles is

$$f = (4a)^{-\frac{1}{2}} (1_{[-A-a, -A+a]} + 1_{[A-a, A+a]}) .$$

Hence the probability density functions of position and momentum are

$$|f|^2 = (4a)^{-1} (1_{[-A-a, -A+a]} + 1_{[A-a, A+a]}) ,$$

$$\hbar^{-1} |F(y/\hbar)|^2 = (4a/\hbar) \operatorname{sinc}^2 2\pi ay/\hbar \cos^2 2\pi Ay/\hbar$$

respectively, where $\operatorname{sinc} \theta = (\sin \theta)/\theta$ for $\theta \neq 0$ and 1 for $\theta = 0$. Notice that $\Delta q = (a(3A^2 + a^2)/3A)^{\frac{1}{2}}$ and $\Delta p = \infty$.

Denote the horizontal momentum of each of the particles by p_0 . For simplicity assume that the particles passing through the slits leave from the centres of the slits. This means that when a particle

arrives at the detecting screen we know that it has followed one of two paths. As we shall see, the period of the interference pattern is $hd/2Ap_0$ so this assumption will have a negligible effect on the pattern if $hd/Ap_0 \gg a$.

The time taken for each particle to cross from the diaphragm to the screen is md/p_0 , where m is the mass of the particle. Hence the pdf of the arrivals of the particles at the screen is

$$(4) \quad \begin{aligned} \phi(x) = & (\theta a/\pi) (\text{sinc}^2 \theta a(x+A) \cos^2 \theta A(x+A) \\ & + \text{sinc}^2 \theta a(x-A) \cos^2 \theta A(x-A)) \end{aligned}$$

where $\theta = 2\pi p_0/hd$. We are principally interested in the cosine terms since it is these which describe the interference pattern characteristic of the double-slit experiment. This phenomenon was first demonstrated by Thomas Young in 1803. In practice $A \gg a$.

Suppose we modify the experiment by first closing one slit and then the other. If we average the two resulting pdf's we obtain from

$$(5) \quad \Psi(x) = (\theta a/2\pi) (\text{sinc}^2 \theta a(x+A) + \text{sinc}^2 \theta a(x-A)) .$$

Notice that the interference terms are no longer present.

By only having one slit open at a time we are imposing conditions which enable us to know through which slit each particle passes. This suggests the conjecture that "the interference pattern appears if and only if we cannot determine the paths of the particles". It is interesting to note that, as predicted by the theory, the interference pattern has been observed even when the time interval between the arrivals of individual particles was around 30,000 times longer than the time for an individual particle to pass through the system [1]. A modern variant uses two lasers instead of two slits [9].

The remainder of the paper is an analysis of a procedure suggested by Einstein to disprove the above conjecture. It will also be seen that this conjecture is, at least qualitatively, equivalent to Heisenberg's uncertainty principle.

FIFTH SOLVAY CONFERENCE

Suppose that we have a way of keeping ϕ as the pdf of the arrivals of the particles but still knowing which slits the particles passed through. In this case the pdf of position of the particles at the diaphragm gives

$$\Delta q = 3^{-\frac{1}{2}} a ,$$

the standard deviation for the single slit experiment. In other words, the uncertainty of position has been reduced, and substantially if $A \gg a$. Since the form of ϕ is a consequence of the distribution of momentum at the diaphragm, this means that the uncertainty of position has been reduced without changing the pdf of momentum. Although this does not defeat the uncertainty principle inequality (1) as it stands (since in this case $\Delta p = \infty$) it certainly undermines the spirit of the general principle.

The above argument was appreciated by Albert Einstein as early as October 1927 for at that time he presented it to the Fifth Physical Conference of the Solvay Institute in Brussels. Furthermore, he put forward a method, a mind-experiment, for resolving the ambiguity of the slits. He suggested that a very delicate mechanism be attached to the screen that was capable of measuring the vertical impulses or kicks of the arriving particles. (Actually his suggestion was to attach it to the diaphragm but the resulting argument is the same.) It is to be so sensitive that it can detect the difference between the

momentum of a particle coming from the top slit and one coming from the bottom. In other words, by virtue of this mechanism we can detect the paths of the particles.

As was so often the case during this period, it was Niels Bohr who supplied the counterargument. Its general thrust is that if we can measure these minute impulses so precisely, the uncertainty of momentum of the screen must be very small. But then, by (1), the uncertainty of its positron must be large, so large, in fact, that it would obliterate the interference pattern. (This is just one of a series of mind-experiments put forward by Einstein in an attempt to locate weaknesses in the quantum theory. For a fascinating account, see Bohr [3]. Another highly-readable introduction to some of these experiments is contained in [7].)

We shall see, however, that this conclusion cannot be inferred from (1) since Δq is too general a measure of spread. Other inequalities will then be brought to bear on the problem with the final conclusion that Bohr was correct, the interference pattern would be lost.

MOMENTUM AND PROBABILITY

Suppose that the state function of the screen in the vertical direction is g . Denote the pdf of momentum by

$$\eta = h^{-1} |G(\cdot/h)|^2,$$

where G is the Fourier transform of g . Suppose that a particle coming from the bottom slit hits the detecting screen at a height x . (The height-measuring scale is assumed to be rigidly fixed with respect to the diaphragm. It is separate from the screen.) The reading of the strength of the impulse should be $(x+A)p_0/d$ since

the time taken to cross from the diaphragm is md/p_0 , m being the mass of an individual particle. But if it is nearer to $(x-A)p_0/d$, that is, if the reading is less than xp_0/d , then it will be interpreted as coming from the top slit. Thus the probability of incorrectly determining that this particle comes from the top slit is

$$\int_{-\infty}^{xp_0/d} \eta(z - (x+A)p_0/d) dz = \int_{-\infty}^{-Ap_0/d} \eta(z) dz .$$

Similarly, if the reading exceeds xp_0/d the particle will be judged as coming from the bottom slit: the probability of this being in error is

$$\int_{Ap_0/d}^{\infty} \eta(z) dz .$$

(This decision rule is plausible given that we do not have any further information on η . It always favours the correct conclusion when η is even with $\eta(x)$ decreasing for positive x .) Thus the probability of correctly interpreting the reading as to the path of the particle is

$$\frac{1}{2} + \frac{1}{2} \int_{|z| < Ap_0/d} \eta(z) dz .$$

This means that correct judgements are made with probability one if and only if

$$(6) \quad \text{supp } \eta \subseteq [-Ap_0/d, Ap_0/d] .$$

From now on we assume that this is the case. Also Δq and Δp will denote the uncertainties of position and momentum of the screen. (We assume $\langle q \rangle$ and $\langle p \rangle$ exist.)

DETECTED PATTERN

We now calculate the pdf of arrivals at the screen in the case that the pdf of the position of the screen is $|g|^2$. Suppose the

screen is displaced a distance u . The probability of an arrival exceeding x is $\int_{x-u}^{\infty} \Phi(s) ds$. Hence, in general,

$$\begin{aligned} \text{prob}(\text{arrival} \geq x) &= \int |g(u)|^2 \left(\int_{x-u}^{\infty} \Phi(s) ds \right) du \\ &= \int |g(u)|^2 \left(\int_x^{\infty} \Phi(s-u) ds \right) du \\ &= \int_x^{\infty} |g|^2 * \Phi . \end{aligned}$$

Hence the pdf of arrivals on the detecting screen is

$$(7) \quad \Phi_g = |g|^2 * \Phi .$$

It is emphasised that this pdf is with respect to a scale *on the screen*. The problem is to show that under the above assumption (6), Φ_g equals, or is at least close to, Ψ as defined in (5).

I. CLASSICAL UNCERTAINTY INEQUALITY

Under assumption (6)

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 \leq \int z^2 \eta(z) dz < (\Delta p_0/d)^2 ,$$

and hence from (1)

$$(8) \quad \Delta q > hd/4\pi\Delta p_0 .$$

The usual response to this is that since the right side of this inequality is of the same order as the period $hd/2\Delta p_0$ of the interference pattern, then the pattern will be obliterated. (See, for example, [3], [2], [7] and [8].) But standard deviations give us meagre information about the fine details. For example, it could be that, as depicted in Figure 3, η consists of two peaks far from the origin but is very small elsewhere. In this way (8) can be satisfied and if the distance between the peaks is a multiple of the period

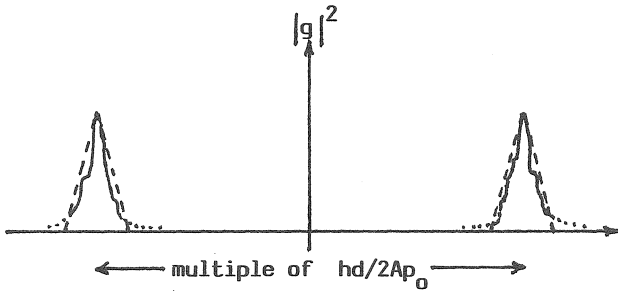


Figure 3

$h d / 2 A p_0$, then there would be no essential effect on the interference pattern. Something stronger is needed to uphold Bohr's conclusion.

II. LOCAL UNCERTAINTY PRINCIPLE INEQUALITIES

Local uncertainty principles assert that if the uncertainty of momentum is small, then not only is the uncertainty of position large, but the probability of being localized at any point is also small. A number of inequalities supporting this principle are developed and applied in Faris [6]. Recently I have extended two of these:

THEOREM 2 [10]. *Suppose that $1 \leq t \leq \infty$ and $\theta \geq 0$.*

(i) *Let (t, θ) satisfy $1/t^\# < \theta < 1/t'$ (where $t^\# = 2t/(t-2)$ and $t' = t/(t-1)$) or $(t, \theta) = (1, 0)$ or $(2, 0)$. There exists a constant K such that*

$$\left(\int_E |F(y)|^2 dy \right)^{1/2} \leq K m(E)^{\theta - 1/t^\#} \| |x - b|^\theta f \|_t$$

for all $f \in L^2(\mathbb{R})$ and measurable $E \subseteq \mathbb{R}$.

(ii) *If $\theta \geq 1/t'$ [except for $(t, \theta) = (1, 0)$] or $\theta \leq 1/t^\#$ [except for $(t, \theta) = (2, 0)$], no such inequality is possible.*

THEOREM 3 [11]. *Suppose $E \subseteq \mathbb{R}$ is measurable and $\theta > 1/2$. Then*

$$\int_E |F(y)|^2 dy < K_{1,m}(E) \|f\|_2^{2-1/\theta} \| |t - b|^\theta f \|_2^{1/\theta}$$

for all $f \in L^2(\mathbb{R})$ and $b \in \mathbb{R}$ where

$$K_1 = \frac{1}{\theta} \Gamma\left(\frac{1}{2\theta}\right) \Gamma\left(1 - \frac{1}{2\theta}\right) (2\theta - 1)^{1/2\theta} \left(1 - \frac{1}{2\theta}\right)^{-1}$$

and $K_{1m}(E)$ is the smallest possible constant. If $\theta \leq \frac{1}{2}$ no such inequality is possible.

Proof. In the one-dimensional situation the most important inequality is that of Theorem 3 with $\theta = 1$. Fortunately, as communicated to me by Henry Landau, it has a very simple proof. Standard completeness arguments show that it is enough to establish the inequality for $f \in S$, the Schwartz space of rapidly decreasing functions. Given $y \in \mathbb{R}$,

$$\begin{aligned} 2|F(y)|^2 &= \int_{-\infty}^y |F^2|' - \int_y^{\infty} |F^2|' \\ &= \int_{-\infty}^y (F\bar{F}' + F'\bar{F}) - \int_y^{\infty} (F\bar{F}' + F'\bar{F}) \end{aligned}$$

and hence

$$\begin{aligned} \|F\|_{\infty}^2 &\leq \int_{-\infty}^y |FF'| + \int_y^{\infty} |FF'| = \int |FF'| \\ &< \|F\|_2 \|F'\|_2 = 2\pi \|f\|_2 \|tf\|_2. \end{aligned}$$

(The last inequality must be strict since $f \in S$.) Thus

$$\int_E |F(y)|^2 dy \leq m(E) \|F\|_{\infty}^2 < 2\pi m(E) \|f\|_2 \|tf\|_2,$$

as required. The full proofs of Theorems 2 and 3 are given in [10] and [11] respectively.

The fact that $K_{1m}(E)$ is the best constant in Theorem 3 can be shown in the following way. Define

$$g(x) = \alpha(\theta)/(1 + |x|^{2\theta}) \quad \text{where} \quad \alpha(\theta) = \theta/\Gamma(1/2\theta)\Gamma(1-1/2\theta).$$

Simple calculations show that

$$\|g\|_1 = 1 ,$$

$$\|g\|_2^2 = (\alpha(\theta)^2/\theta) B(1+1/2\theta, 1-1/2\theta) ,$$

$$\| |t|^\theta g \|_2^2 = (\alpha(\theta)^2/\theta) B(1+1/2\theta, 1-1/2\theta) .$$

Define $g_n = ng(n^*)$ for $n \in \mathbb{Z}^+$. Then (g_n) is an approximate identity with

$$\frac{\int_E |G_n|^2}{\|g_n\|_2^{2-1/\theta} \| |t|^\theta g_n \|_2^{1/\theta}} = K_1 \int_E |G_n|^2 .$$

As $n \rightarrow \infty$, $G_n(y) \rightarrow 1$ for each y so that $\int_E |G_n|^2 \rightarrow m(E)$ by the dominated convergence theorem, demonstrating the sharpness of the constant $K_1 m(E)$. (Of course we can have equality in Theorem 3 in a trivial sense by supposing that E has infinite measure.)

DISCUSSION. In general, local uncertainty principles are stronger than global ones. For example, it follows from Theorem 3 with $\theta = 1$ that

$$\begin{aligned} \|F\|_2^2 &= \int_{|y| \leq b} |F|^2 + \int_{|y| > b} |F|^2 \\ &< 4\pi b \|f\|_2 \|xf\|_2 + b^{-2} \|yF\|_2^2 . \end{aligned}$$

Setting

$$b = (\|yF\|_2^2 / 2\pi \|f\|_2 \|xf\|_2)^{1/3}$$

gives

$$\|F\|_2^2 < 4\pi (2^{1/3} + 2^{-2/3})^{3/2} \|xf\|_2 \|yF\|_2$$

which has the same general form as (3).

Theorem 3 with $\theta = 1$ applied to the double-slit experiment under assumption (6) gives

$$\|g\|_\infty^2 < 2\pi h^{-1} \Delta p < 2\pi A p_0 / h d ,$$

where g is the state function of the screen. Hence, if g is a

depicted in Figure 3, the bases of the triangles would have to exceed $(hd/2\pi Ap_0)^{1/2}$. This does not help much since $hd/Ap_0 \gg 1$ in practice so that the period of the interference pattern $hd/2Ap_0$ greatly exceeds $(hd/2\pi Ap_0)^{1/2}$.

We can do better with Theorem 2. Taking $t = 2$, $\theta = \frac{1}{4}$ and E the base of one of the triangles,

$$\begin{aligned} (\frac{1}{2})^{1/2} &= \left(\int_E |g|^2 \right)^{1/2} \leq Km(E)^{1/4} \| |y|^{1/4} G \|_2 \\ &\leq Km(E)^{1/4} (Ap_0/hd)^{1/4} \end{aligned}$$

since $\text{supp } G \subseteq [-Ap_0/hd, Ap_0/hd]$. This implies

$$m(E) \geq (4K^4)^{-1} hd/Ap_0.$$

In other words, the lengths of the bases of the triangles in Figure 3 are of the same order as the period of the interference pattern so, at least, the pattern would be markedly reduced. (One estimate for K is $1+2(1-2\theta)^{-1/2}$ when $t = 2$ [10].)

However, this whole approach becomes less useful if $|g|^2$ is made up of many low peaks separated by integer multiples of the wavelength of the interference pattern.

III. CONSTANCY VERSUS CONCENTRATION

The essence of the family of inequalities in this section is that the more a function is concentrated, the less variable is its transform. Suppose that f is a one-dimensional state. Given $\alpha \in (0,1)$, define

$$W = W_\alpha(f) = \min \left\{ u : \int_{-u}^u |f(x)|^2 dx = \alpha \right\},$$

while for $\beta \in (0,1)$ define

$$w = w_\beta(F) = \min \left\{ v : \left| \int \overline{F(y)} F(y-v) dy \right| = \beta \right\}.$$

Uffink and Hilgevoord [14] and myself [12] independently obtained lower bounds for the product $W_\alpha \dots$ with the physicists getting the sharper result! Work is under way for a paper containing all the details but the central result is:

THEOREM 4. *Suppose $\alpha, \beta \in (0, 1)$. Then*

$$W_\alpha(f) w_\beta(F) > ((2\alpha - \beta - 1)/2\pi^2\alpha)^{\frac{1}{2}}$$

for all states f provided $2\alpha > \beta + 1$. Further, if $2\alpha \leq \beta + 1$, there is no positive constant K so that $W_\alpha(f) w_\beta(F) \geq K$ for all states.

Returning to the double-slit experiment, replace f by G , where g is the state function of the screen. By letting $\alpha \rightarrow 1$ and observing that

$$\text{supp}(h^{-1}|G(\cdot/h)|^2) \subseteq [-Ap_0/d, Ap_0/d],$$

we arrive at

$$w_\beta(g) \geq ((1 - \beta)/2)^{\frac{1}{2}} hd/\pi Ap_0$$

for $\beta \in (0, 1)$. This means, for example, that g cannot have support in disjoint intervals $(E_n)_{n=1}^\infty$ where

(i) $m(E_n) \leq \lambda hd/\pi Ap_0$ for all n with $0 < \lambda < 1/\sqrt{2}$,

(ii) the distance between adjacent pairs of intervals exceeds $\lambda hd/\pi Ap_0$.

For if we had such a function, letting $\beta = 1 - 2\lambda^2$ gives $w_\beta(g) = 0$. In particular, the pdf of position $|g|^2$ cannot be supported in intervals of length $\lambda hd/\pi Ap_0$ with centres hd/Ap_0 , the period of the interference pattern, apart.

However, it is clear that we are still a long way from establishing that the interference pattern is destroyed. This will be done in the next section using Poisson summation.

IV. POISSON SUMMATION

As before, g is the state function of the screen and we assume that $\text{supp } \eta \subseteq [-Ap_0/d, Ap_0/d]$, where η is the corresponding pdf of momentum. Hence

$$(9) \quad \text{supp } G \subseteq [-Ap_0/hd, Ap_0/hd] .$$

Define $g^\#$ by

$$g^\#(x) = \sum_{n \in \mathbb{Z}} |g|^2(x + nhd/2Ap_0) .$$

Considering $g^\#$ as a function on $[0, hd/2Ap_0]$, $g^\# \in L^1[0, hd/2Ap_0]$ since $|g|^2 \in L^1(\mathbb{R})$. Its Fourier coefficients are:

$$\begin{aligned} c_k &= (2Ap_0/hd) \int_0^{hd/2Ap_0} g^\#(x) \exp(-4\pi i x k Ap_0/hd) dx \\ &= (2Ap_0/hd) \int |g|^2(x) \exp(-4\pi i x k Ap_0/hd) dx \\ &= |g|^2 \hat{\ } (2k Ap_0/hd) . \end{aligned}$$

In view of (9) it follows that

$$\text{supp}(|g|^2 \hat{\ }) \subseteq [-2Ap_0/hd, 2Ap_0/hd] .$$

Furthermore, $|g|^2 \hat{\ }$ is continuous since $|g|^2 \in L^1$ so that it is 0 for $|x| \geq 2Ap_0/hd$. Hence $c_k = 0$ for all integers $k \neq 0$ with the consequence that

$$g^\#(x) = c_0 = 1 \quad (\text{almost everywhere}) .$$

Hence, modulo the period of the interference pattern, $hd/2Ap_0$, g is a constant almost everywhere. In general terms this means that g kills off the interference terms $\cos^2(2\pi Ap_0/hd)(x+A)$ and $\cos^2(2\pi Ap_0/hd)(x-A)$ announced in (4).

A stronger statement of this phenomenon is obtained by letting $a \rightarrow 0$. As explained above, the pdf of arrivals at the screen is given by (7), namely $\phi_g = |g|^2 * \phi$. Suppose that the rate of

particles passing through the slits is a^{-1} so that $a^{-1}\Phi_g$ is the density distribution of arrivals at the screen. In other words, the expected number of arrivals in a set E is $\int_E a^{-1}\Phi_g$. Working within S' , the space of tempered distributions, we show that

$$(10) \quad \lim_{a \rightarrow 0+} a^{-1}\Phi_g = 2p_0/hd (= \lim_{a \rightarrow 0+} a^{-1}\psi)$$

for all states g satisfying (6). The effect of letting a tend to 0 is to remove the influence of the sinc-terms in Φ since $\text{sinc}(2\pi p_0/hd)a(x \pm A) \rightarrow 1$ as $a \rightarrow 0$. Thus

$$a^{-1}\Phi \rightarrow (2p_0/hd) (\cos^2(2\pi A p_0/hd) a(x+A) + \cos^2(2\pi A p_0/hd) (x-A))$$

as $a \rightarrow 0$. Next, since $F^{-1}(\delta_{-b} + 2\delta_0 + \delta_b) = 4 \cos^2 \pi b x$, where δ_c is the point measure at c , the Fourier transform of the preceding limit is

$$(p_0/hd) \cos 2\pi A y (\delta_{-2A p_0/hd} + 2\delta_0 + \delta_{2A p_0/hd}) .$$

Hence

$$\begin{aligned} \lim_{a \rightarrow 0} a^{-1}\Phi_g &= F^{-1}((\lim_{a \rightarrow 0} a^{-1}\Phi)^\wedge |g|^{2^\wedge}) \\ &= (p_0/hd) F^{-1}(\cos 2\pi A y (\delta_{-2A p_0/hd} + 2\delta_0 + \delta_{2A p_0/hd}) |g|^{2^\wedge}) . \end{aligned}$$

But $|g|^{2^\wedge} = 0$ for $|x| \geq 2A p_0/hd$ and $|g|^{2^\wedge}(0) = 1$ so that

$$\lim_{a \rightarrow 0} a^{-1}\Phi_g = (p_0/hd) F^{-1}(2\delta_0) = 2p_0/hd ,$$

as asserted. Thus in the limit as $a \rightarrow 0$, we see that Bohr was correct when he asserted that knowledge of the paths of the particles precludes the appearance of an interference pattern.

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