# A PIECEWISE LINEAR THEORY OF MINIMAL SURFACES IN 3-MANIFOLDS

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## Introduction

In an impressive series of papers, Meeks and Yau [MYi, $1 \leq i \leq 5$], Meeks, Simon and Yau [MSY], Freedman, Hass and Scott [FHS], Scott [S], and Meeks and Scott [MS] introduced and used least area surfaces in the investigation of topological problems about 3-manifolds. This has lead to the solution of many outstanding questions in the topology of 3-manifolds. An example is the positive solution of the Smith conjecture (see [SC]), in which the results of Meeks and Yau [MY5] played an important role.

In [JR1], we used least weight normal surfaces to obtain the equivariant decomposition theorems of 3-manifolds in [MYi, $1 \leq i \leq 5$] and [MSY]. These least weight normal surfaces share many of the same useful properties as least area surfaces. However since the Meeks-Yau exchange and roundoff trick cannot be directly applied to normal surfaces, we were unable to recapture the more difficult applications and properties of least area surfaces in [S], [MS] and [FHS], by using least weight normal surfaces.

Here we develop the idea of least weight normal surfaces to obtain piece-wise linear (PL) minimal surfaces in 3-manifolds. This theory has several advantages over the classical area of analytic minimal surfaces, especially with regard to the study of the topology of 3-manifolds. Firstly, to establish existence of PL minimal surfaces, there is no necessity to appeal to deep results from partial differential equations and geometric measure theory, as in the analytic case. (See Hass-Scott [HS] for a new uniform treatment of existence theory for least area surfaces, using only Morrey's solution of Plateau's problem in Riemannian 3-manifolds). For PL minimal surfaces, it suffices to use the short classical PL technique of Kneser [K], plus a little elementary analysis.

Next, PL minimal surfaces are explicitly computable, by the method of Haken [H] for normal surfaces. By contrast, precise descriptions of analytic minimal surfaces are usually rather difficult to obtain. Finally there is a local uniqueness property for PL minimal surfaces (see Theorem 2). There is no analogous result in the analytic case. This local uniqueness leads to a local version for PL minimal surfaces of the properties of least area surfaces established in [FHS]. In particular, PL minimal surfaces have the smallest number of self-intersections and intersections in normal homotopy classes.

PL minimal surfaces are defined by choosing a nice Riemannian metric on the 2-skeleton $\mathfrak{T}^{(2)}$ of a given triangulation $\mathfrak{T}$ of some 3-manifold M. The idea of putting such a metric on $\mathfrak{T}^{(2)}$ arose from the analysis in [JR1] of the intersections of least weight normal surfaces as spanning arcs crossing in 2-simplices of the 2-skeleton. For details of the results in this paper, see [JR2].

Normal and PL minimal surfaces.

A <u>surface</u> f in a 3-manifold M will always refer to a proper immersion $f\colon (F, \partial F) \to (M, \partial M)$, where $\partial$ denotes boundary and possibly $\partial F$ and $\partial M$ are empty. There are seven properly embedded disks in a 3-simplex called <u>disk types</u>. These consist of four triangular disks, which separate a vertex from its opposite face and three disks with quadrilateral boundaries, which separate a pair of opposite edges of the 3-simplex. A <u>normal</u> surface f in M intersects each 3-simplex of $\mathfrak{T}$ in a finite set of such disk types.
Let $\mathfrak{T}^{(i)}$ denote the i-skeleton of $\mathfrak{T}$. The <u>weight</u> of f is the number of points in $f^{-1}(\mathfrak{T}^{(1)})$.

Remark. A normal surface can be thought of as a minimal surface if all the area is concentrated near $\mathcal{T}^{(1)}$, by choosing a suitable Riemannian metric on M.

A normal homotopy is just a homotopy through normal surfaces. Then any normal surface f determines a normal homotopy class which is denoted $\mathcal{N}(f)$.

To introduce the concept of PL minimal surfaces, we now construct a Riemannian metric on $\mathcal{T}^{(2)}$, by identifying each 2-simplex with an ideal hyperbolic 2-simplex in the hyperbolic plane. The 2-simplices can then have common edges identified by isometries. If a group G of simplicial homeomorphisms is given, such that the fixed set of any member of G is a subcomplex, then we can choose the metric on $\mathcal{T}^{(2)}$ so that G acts isometrically.

Given a normal surface $f: F \to M$, we define its length $\ell$ to be the total length of all the arcs in which $f(F)$ meets the 2-simplices of $\mathcal{T}^{(2)}$. We will call these the arcs of f. The PL area of f is defined to be the pair $(\omega, \ell)$, lexicographically ordered. Finally f is PL minimal if its length $\ell$ is stationary for small variations of f. Let $f_s: F \to M$ be a smooth family of (normal) surfaces, where $s \in (-\delta, \delta)$ and $f_0 = f$. Then f is PL minimal if the derivative of the function $\ell(f_s)$ is always zero.

A normal surface $f: F \to M$ is called PL least area if f has smallest PL area amongst all normal surfaces homotopic to f. This will be most useful in the following cases:

f is called $\pi_1$-injective if both the maps $f_*: \pi_1(F) \to \pi_1(M)$ and $f_\#: \pi_1(F, \partial F) \to \pi_1(M, \partial M)$ are one-to-one, with $\pi_1(F) \neq \{1\}$. If F is a disk or 2-sphere then f is essential if either $f: (D, \partial D) \to (M, \partial M)$ is non-zero in $\pi_2(M, \partial M)$ or $f\colon S^2 \to M$ is non-trivial in $\pi_2(M)$ or $f\colon S^2 \to M$ is an embedding with $f(S^2)$ bounding a fake 3-ball, but not a 3-ball in M.

We call a 3-manifold M $P^2$-irreducible if any embedded 2-sphere bounds a 3-ball and there are no embedded two-sided projective planes in M. A surface is called two-sided if it has a trivial normal bundle in M.

The energy E of a normal surface f is defined as the sum of the squares of the lengths of the arcs of f. Energy has the nice property that it is a convex function on $\mathcal{N}(f)$ and this implies the uniqueness of PL minimal surfaces in normal homotopy classes. We would like to thank Bill Thurston for bringing energy to our attention.

Finally we describe the mean curvature field H of a normal surface f. Let $\alpha$ be an arc of f and let $\beta$ be a component of $f^{-1}(\alpha)$. If $y \in \operatorname{int}\beta$ and $x = f(y)$ then we define $H(y) = \nabla_T T(x)$, where $\nabla$ is hyperbolic covariant differentiation and T is the tangent vector field $\alpha'$. We assume without loss of generality that $|T| = 1$. If $y \in f^{-1}(\mathcal{T}^{(1)})$, let $\alpha_1, \ldots, \alpha_k$ be the arcs of f with $y \in f^{-1}(\alpha_i)$, $1 \leq i \leq k$. We can suppose that $f(y) = x = \alpha_i(0)$, for $1 \leq i \leq k$, and can define $H(y) = \sum_{i=1}^{k} \langle T_i, V\rangle V$, where $T_i = \alpha_i'(0)$ and V is a unit vector tangent to the edge in $\mathcal{T}^{(1)}$ at x.

Properties of PL minimal surfaces

A linking 2-sphere is the normal surface which is the boundary of a small regular neighbourhood of a vertex in $\mathcal{T}^{(0)}$.

Theorem 1. For any normal surface f which is not a linking 2-sphere, there is a PL minimal surface in $\mathcal{N}(f)$.

Next we consider first and second variation of length and energy for normal surfaces. Let $f_s: F \to M$ be a small variation of normal surfaces, where $s \in (-\delta, \delta)$ and $f_0 = f$. Let $\ell(s) = \ell(f_s)$. We will denote the arcs of $f_s$ by $\alpha_i^s$, $1 \leq i \leq m$, with $\alpha_i^0$ denoted by $\alpha_i$. By transversality, since $\delta$ is small, m is independent of s. Let $T_i = \alpha_i'$ and assume $|T_i| = 1$. Also let

$$V_i = (\alpha_i^s)_*\left( \frac{\partial}{\partial s} \right)\Big|_{s=0}$$

be the variation vector field and let $\ell_i$ denote $\ell(\alpha_i)$. Then the first variation formula is:

$$\ell'(0) = \sum_{i=1}^{m} \langle V_i, T_i \rangle \Big|_0^{\ell_i} - \sum_{i=1}^{m} \int_0^{\ell_i} \langle V_i, \nabla_{T_i} T_i \rangle dt.$$

This shows immediately that f is PL minimal if and only if the mean curvature H is zero. Also if $E(s) = E(f_s)$, then $E'(0) = \ell'(0)$.

To obtain a nice expression for second variation, we can assume that $V_i$ at an edge e of $\mathcal{T}^{(1)}$ is a unit tangent vector field to e. Hence $\nabla_{V_i} V_i = 0$ along $\mathcal{T}^{(1)}$. Also the Gaussian curvature of the hyperbolic metric is $-1$. Consequently second variation of length and energy are:

$$\ell''(0) = \sum_{i=1}^{m} \int_0^{\ell_i} (|\nabla_{T_i} V_i|^2 + |V_i \times T_i| - (T_i \langle V_i, T_i \rangle)^2) dt$$

and $$E''(0) = \sum_{i=1}^{m} \int_0^{\ell_i} (|\nabla_{T_i} V_i|^2 + |V_i \times T_i|) dt.$$

Since E is convex, it has a unique minimum in $\mathcal{N}(f)$. This establishes:

Theorem 2. There is precisely one PL minimal surface in $\mathcal{N}(f)$, for any normal surface f which is not a linking 2-sphere.

The local behaviour at a point x of "common tangency" of two PL minimal surfaces $f_1$ and $f_2$ can be analysed, as in the analytic case (cf. e.g. [B]). If x is in $\mathfrak{T}^{(1)}$, we obtain a generalised saddle picture. If x is in $\mathfrak{T}^{(2)}$ but not in $\mathfrak{T}^{(1)}$, tangency should be interpreted more widely since we are working in a PL setting. In this case we obtain that the arcs of $f_1$ and $f_2$ through x coincide. The behaviour of $f_1$ and $f_2$ in $\mathfrak{T}^{(3)} - \mathfrak{T}^{(2)}$ is not of interest. (PL minimal surfaces are really defined only by their points in $\mathfrak{T}^{(2)}$). Also barriers for PL minimal surfaces, such as convex boundaries, can be constructed as in e.g. [MY3].

The exchange and roundoff trick (cf. [MY1] and lemma 1.2 of [FHS]) works in the PL case. We have for example:

Lemma. Suppose $f_1$, $f_2$ are embedded PL least area surfaces in their homotopy classes and $f_1$ meets $f_2$ transversely, with $f_1(F_1) \cap f_2(F_2)$ transverse to $\mathfrak{T}$. Then there are no product regions $R \times [0, 1] \subset M$, where $R \times \{0\} \subset f_1(F_1)$ and $R \times \{1\} \cup \partial R \times [0, 1] \subset f_2(F_2)$.

Often, the exchange and roundoff trick must be applied where $f_1$ and $f_2$ may not be transverse, or their intersection may cross $\mathfrak{T}$ non transversely. To avoid this we can use the Meeks-Yau trick (cf. [MY1] and lemma 1.3 of [FHS]). The idea is to perturb $f_1$ to $f_1^*$, increasing length by $\varepsilon$, so that $f_1^*$ and $f_2$ have the desired transversality properties. If there are product regions, then at least $2\varepsilon$ in length is saved by exchange and roundoff, a contradiction.

Applications of PL minimal surfaces

The basic existence result for PL least area surfaces is:

Theorem 3 (cf. Theorems 3.1 and 7.2 of [FHS]). Let M be a 3-manifold which covers a compact 3-manifold.

(1) Suppose M is $P^2$-irreducible and let $f: F \to M$ be a $\pi_1$-injective surface. Then there exists a PL least area surface in the homotopy class of f.

(2) Suppose $\pi_2(M, \partial M) \neq \{1\}$ (or $\pi_2(M) \neq \{1\}$, respectively). Then there exists an essential PL minimal disk (or non-contractible 2-sphere) which has smallest PL area amongst all such disks (or 2-spheres respectively).

Then we can obtain the results of [MYi, $1 \leq i \leq 5$], [FHS], [S] and [MS] using PL least area surfaces. Finally to obtain the main application of [MSY], i.e. that any covering of a $P^2$-irreducible 3-manifold is $P^2$-irreducible, we need to show that PL least area essential 2-spheres can be found if $\pi_2(M) = \{1\}$ but M contains fake balls. This follows from:

Theorem 4. Let M be a compact 3-manifold. Suppose f and f' are normal surfaces and g, g' are the PL minimal surfaces in $\mathcal{N}(f)$, $\mathcal{N}(f')$ respectively. Then the number of self-intersections of g is smallest for surfaces in $\mathcal{N}(f)$ and the number of intersections of g and g' is the least for pairs of surfaces in $\mathcal{N}(f)$ and $\mathcal{N}(f')$.

Remarks. 1. For a precise description of how to count intersections and self-intersections, the reader is referred to [JR2] and [FHS].

2. Note that it is not necessary to include any homotopy assumptions about f and f'. In [FHS], the hypotheses are that the surfaces

are $\pi_1$-injective and two-sided for the analogous theorems.

Corollary 1. Suppose f is an embedding. Then g is either an embedding or a double cover of an embedded surface. In the latter case, the image of f bounds a twisted I-bundle over a non-orientable surface isotopic to the image of g.

Corollary 2. Assume f and f' have disjoint images. Then g and g' have images which are either disjoint or the same. In the latter case, g and g' are covers of embeddings.

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