A NOTE ON BRANCHED STABLE TWO-DIMENSIONAL MINIMAL SURFACES

Mario J. Micallef

§1. INTRODUCTION AND STATEMENT OF RESULT

In [3], D. Fischer-Colbrie and R. Schoen investigate the properties of stable minimal surfaces in 3-manifolds of non-negative scalar curvature. As a special case, they prove that, the only complete, stable, oriented, immersed minimal surfaces in $\mathbb{R}^3$ are planes, a result which was also proved by do Carmo and Peng [2]. For applications, it is useful to know whether this theorem still holds if the minimal surface has branch points (an argument given in §3 shows that the usual second variation of area formula is valid even in the presence of branch points). The answer, in general, is no: there exist branched minimal surfaces in $\mathbb{R}^3$ of the conformal type of the disk whose Gauss image lies in a disk of arbitrarily small radius in S^2 (see, for instance, [5, page 73]); these are then stable by a theorem in [1] (see also Remark 5 in [3]). In this note we show, however, that if, as a Riemann surface, the minimal surface is of parabolic type (i.e. it admits no non-constant positive superharmonic functions), the stability of the minimal surface in $\mathbb{R}^3$ implies that it is a plane. This is actually a special case of the following

THEOREM. *Let* $F : M^2 \to \mathbb{R}^4$ *be a (possibly branched) stable minimal immersion of an open oriented surface* M^2 *without boundary. Then* F *is holomorphic with respect to an orthogonal complex structure on* $\mathbb{R}^4$ *if any one of the following conditions holds:*

(i) M *is parabolic in the conformal structure induced by* F

(ii) M *is complete and the complexified normal bundle of* M *(denoted by* $N_{\mathbb{C}}M$*) admits a square integrable holomorphic section (see Remark (b) below for the precise meaning of this).*

Proof. The theorem was proved by the author in [4] in the case that the minimal surface is unbranched. The same proofs work even if the minimal surface is branched upon utilising the proposition in §2 below and upon observing that the usual formula for the second variation of area is still valid for branched immersions, a fact which is proved in §3 below.

Remarks (a) Condition (i) is the more useful one for applications: for example, M is parabolic if it has quadratic area growth or if it is a graph over all of $\mathbb{R}^2$.

(b) Since M is oriented, so is its normal bundle NM. We may therefore define an orthogonal complex structure on NM by rotation by 90°. This implies that $N_{\mathbb{C}}M = L \oplus \bar{L}$ where L is the complex line bundle $(N_{\mathbb{C}}M)^{1,0}$ whose fibres are locally spanned by $e_3 - ie_4$, $\{e_3, e_4\}$ being a local oriented orthonormal frame for NM. (Note that at branch points, M still has a well defined tangent space in $\mathbb{R}^4$ and therefore also a well-defined normal space.) A section s of $N_{\mathbb{C}}M$ is defined to be holomorphic if and only if $\nabla^{\perp}_{\bar{z}} s = 0$ where $\nabla^{\perp}$ is covariant differentiation in NM and z is a local complex co-ordinate on M. In [4], it is proved that if one of the projections of the Gauss map onto each of the factors of $S^2 \times S^2 (= G_{2,4})$ omits an open set, then one of L and $\bar{L}$ admits a square integrable holomorphic section.

§2. A GENERAL RESULT FOR THE OPERATOR $\Delta - q$.

Let $(M, (ds)^2)$ be an n-dimensional non-compact Riemannian manifold without boundary, and let $q : M \to \mathbb{R}$ be smooth. For every bounded domain $D \subset M$, let

$$\nu_1(D) = \inf_{f \in C_0^\infty(D)} \frac{\int_M (|\nabla f|^2 + qf^2)\, dvol}{\int_M f^2\, dvol}$$

where dvol is the element of volume for the metric $(ds)^2$.

THEOREM. [3, Theorem 1, page 201]. *The following conditions are equivalent:*

(i) $\nu_1(D) \geq 0$ *for every bounded domain* $D \subset M$

(ii) $\nu_1(D) > 0$ *for every bounded domain* $D \subset M$

(iii) there exists a positive function g , *satisfying* $\Delta g - qg = 0$ *on* M .

We now point out an easy generalization of this theorem. Let M be a Riemann surface. A metric $(ds)^2$ on M will be called *weakly conformal* if locally, $(ds)^2 = \lambda(z)|dz|^2$, where $\lambda \geq 0$, $\lambda = 0$ only at isolated points and z is a local complex co-ordinate on some open set in M . The element of area dA on M is, of course, given by $\lambda\, dx \wedge dy$ where $z = x + iy$. The gradient of a function $f : M \to \mathbb{R}$ is, of course, locally given by $\nabla f = \lambda^{-\frac{1}{2}}(f_x \partial_x + f_y \partial_y)$ on the set where $\lambda > 0$. Note that the Dirichlet integral of f , $\int_M |\nabla f|^2\, dA$ makes sense even though ∇f blows up at "branch points", i.e. the points where $\lambda = 0$.

PROPOSITION. *The above theorem is still true for an open Riemann surface* M *without boundary equipped with a weakly conformal metric.*

REMARK. Near points where $\lambda = 0$, the equation $\Delta g - qg = 0$ is to be interpreted as $\Delta_E g - \lambda q g = 0$ where $\Delta_E = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Proof. We only have to check that (ii) $\Rightarrow$ (iii) ; the rest is self-evident. Let $(d\tilde{s})^2$ be any metric on M for which the corresponding conformal structure is the same as that which M has as a Riemann surface, i.e. locally $(d\tilde{s})^2 = \mu(z)|dz|^2$ where $\mu > 0$. Then $(ds)^2 = \rho(d\tilde{s})^2$ where locally, $\rho = \frac{\lambda}{\mu}$. For any bounded domain $D \subset M$, let

$$\tilde{\nu}_1(D) = \inf_{f \in C_0^\infty(D)} \frac{\int_M (|\tilde{\nabla} f|^2 + q\rho f^2)d\tilde{A}}{\int_M f^2 \, d\tilde{A}}$$

where $d\tilde{A}$ is the element of area for the metric $(d\tilde{s})^2$. Then $\tilde{\nu}_1(D) \geq 0$ if and only if $\nu_1(D) \geq 0$. The above theorem then implies that there exists $g > 0$ satisfying $\tilde{\Delta} g - \rho q g = 0$ on M, i.e. $\Delta g - qg = 0$ on M. The proof is complete.

§3. THE SECOND VARIATION OF AREA FORMULA FOR BRANCHED MINIMAL IMMERSIONS.

In this section we show that the usual second variation of area formula is still valid for branched minimal immersions. Let p be a branch point of F and let U be a neighbourhood of p which contains no other branch point. (This is possible because branch points are isolated.) By choosing U sufficiently small, we may define a local complex co-ordinate z on U with respect to which the metric is

$\lambda(z)|dz|^2$. We may also suppose that $z(U) \supset \mathbb{B}(0, 1) = \{z \in \mathbb{C} \mid |z| < 1\}$. For each $1 > \theta > 0$, define a Lipschitz function f_θ by

$$f_\theta(z) = \begin{cases} 1 & \text{if } |z| > \theta \\ \dfrac{\log |z|/\theta^2}{\log 1/\theta} & \text{if } \theta^2 \le |z| \le \theta \\ 0 & \text{if } |z| < \theta^2 . \end{cases}$$

If s is a section of NM defined over U, then, for any $\delta > 0$, we have:

$$\begin{aligned} \int_{U\setminus z^{-1}(\mathbb{B}(0,\theta))} |\nabla^\perp s|^2 dA \le \int_U |\nabla^\perp (f_\theta s)|^2 dA &\le (1 + 1/\delta) \int_{z(U)} |\nabla_E f_\theta|^2 |s|^2 dx\,dy \\ &\quad + (1+\delta) \int_{z(U)} f_\theta^2 |\nabla_E^\perp s|^2 dx\,dy \\ &\le \frac{c(\delta, s)}{\log 1/\theta} + (1+\delta)\int_U f_\theta^2 |\nabla^\perp s|^2 dA \end{aligned}$$

In the above, $z = x + iy$ and ∇_E is the Euclidean gradient so that $|\nabla_E^\perp s|^2 = |\nabla^\perp_{\partial/\partial x} s|^2 + |\nabla^\perp_{\partial/\partial y} s|^2$ and $|\nabla_E f_\theta|^2 = \left(\frac{\partial f_\theta}{\partial x}\right)^2 + \left(\frac{\partial f_\theta}{\partial y}\right)^2$. By letting $\theta \to 0$ and then letting $\delta \to 0$ in the above inequalities we see that

$$(*) \qquad \int_U |\nabla^\perp (f_\theta s)|^2 dA \to \int_U |\nabla^\perp s|^2 dA \quad \text{as} \quad \theta \to 0 .$$

Given a compactly supported section s of NM, let s_θ be the section obtained by cutting s down to zero on the annuli $B(p_i,\theta)\setminus B(p_i,\theta^2)$ by means of the function f_θ defined above, where $\{p_i\}$ is the set of branch points of F. It is legitimate to apply the usual second variation of area formula to s_θ . By letting $\theta \to 0$ and using (*) we see that the usual second variation of area formula is also valid for s .

BIBLIOGRAPHY.

[1] Barbosa, J.L. and do Carmo, M., On the size of a stable minimal surface in $\mathbb{R}^3$, Amer. J. Math. 98 (2) (1976) 515-528.

[2] do Carmo, M. and Peng, C.K., Stable minimal surfaces in $\mathbb{R}^3$ are planes, Bull. Amer. Math. Soc. (New Series), 1 (1979) 903-906.

[3] Fischer-Colbrie, D. and Schoen, R.M., The structure of Complete Stable Minimal Surfaces in 3-Manifolds of Non-Negative Scalar Curvature, Comm. Pure and Appl. Math., 33 (1980) 199-211.

[4] Micallef, M.J., Stable Minimal Surfaces in Euclidean Space, J. Diff. Geom., 19 (1984) 57-84.

[5] Osserman, R., A Survey of Minimal Surfaces, Van Nostrand Reinhold, New York, 1969.

Department of Mathematics
3220 Angell Hall
University of Michigan
Ann Arbor, Michigan 48109-1003
U.S.A.