BOUNDARY REGULARITY FOR SOLUTIONS OF QUASI-LINEAR ELLIPTIC EQUATIONS

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1. INTRODUCTION

We consider the boundary regularity of a classical solution $u(x) \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ to the Dirichlet problem of a class of quasi-linear elliptic equations:

$$(1.1) \qquad Q(u) \equiv a_{ij}(x,u,Du)D_{ij}u = 0 \quad \text{in } \Omega , \qquad u = \varphi \quad \text{on } \partial\Omega ,$$

where Ω is a bounded C^2 domain in $\mathbb{R}^n$, $n \geq 2$ and $\varphi \in C^0(\partial\Omega)$ has some modulus of continuity β. Here we use the usual summation convention for repeated indices.

We refer to [GT], [JS] for the case when $\varphi \in C^{2,\alpha}(\partial\Omega)$, [GG], [G], [Li 1] for $\varphi \in C^{1,\alpha}(\partial\Omega)$, [Li 3] for φ having $D\varphi$ Dini continuous and [Li 2], [Sl] for $\varphi \in C^{0,1}(\partial\Omega)$.

We shall mainly discuss how the order of non-uniformity (h) and the geometry (convexity) of Ω affect the regularity of a solution of (1.1) near the boundary. As was remarked in [B], when $0 \leq h < 1$, the operator Q behaves very similarly to the Laplace operator (where $h = 0$) ; when $1 \leq h \leq 2$, some convexity (or some generalized convexity) condition has to be imposed on Ω. A typical representative of the latter class is the minimal surface operator (where $h = 2$). Since this is discussed in

G. Williams' article in these proceedings we shall concentrate on $0 \le h < 2$. For simplicity we shall not state the results in their full generality and refer the interested reader to the articles listed in the references.

2. NOTATIONS AND DEFINITIONS

We shall always assume Ω to be a bounded C^2 domain in $\mathbb{R}^n$, $n \ge 2$ and $a_{ij}(x,z,p) \in C^1(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. Let

(2.1) $\beta(t) \in C^2(0,\infty) \cap C^0[0,\infty)$ with $\beta(0) = 0$, $\beta'(t) > 0$ in $(0,\infty)$, $\beta''(t) \le 0$ in $(0,\infty)$,

(2.2) $\lambda(x,z,p)$, $\Lambda(x,z,p)$ = the minimum, maximum eigenvalues of $[a_{ij}(x,z,p)]$,

(2.3) $E(x,z,p) = a_{ij}(x,z,p)p_i p_j$,

(2.4) $E^*(x,z,p) = |p|^{-2}E(x,z,p)$ for $p \neq 0$,

(2.5) $T(x,z,p) = \sum_{i=1}^{n} a_{ii}(x,z,p)$,

(2.6) $|u|_{0;\Omega} = \sup\{|u(x)| : x \in \Omega\}$,

(2.7) $[u]'_{\alpha;\Omega} = \inf\{H : |u(x) - u(y)| \le H|x - y|^\alpha$ for all $x \in \Omega$, $y \in \partial\Omega\}$ for $0 < \alpha \le 1$,

(2.8) $[u]_{\alpha;\Omega} = \inf\{H : |u(x) - u(y)| \le H|x - y|^\alpha$ for all $x, y \in \Omega\}$ for $0 < \alpha \le 1$,

(2.9) $H_\alpha(\Omega)$ = the set of all functions u on Ω for which $|u|_{0;\Omega} + [u]_{\alpha;\Omega}$ is finite.

(2.10) $H_\alpha(\partial\Omega)$ can be defined in the standard way by covering $\partial\Omega$ with open balls and straightening the boundary.

DEFINITION 1

If $\dfrac{\Lambda(x,z,p)}{\lambda(x,z,p)} = O(|p|^h)$ as $|p| \to \infty$ uniformly on $(x,z) \in \overline{\Omega} \times [-M,M]$ for each $M > 0$, then h is called *the order of non-uniformity* of the operator Q.

REMARK 1

If the order of non-uniformity is h, then $E(x,z,p) \geq C|p|^{2-h} \cdot \Lambda(x,z,p)$ for some $C > 0$ as $|p| \to \infty$. We shall always write $k = 2 - h$.

3. STATEMENTS OF RESULTS

CASE 1: GENERAL DOMAIN, $0 \leq h < 1$

REMARK 2

It is reasonable to consider only $0 \leq h < 1$ because for general domains, $1 \leq h \leq 2$, a solution may not even exist. See e.g. [La 2].

THEOREM 1

Let $\varphi \in H_\alpha(\partial\Omega)$, $\alpha \in (0,1)$ *and* $0 \leq h < 1$. *Then*

(3.1) $$[u]'_{\alpha\gamma} < \infty$$

where $\gamma = \dfrac{2-h}{2-\alpha h}$.

PROOF. Theorem 3.1 of [Li 2].

THEOREM 2

Let $0 < h < 1$. *Suppose for some neighbourhood* U *of* $x_0 \in \partial\Omega$,

(3.2) $\varphi(x) - \varphi(x_0) \leq \beta(|x - x_0|)$ *for all* $x \in U \cap \partial\Omega$.

Then there exists a constant $\mathbb{C}$ *depending on* φ, Ω, h *such that*

(3.3) $u(x) - \varphi(x_0) \leq C\beta(C|x - x_0|^{\frac{1}{1+h}})$ *for all* $x \in \Omega$.

If $h = 0$, *the exponent* $\frac{1}{1+h}$ *in* (3.3) *can be replaced by any* θ , $0 < \theta < 1$, *with* C *depending on* θ *as well.*

CASE 2: CONVEX DOMAINS , $0 \leq h < 2$

THEOREM 3

Let Ω *be convex,* $\varphi \in H_\alpha(\partial\Omega)$, $\alpha \in (0,1)$. *Suppose* $0 \leq h < 2$. *Then*

(3.4) $[u]'_{\alpha\gamma} < \infty$

where $\gamma = \frac{2-h}{2-\alpha h}$.

PROOF. Theorem 3.4 of [Li 2].

THEOREM 4

Let Ω *be convex,* $1 \leq h < 2$. *Suppose for some neighbourhood* U *of* $x_0 \in \partial\Omega$ *,*

$$(3.5) \qquad \varphi(x) - \varphi(x_0) \leq \beta(|x - x_0|) \quad \text{for all} \quad x \in U \cap \partial\Omega .$$

Then there is a constant C *depending only on* φ*,* Ω *and* h *such that*

$$(3.6) \qquad u(x) - \varphi(x_0) \leq C\beta(C|x - x_0|^{\frac{2-h}{2}}) .$$

CASE 3: STRICTLY CONVEX DOMAINS , $0 \leq h \leq 2$

DEFINITION 2

Ω is said to satisfy *an enclosing sphere condition* at a point $x_0 \in \partial\Omega$ if there exists a ball $B = B_R(y) \supseteq \Omega$ with $x_0 \in \partial B$. The domain Ω is said to be R-*uniformly convex* if it satisfies an enclosing sphere condition at each boundary point with a ball of fixed radius $R > 0$.

THEOREM 5

Suppose there is a neighbourhood U *of* x_0 *such that at each point* $y \in U \cap \partial\Omega$ *, there is an enclosing ball of fixed radius* $R > 0$ *,* $B_R \supseteq U \cap \Omega$ *with* $y \in \partial B_R$. *If* $0 \leq h \leq 2$ *and*

$$(3.7) \qquad \varphi(x) - \varphi(x_0) \leq \beta(|x - x_0|) \quad \text{for all} \quad x \in U \cap \partial\Omega ,$$

then there is a constant C *depending on* φ *,* Ω *such that*

(3.8) $\quad u(x) - \varphi(x_0) \le C\beta(C|x - x_0|^{\frac{1}{2}})$.

CORROLLARY 6 ([Li 2])

Let Ω *be* R-*uniformly convex for some* $R > 0$ *and* $0 \le h \le 2$.

If $\varphi \in H_\alpha(\partial\Omega)$, $\alpha \in (0,1]$, *then* $[u]'_{\frac{\alpha}{2}} < \infty$.

4. SOME PROOFS

We shall indicate how to prove theorems 2, 4 and 5. For convenience we may assume that $x_0 = 0$, $\varphi(x_0) = 0$ and the $(x_1, \ldots, x_{n-1})$-plane is tangent to $\partial\Omega$ at x_0 . Let $d(x)$ be the distance of $x \in \Omega$ from $\partial\Omega$. Since $\partial\Omega$ is assumed to be C^2 , $d(x)$ is C^2 in some neighbourhood Γ of $\partial\Omega$. See [GT] Appendix.

We take

(4.1) $\quad W = [d(x)^2 + |x'|^2]^{\frac{1}{2}}$, where $x' = (x_1, \ldots, x_{n-1})$,

(4.2) $\quad N = \{(x', x_n) \in \mathbb{R}^n : |x'| < \delta$ and $0 < d(x) < \frac{2}{J}(\delta^{\frac{1}{\theta}} - W^{\frac{1}{\theta}})\}$,

(4.3) $\quad v(s) = K\beta(s^\theta)$ for $s \ge 0$, for some $0 < \theta < 1$,

(4.4) $\quad f(x) = Jd(x) + 2W^{\frac{1}{\theta}}$ for $x \in \overline{N}$.

Hence

(4.5) $\quad w(x) = v(f(x)) = K\beta([Jd(x) + 2W^{\frac{1}{\theta}}]^\theta)$.

$\delta > 0$ will be chosen to be small while J, K big. On $\partial\Omega$, $d(x) \equiv 0$, we have

(4.6) $$w(x) = K\beta(2^{\theta}|x'|) \ge K\beta(|x|) \ge \varphi(x) \quad \text{for all} \quad |x'| < \delta \ , \ \delta \text{ small} \ .$$

Choose $K > 0$ sufficiently large so that

(4.7) $$K\beta(2^{\theta}\delta) \ge \sup\{|u(x)| \ : \ x \in \Omega\} \ .$$

Then $u \le w$ on ∂N . By the comparison principle, we only need to show

(4.8) $$\overline{Q}(w) \equiv a_{ij}(x,u(x) \ , \ Dw)D_{ij}w \le 0 \quad \text{in} \quad N$$

to conclude that $u \le w$ in N and hence our theorems. We use $\overline{Q}$ instead of Q to avoid the direct dependence on the w variable so that the usual comparison principle can be applied. See [GT] p. 207. It is easy to compute that

(4.9) $$\begin{aligned} \overline{Q}(w) &= \frac{v''(f)}{v'(f)^2} E(x,u(x) \ , \ Dw) + v'(f)a_{ij}(x,u(x) \ , \ Dw)D_{ij}f \\ &\le \frac{v''(f)}{v'(f)^2} C_1 \wedge |Dw|^k + v'(f)a_{ij}D_{ij}f \qquad \text{(using remark 1)} \\ &\le v'(f) \wedge \left\{ \frac{v''(f)}{v'(f)^{3-k}} C_1 J^k + J \cdot \frac{a_{ij}D_{ij}d(x)}{\wedge} + C(n,\theta)W^{\frac{1}{\theta} - 2} \right\} \end{aligned}$$

where $C(n,\theta)$ = a constant depending on n , θ . Recall $k = 2 - h$. We first note that

(4.10) $$a_{ij}D_{ij}d(x) \le a_{ij}D_{ij}d(y) \ .$$

where $y = y(x)$ = the point on $\partial\Omega$ nearest to x . In fact, in terms of a principal coordinate system at $y = y(x)$, we have

(4.11) $$[D^2 d(x)] = \text{diag}\left[\frac{-\kappa_1}{1-\kappa_1 d(x)} \ , \ \ldots \ , \ \frac{-\kappa_{n-1}}{1-\kappa_{n-1} d(x)} \ , \ 0\right]$$

where κ_i's are principal curvatures of $\partial\Omega$ at y. (See [GT] Appendix Lemma 2.)

CASE 1: General domain, $0 < h < 1$ (or $1 < k < 2$). In this case, all we can say about the second term is

$$\frac{Ja_{ij}D_{ij}d(x)}{\Lambda} \le JC(n)|D^2d|_{0;\Gamma} \tag{4.12}$$

In order to make $\overline{Q}(w) \le 0$, we make use of the first term

$$\frac{C_1 v''(f)J^k}{v'(f)^{3-k}} \le \frac{C_1 J^k(\theta-1)f^{1-k}}{v(f)^{2-k}} \tag{4.13}$$

$$\le \frac{C_1 J^k(\theta-1)(2JW)^{1-k}}{v(f)^{2-k}}$$

$$= \frac{C_1(\theta-1)J^1 2^{1-k}W^{1-k}}{v(f)^{2-k}}$$

Take $\theta = \frac{1}{3-k}$ so that

$$\frac{1}{\theta} - 2 = 1-k \tag{4.14}$$

By choosing $J > 0$ sufficiently large, we ensure that $\overline{Q}(w) \le 0$ in $\mathcal{N}$. For the case $h = 0$ or $k = 2$, take $\theta \in (0,1)$.

CASE 2: Convex Domain, $1 \le h < 2$ or $0 < k \le 1$. Since Ω is convex, we have

$$a_{ij}D_{ij}d(x) \le a_{ij}D_{ij}d(y) \le 0\ . \tag{4.15}$$

Hence

$$(4.16)\qquad \overline{Q}(w) \le v'(f)\,\Lambda \left\{ \frac{v''(f)}{v'(f)^{3-k}} C_1 J^k + C(n,\theta) W^{\frac{1}{\theta}-2} \right\} .$$

Now $1 - k \ge 0$ and

$$(4.17)\qquad \frac{C_1 v''(f) J^k}{v'(f)^{3-k}} \le \frac{C_1(\theta-1) J^k f^{1-k}}{v(f)^{2-k}}$$

$$\frac{C_1(\theta-1) J^k (2W^{\frac{1}{\theta}})^{1-k}}{v(f)^{2-k}} = \frac{C_1(\theta-1) 2^{1-k} J^k W^{\frac{1}{\theta}(1-k)}}{v(f)^{2-k}} .$$

We take $\theta = \frac{k}{2}$ so that

$$(4.18)\qquad \frac{1}{\theta}(1-k) = \frac{1}{\theta} - 2$$

and argue as before.

REMARK 3

If we consider the case $1 < k < 2$, then $1 - k < 0$ and we are back to (4.13).

CASE 3: R-uniformly convex domain, $0 \le h \le 2$ or $0 \le k \le 2$. Since Ω is R-uniformly convex, we have

$$(4.19)\qquad \frac{a_{ij} D_{ij} d(x)}{\Lambda} \le \frac{a_{ij} D_{ij} d(y)}{\Lambda} \le \frac{-1}{R}$$

Take $\theta = \frac{1}{2}$ so that $\frac{1}{\theta} - 2 = 0$ and argue as before. Q.E.D.

REMARK 4

For the case of the minimal surface equation, we have

(4.20) $$Q(u) = \Delta u - (1 + |Du|^2)^{-1} D_i u D_j u D_{ij} u$$

and $$a_{ij}(x,z,p) = a_{ij}(p) = \delta_{ij} - (1 + |p|^2)^{-1} p_i p_j \;.$$

The crucial curvature term is then

(4.21) $$a_{ij}(Dw) D_{ij} d(x) = (\delta_{ij} - (1 + |Dw|^2)^{-1} D_i w D_j w) D_{ij} d(x)$$

$$= \Delta d(x) - (1 + |Dw|^2)^{-1} v'(f)^2 \; D_i f D_j f D_{ij} d(x)$$

Since $Df(x) \approx Dd(x)$ and $|Dd(x)| \equiv 1$, $D_i d D_{ij} d(x) = 0$, the dominant term is $\Delta d(x)$. Since

(4.22) $$\Delta d(x) \leq \Delta d(y) = \kappa_1(y) + \ldots + \kappa_{n-1}(y)$$

$$= (n-1) \cdot \text{the mean curvature of } \partial\Omega \text{ at } y$$

convexity of Ω is not exactly the most suitable geometric condition. In fact we have the following classical result:

THEOREM ([JS])

The Dirichlet problem for the minimal surface equation is solvable with <u>*every*</u> *arbitrary boundary function* $\varphi \in C^0(\partial\Omega)$ *if and only if* $\partial\Omega$ *has non-negative mean curvature (wrt inward normal) everywhere.*

REMARK 5

Of course, geometrically the most interesting case is when $k = 2$ which includes in particular the Euler-Lagrange equation of elliptic parametric integrals. When $\partial\Omega$ is only assumed to have *non-negative* mean curvature, the boundary regularity question for the *minimal surface equation* has been thoroughly discussed in [W3]. But the general case is still open.

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