BASIC MEASURES

Gavin Brown

1. INTRODUCTION

We are all familiar with convolution as a smoothing operation. An example of this is the classical theorem of Steinhaus that

(1.1) $|E| > 0 \Rightarrow E + E$ contains an interval.

One simple way of proving (1.1) is to consider the convolution of the indicator function of E with itself.

Now let C denote Cantor's middle third set and let μ_c be a probability measure evenly distributed over C. Since $|C| = 0$, it is obvious that

(1.2) $\mu_c \perp \lambda$ (where λ denotes Lebesgue measure).

Less obviously

$$\mu_c * \mu_c \perp \lambda,$$

despite the fact that $C + C$ fills out an interval. Indeed some support sets of convolution powers of μ_c must be quite small because

(1.4) $\mu_c^n \perp \mu_c^m \perp \lambda, \quad n \neq m.$

These results (which can be found in [2]) used all of C, and make it surprising that in the opposite direction (see [3])

$$(1.5) \qquad |E+F| \geq \mu_c(E)^{\alpha}\, \mu_c(F)^{\alpha},$$

where $\qquad \alpha = \log 3/\log 4.$

In general, we say that a measure is <u>basic</u> if

$$(1.6) \qquad \mu(E) > 0 \Rightarrow \mathrm{Gp}(E) = \mathbb{R}.$$

Here

$$\mathrm{Gp}(E) = \{n_1e_1 + \dots + n_je_j : j \in \mathbb{N},\ n_i \in \mathbb{Z},\ e_i \in E,\ i = 1, \dots, j\}$$

is the group generated by E. (This is a purely algebraic definition - there is no question of topological closure.)

Observe that Steinhaus's theorem (1.1) shows that Lebesgue measure is basic while (1.5) shows that μ_c is basic.

What follows is an account of some recent results on basic measures obtained with a variety of co-authors. I hope to show that this is an interesting area of neo-classical measure theory with scope for further development.

2. BANACH ALGEBRA BACKGROUND

Bill Moran and I were led to the notion of basic measures by the consideration of radical objects in Banach algebras. This is not as far-fetched as, at first, it seems! Questions like those in the introduction are naturally discussed within the convolution algebra, $M(\mathbb{R})$, of all regular bounded Borel measures on the line which is a Banach algebra under the total variation norm.

Let us ask the naive question, "Which are the nice measures in $M(\mathbb{R})$?" In many situations $L^1(\mathbb{R})$ (imbedded as the collection of all absolutely continuous measures) constitutes an appropriate answer. However Hewitt and Zuckerman noted in [7] that there exist singular measures with absolutely continuous convolution squares, so it is natural to enlarge the class of non-pathological measures by passing from the ideal $L^1(\mathbb{R})$ to the ideal $\mathrm{Rad}\, L^1(\mathbb{R})$ defined by

$$(2.1)\quad \mathrm{Rad}\, L^1(\mathbb{R}) = \{\mu \in M(\mathbb{R}) : \phi \in \Delta,\ \phi(L^1(\mathbb{R})) = 0 \Rightarrow \phi(\mu) = 0\},$$

where Δ is the set of all complex homomorphisms of $M(\mathbb{R})$.

We may also write

$$(2.2)\qquad \mathrm{Rad}\, L^1(\mathbb{R}) = \cap \{\ker \phi : \phi \in \Delta_1\},$$

where Δ_1 comprises all complex homomorphisms of $M(\mathbb{R})$ which are <u>not</u> of the form $\mu \to \int e^{ixy}\, d\mu(x)$, for fixed real y. (A simple example of a member of Δ_1 is the map $\mu \to \mu_d(\mathbb{R})$, which sends μ to the total mass of its discrete part.)

Moran and I introduced a still larger radical, the <u>symmetric Raikov radical</u>, $S(\mathbb{R})$, defined by

$$(2.3)\qquad S(\mathbb{R}) = \cap \{\ker \phi : \phi \in \Delta_2\}$$

where Δ_2 comprises the "Raikov homomorphisms" corresponding to symmetric Raikov systems. The reader who is prepared to take these definitions on trust can proceed to Theorem 1 without loss of continuity. The intervening paragraphs give the details of the definition of $S(\mathbb{R})$.

A <u>symmetric Raikov system</u> (cf. [6]) is a family $\mathcal{Y}$ of F_σ-subsets of $\mathbb{R}$ containing all singletons, closed under countable unions and under passage to F_σ-subsets, and having the fundamental property:

$$E, F \in \mathcal{Y} \Rightarrow E - F \in \mathcal{Y}.$$

Given a symmetric Raikov system, $\mathcal{Y}$, we define

$$(2.4) \qquad \begin{aligned} A_{\mathcal{Y}} &= \{\mu \in M(\mathbb{R}) : \mu \text{ is concentrated on a set of } \mathcal{Y}\} \\ I_{\mathcal{Y}} &= \{\mu \in M(\mathbb{R}) : \mu(F) = 0 \quad (F \in \mathcal{Y})\} \end{aligned}$$

The sets $A_{\mathcal{Y}}$, $I_{\mathcal{Y}}$ in (2.4) are respectively a closed subalgebra and closed ideal, and

$$(2.5) \qquad M(\mathbb{R}) = A_{\mathcal{Y}} \oplus I_{\mathcal{Y}}.$$

The <u>Raikov homomorphism</u> associated with $\mathcal{Y}$ is the map $\mu \to \mu_{\mathcal{Y}}(\mathbb{R})$, where $\mu_{\mathcal{Y}}$ is the result of projection on the first summand in (2.5). For members of Δ_2 we use only those Raikov homomorphisms corresponding to systems $\mathcal{Y}$ which are <u>proper</u> in the sense that not all F_σ-subsets are included. Note that a system $\mathcal{Y}$ which contains any set of positive Lebesgue measure cannot be proper. (Use (1.1)) Note also that our earlier example of a member of Δ_1 is also a member of Δ_2 (corresponding to the Raikov system of all countable subsets of $\mathbb{R}$).

It turns out (see [3]) that

Theorem 1 *A measure is basic if and only if it belongs to the symmetric Raikov radical.*

What makes Theorem 1 interesting is the fact that measures such as μ_c which do not belong to Rad $L^1(\mathbb{T})$ are basic and hence belong to $S(\mathbb{T})$. Moran and I conjectured in 1974 that μ_c is basic but were unable to find a proof. The result was demonstrated by Talagrand, [8], and Woodall, [9]. We subsequently exploited a combinatorial inequality of Woodall to obtain the sharp result (1.5).

3. RIESZ PRODUCT MEASURES

It will be convenient, in this section, to switch from the real line, $\mathbb{R}$, to the circle group, $\mathbb{T}$. The example of Hewitt and Zuckerman mentioned in the last section was, in fact, a Riesz product on the circle, and in this section I would like to discuss

THEROEM 2 *Riesz products are basic measures.*

The typical Riesz product takes the form of a weak* limit

$$\mu = \lim_{N\to\infty} \prod_{k=1}^{N} (1 + r_k \cos(2\pi n_k t)).\lambda \tag{3.1}$$

where $-1 \leqslant r_k \leqslant 1$ and $n_{k+1}/n_k \geqslant 3$, for all k.

We do not know if Theorem 2 is true in full generality. Recently Moran, Tijdeman, and I obtained a very indirect proof (see [4]) under the stronger lacunary hypothesis

$$\sup_t \lim_k \inf(n_{k+1}/n_k)^{1/t} > 3. \tag{3.2}$$

It seems very likely that (3.2) is a property of our proof and that a clumsier but more powerful proof is required!

We use hypothesis (3.2) in showing

LEMMA 1 *Let* μ *be a Riesz product satisfying (3.2). Then there exist positive integers* $p_1, p_2, \ldots, p_r$ *such that*

$$p_1^*\mu * p_2^*\mu * \ldots * p_r^*\mu \in L^1(\mathbb{T}).$$

(Here $p^*\mu$ *is the measure defined by)*

$$\int f(t)dp^*\mu(t) = \int f(pt)d\mu(t) \qquad (f \in C(\mathbb{T}))).$$

Any measure satisfying the conclusion of Lemma 1 must be basic. The proof of that fact uses what is at first sight a bizarre characterization of basic measures in terms of (one-dimensional) characters of the discrete circle group. In fact let P denote the collection of all positive measures μ in $M(\mathbb{T})$ with the property that there exists some character of the discrete circle group which is μ-measurable but which is not a character of the circle group with its usual topology (i.e. is not of the form $t \to \exp 2\pi int$, for fixed $n \in \mathbb{Z}$). It is not difficult to prove that the basic measures are precisely those measures singular to every measure in P. (This is consistent with the search in §2 for a class of non-pathological measures.)

LEMMA 2 *Suppose that* $p_1^*\mu * p_2^*\mu * \ldots * p_r^*\mu$ *belongs to* $L^1(\mathbb{T})$ *then* μ *is a basic measure.*

Proof. In view of the characterization of basic measures as $P^{\perp}$, it will suffice to consider an arbitrary positive measure ν absolutely continuous with respect to μ and an arbitrary character χ of the discrete group such that χ is ν-measurable; then prove that χ is necessarily continuous. Certainly $p_1^*\nu * p_2^*\nu * \ldots * p_r^*\nu$ is absolutely continuous with respect to the corresponding measure involving μ and hence with respect to Haar measure. Moreover χ being a ν-measurable character is $p_j^*\nu$-measurable for each j. Also a character which is measurable with respect to a finite family of measures is measurable with respect to their convolution product. It follows that χ is $p_1^*\nu * p_2^*\nu * \ldots * p_r^*\nu$-measurable and hence Haar measurable. We finish the proof by noting that every Haar measurable character is necessarily continuous - the Steinhaus theme again!

Let me repeat that I would like to see a proof of Theorem 2 which does not proceed via Lemma 2 - and which extends to a wider class of measures than Riesz products.

4. COMBINATORIAL METHODS

The methods of §3 give no hint of how many summands are necessary to produce a set of positive Lebesgue measure from a set of positive μ-measure, given that μ is basic. A result like (1.5) does that and more, so the proof requires much more in the way of technical estimates.

There are two ways to proceed. On the one hand it is possible to concentrate on the basic property and try to handle large classes of measures of infinite convolution type. This is the approach adopted in [5]. On the other hand one can aim for sharp results like (1.5) and face up to proving the necessary combinatorial estimates. This can be surprisingly difficult. In particular Keane, Moran, Pearce, and I have recently (see [1]) extended (1.5) to the corresponding result where the ratio of dissection is no longer 3 as for μ_c but can be any positive integer $m + 1$ (exceeding 3).

THEOREM 3 *Suppose that* $0 \leqslant x_m \leqslant x_{m-1} \leqslant \ldots \leqslant x_1 \leqslant 1$ *and let* $\alpha = \log(m + 1)/m \log 2$. *Then*

$$x_1^\alpha x_2^\alpha \ldots x_m^\alpha + x_1^\alpha x_2^\alpha \ldots x_{m-1}^\alpha (1 - x_m)^\alpha + \ldots$$
$$+ (1 - x_1)^\alpha (1 - x_2)^\alpha \ldots (1 - x_m)^\alpha \geqslant 1.$$

Even the special case of Theorem 3 in which all x_i are equal is not at all trivial. A simple substitution reduces this to

(4.1) For $0 \leqslant \alpha \leqslant \log(m + 1)/m \log 2$ and $b \geqslant 0$,

$$1 + b^\alpha + b^{2\alpha} + \ldots + b^{m\alpha} \geqslant (1 + b)^{m\alpha}.$$

Inequality (4.1), which is proved in [1], looks both obvious and classical. To the best of my belief it is neither!

REFERENCES

[1] G. Brown, M. Keane, W. Moran, and C. Pearce, "A combinatorial inequality with application to Cantor measures", preprint.

[2] G. Brown and W. Moran, "Bernoulli measure algebras", Acta Math. 152 (1974), 77-109.

[3] G. Brown and W. Moran, "Raikov systems and radicals in convolution measure algebras", J. London Math. Soc. (2) 28 (1983), 531-542.

[4] G. Brown, W. Moran, and R. Tijdeman, "Riesz products are basic measures", J. London Math. Soc. (2) 30 (1984), 105-109.

[5] G. Brown and J.H. Williamson, "Coin-tossing and sum sets", in preparation.

[6] I.M. Gelfand, D.A. Raikov, and G.E. Silov, *Commutative normed rings* (Chelsea, New York, 1964).

[7] E. Hewitt and H.S. Zuckerman, "Singular measures with absolutely continuous convolution squares", Proc. Camb. Phil. Soc. 62 (1966), 399-420, Corrigendum ibid 63 (1967), 367-368.

[8] M. Talagrand, "Sommes vectorielles d'ensembles de mesure nulle", Ann. Inst. Fourier (Grenoble) 26 (1976), 137-172.

[9] D.R. Woodall, "A theorem on cubes", Mathematika 24 (1977), 60-62.

School of Mathematics
University of New South Wales
Kensington N.S.W. 2033
AUSTRALIA.