SOME THEOREMS ON ORLICZ-SOBOLEV SPACES, AND AN APPLICATION TO NEMITSKY OPERATORS

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1. INTRODUCTION

We are concerned here with the problem of extending, to Orlicz-Sobolev spaces, certain theorems of Marcus and Mizel on Nemitsky operators on Sobolev spaces. (See [5].)

Marcus and Mizel's proofs rely upon, in particular,

(i) Gagliardo's characterisation of the Sobolev space $W_{1,p}$ in terms of absolute continuity; and

(ii) bounds and limits of difference quotients in Sobolev spaces.

We shall give suitable extensions of (i) and (ii) to Orlicz-Sobolev spaces in §§ (2) and (3) below, which enables us to give an extension of the theorems of Marcus and Mizel. (See § 4.)

2. ORLICZ-SOBOLEV SPACES AND THE SPACES $A(\Omega)$

Throughout this paper, Ω denotes a domain in R^n.

Since all the definitions of both Orlicz and Orlicz-Sobolev spaces which occur in the statements of our theorems can be found in [1], we shall not repeat them here. For the spaces $A(\Omega)$ (i.e., Beppo Levi spaces), we shall follow [5]. (With a minor difference in notation, essentially that, in denoting certain equivalence classes, we use "~" instead of a dash, to avoid an obvious source of confusion.)

Thus $A(\Omega)$ denotes the class of real measurable functions u on Ω such that, for almost every line τ parallel to any co-ordinate axis, u is locally absolutely continuous on $\tau\cap\Omega$. $\tilde{A}(\Omega)$ denotes the class of functions u such that u coincides almost everywhere in Ω with a function $\tilde{u}$ in $A(\Omega)$. For $u \in \tilde{A}(\Omega)$, $\tilde{D}_j u$ (or $\tilde{D}_{x_j}u$), the *strong approximate derivative* of u with respect to x_j, denotes any member of the equivalence class of functions measurable on Ω which contains the classical partial derivative $D_j\tilde{u}$. We shall use $\partial_j u$ or $\partial_{x_j}u$ to denote a weak derivative. Our extension of Gagliardo's theorem is:

THEOREM 1 *Let M be an N-function, and suppose Ω is a bounded domain in R^n with the cone property. Then a function u defined on Ω belongs to $W^1L_M(\Omega)$ if and only if*

(a) $u \in \tilde{A}(\Omega)$;

(b) $\tilde{D}_j u \in L_M(\Omega)$, $j = 1, \ldots, n$.

Moreover, if $u \in W^1L_M(\Omega)$, then $\tilde{D}_j u = \partial_j u$ almost everywhere in Ω.

Using Theorem 1 (instead of Gagliardo's Theorem), we obtain the following version of a chain rule due to Serrin.

THEOREM 2 *Let f: $R \to R$ be locally absolutely continuous, let M be an N-function and suppose $u \in W_{1,1}^{loc}(\Omega)$. Then $f \circ u \in W^1L_M(\Omega)$ if and only if*

(i) $(f' \circ u)\partial_j u \in L_M(\Omega)$, $j = 1, \ldots, n$,

where we make the following convention:

(*) *the product is zero if the term on the right is zero.*

Moreover, if (i) holds,

$\partial_j(f\circ u) = (f'\circ u)\,\partial_j u$, $j = 1,\ldots,n$, *almost everywhere in* Ω.

3. DIFFERENCE QUOTIENTS IN ORLICZ-SOBOLEV SPACES

Definition. For $u : \Omega \to R$, e_j, $1 \leq j \leq n$, the standard basis for R^n, and $x \in R^n$, we define the difference quotient in the direction e_j by

$$\delta_h^j u(x) = \frac{u(x+he_j) - u(x)}{h}, \quad h \neq 0, \text{ whenever } x \text{ and } x + he_j \in \Omega.$$

Using arguments similar to those used to establish the analagous results for Sobolev spaces, (see [2]), we can prove the following:

THEOREM 3. *Suppose* Ω *is a bounded, and that* Ω' *is an open set such that* $\Omega' \subset\subset \Omega$. *Then if* $0 < |h| < dist(\Omega', bdry\ \Omega)$, *and if* $u \in W^m E_M(\Omega)$ *for some* $m \geq 1$,

$$\| \delta_h^j u \|_{m-1,M,\Omega'} \leq \| u \|_{m,M,\Omega} \quad .$$

Further, if there exists a number C *such that* $\| \delta_h^j u \|_{m,M,\Omega'} \leq C$, $1 \leq j \leq n$, *for every open* $\Omega' \subset\subset \Omega$ *and all* h *sufficiently small, then* $u \in W^{m+1} L_M(\Omega)$ *and* $\| \partial_j u \|_{m,M,\Omega} \leq C$, $1 \leq j \leq n$.

4. NEMITSKY OPERATORS

Definition. A function $g : \Omega \times R^m \to R$ is said to be a *generalised locally absolutely continuous* (briefly g.l.a.c.) *Caratheodory function* if:

(i) There exists a null subset N of Ω such that for every fixed $x \in \Omega \setminus N$ we have

(a) $g(x,\cdot)$ is separately continuous in R^m;

(b) for every line τ parallel to one of the axes in R^m, $g(x,\cdot)|_\tau$ is locally absolutely continuous.

(ii) For each fixed $t \in R^m$, $g(\cdot,t) \in \tilde{A}(\Omega)$.

The *Nemitsky operator* G is then defined on functions $u : \Omega \to R^m$ by $(Gu)(x) = g(x, u(x))$.

Our extension of Marcus and Mizel's theorem (including a corollary) is then:

THEOREM 4. *Let Ω be a bounded domain in R^n having the cone property, and let g be a g.l.a.c. Caratheodory function on $\Omega \times R^m$. Let P, Q_k and $Q_k\dagger$, $k = 1,\ldots,m$, be N-functions having the following properties:*

(i) P and Q_k, $k = 1,\ldots,m$, satisfy the Δ_2 condition;

(ii) $P \prec Q_k$, $k = 1,\ldots,m$;

(iii) there exist complementary N-functions R_k and $\tilde{R}_k$ such that the inequalities

$$R_k(s) \leqslant P^{-1}[Q_k(\alpha_k s)]$$

and

$$\tilde{R}_k(s) \leqslant P^{-1}[Q_k\dagger(\beta_k s)]$$

are satisfied for $s \geqslant c_k$, where α_k, β_k, c_k, $k = 1,\ldots,m$, are constants.

Suppose a, b, a_k, $b_{k,j}$ are functions such that for every fixed $t \in R^m$

(iv) $|\tilde{D}_{x_i} g(x,t)| \leqslant a(x) + b(t)$ a.e. in Ω, $i = 1,\ldots,n$; and the inequality

(v) $\left|\frac{\partial g(x,t)}{\partial t_k}\right| \leq a_k(x) + \sum_{j=1}^{m} b_{k,j}(t_j)$, $k = 1,\ldots,m$,

holds at every point $(x,t) \in (\Omega \setminus N) \times \mathbb{R}^m$ *at which the derivative exists in the classical sense. (Here* N *is the null set of the definition above.)*

Furthermore, a, b, a_k *and* $b_{k,j}$ *have the properties (vi) - (x) listed below:*

(vi) $0 \leq a \in L_P(\Omega)$;

(vii) b *is non-negative and separately continuous in* $\mathbb{R}^m$;

(viii) $0 \leq a_k \in L_{Q_k^\dagger}(\Omega)$, $k = 1,\ldots,m$;

(ix) $0 \leq b_{k,j}$ *is an extended real valued Borel function on* $\mathbb{R}$, $k,j = 1,\ldots,m$;

(x) $b_{k,k} \in L_1^{loc}(\mathbb{R})$, $k = 1,\ldots,m$.

Let $u_k \in W^1L_{Q_k}(\Omega)$, $k = 1,\ldots,m$, *let* $\mathrm{u} = (u_1,\ldots,u_m)$, *and suppose that*

(xi) $b \circ \mathrm{u} \in L_P(\Omega)$;

(xii) $b_{k,j} \circ u_j \in L_{Q_k^\dagger}(\Omega)$ $k,j = 1,\ldots,m$, $k \neq j$;

and, with the convention (*)

(xiii) $[b_{k,k} \circ u_k]\partial_i u_k \in L_P(\Omega)$, $k = 1,\ldots,m$, $i = 1,\ldots,n$.

Then $G\mathrm{u}$ *belongs to* $W^1L_P(\Omega)$, *and, with the convention (*),*

$$|\partial_i(G\mathrm{u}(x)| \leq a(x) + (b \circ \mathrm{u})(x) + \sum_{k=1}^{m}\left[a_k(x) + \sum_{j=1}^{m}(b_{k,j} \circ u_j)(x)\,|\partial_i u_k(x)|\right],$$

almost everywhere in Ω, $i = 1,\ldots,n$.

NOTE. Families of N-functions satisfying (i), (ii), and (iii) can be constructed from standard N-functions (such as those listed in [4]), using the following:

Proposition. *Let P and R be N-functions satisfying the Δ_2 condition, and let $Q = P \circ R$, $Q^\dagger = P \circ \tilde{R}$. Then Q and $Q^\dagger$ are N-functions having the properties:*

(i) *Q satisfies the Δ_2 condition;*

(ii) $P \prec Q$;

(iii) $R = P^{-1} \circ Q$, $\tilde{R} = P^{-1} \circ Q^\dagger$.

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