ON THE STRUCTURE OF BRANCH POINTS OF MINIMIZING DISKS

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§1. Introduction and statement of results.

Let B be the closed unit disk in C. A smooth map F of B into an n-dimensional manifold N is said to have a <u>branch point of order $Q - 1$</u> at $0 \in$ B if there exist local co-ordinates $(x_1, \ldots, x_n)$ of a neighborhood of $F(0)$ with respect to which F takes the form

$$x_1 + \sqrt{-1}\, x_2 \;=\; z^Q + o(|z|^Q)$$

$$x_k \;=\; o(|z|^Q), \quad 3 \leq k \leq n,$$

where Q is an integer ≥ 2.

Branch points fall into two categories, true and false. A branch point of order $Q - 1$ at $0 \in$ B is <u>false</u> if there exists an immersion $\tilde{F} : B \to N$ and $\psi : B \to B$ of degree Q with $\psi(0) = 0$ such that $F = \tilde{F} \circ \psi$. A branch point is <u>true</u> if it is not false. Thus, the image of a map with a false branch point is a smooth submanifold of N, whereas the image of a map with a true branch point is singular in the usual sense of differential geometry.

Branch points arise very naturally in the theory of minimal surfaces. They are the simplest type of singularity that a minimal surface could possess. A recent spectacular result of Sheldon Chang [C] shows that in fact they are the <u>only</u> possible singularities of area-minimizing two dimensional integral currents. For a history of the study of branch points we refer to [02]. The results most closely related to the ones in this article are the following.

THEOREM 1 *(Osserman [01]) Let $F : B \to \mathbb{R}^3$ define a minimal surface which has a true branch point at 0. Then, given an arbitrarily small neighborhood V of 0, there exists a piecewise smooth map $\overline{F} : B \to \mathbb{R}^3$ which agrees with F on B\V and such that Area $(\overline{F}(V)) < $ Area $(F(V))$.*

<u>Corollary</u> The Douglas solution to Plateau's problem in $\mathbf{R}^3$ defines a regular surface in the interior.

THEOREM 2 *(Federer [F], pp. 652-653). The surface defined by $z \mapsto (z^2, z^3) : B \to C^2 = \mathbf{R}^4$ is an area minimizing disk which must therefore be the Douglas solution for the curve $\Gamma = \{(e^{2i\theta}, e^{3i\theta}) \mid 0 \le \theta < 2\pi\}$. In contrast to the above corollary, we see that the Douglas solution in $\mathbf{R}^n$, $n \ge 4$, may very well be singular in the interior.*

Federer's result is much more general. He shows that, in fact, any complex subvariety of a Kähler manifold minimizes volume in its homology class. Motivated by these results, we have proved Theorem 3 below. In §2 we outline the proof of this theorem; details will appear elsewhere.

THEOREM 3 *Let $F : B \to N$ be a branched minimal immersion of the unit disk with a true branch point at $0 \in B$. We assume that $F(B)$ lies in a single co-ordinate chart and that with respect to this chart, F is of the form $(z^Q, f(z))$ where $z \epsilon B$, $\mathbf{R}^n = \mathbf{R}^2 \times \mathbf{R}^{n-2} = \mathbf{C} \times \mathbf{R}^{n-2}$; $f : B \to \mathbf{R}^{n-2}$, $\mid f(z) \mid = o(\mid z \mid^Q)$; Q is an integer ≥ 2. Then, the Taylor expansion of f has a first term $p(z)$ whose degree is not divisible by Q. If F minimizes area among disks on some arbitrarily small neighborhood V of $0 \in B$, then p is harmonic and therefore $p(z) = a_\mu z^\mu + \overline{a}_\mu \overline{z}^\mu$ for some $a_\mu \epsilon \mathbf{C}^{n-2}$. Moreover, $a_\mu \cdot a_\mu = 0$ where, if $v, w \epsilon \mathbf{C}^{n-2}$, $v \cdot w = \sum_{i=1}^{n-2} v_i w_i$.*

<u>Remarks:</u>

(i) The requirement that F is of the form $(z^Q, f(z))$ with $\mid f(z) \mid = o(\mid z \mid^Q)$ places no restriction on F, i.e. this form for F can always be achieved by suitable change of local co-ordinate (see, for example, [G], Lemma 2.2 on page 279).

(ii) If $n = 3$, then $a_\mu = 0$ and we recover Theorem 1.

(iii) If $n = 4$, then $a_\mu = \alpha(1, \pm\sqrt{-1})$ for some $\alpha \epsilon \mathbf{C}$. In this case the map $z \mapsto (z^Q, a_\mu z^\mu + \overline{a}_\mu \overline{z}^\mu)$ is holomorphic with respect to the complex structure on

$\mathbf{R}^4$ defined by the matrix

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \\ 0 & 0 & \mp 1 & 0 \end{pmatrix}$$

This, together with Sheldon Chang's work, provides a partial answer to Yau's question [Y, page 54] on comparing the singularities of a minimizing surface in $\mathbf{R}^4$ to the singularities of a complex curve in $\mathbf{C}^2$.

(iv) The minimizing assumption cannot be replaced by stability with respect to variations by means of compactly supported sections of the normal bundle of F. This is because the usual second variation of area formula is still valid, even in the presence of branch points (see, for example, [Mi]).

§2. Outline of proof of Theorem 3.

Let $M \subset \mathbf{B} \times \mathbf{R}^m$ be a closed two-dimensional disk for which $\pi : M \to \mathbf{B}$ is a branched covering of $\mathbf{B}$ of degree Q. ($\pi : \mathbf{B} \times \mathbf{R}^m \to \mathbf{B}$ is, of course, the obvious projection.) We shall define what it means for M to minimize Dirichlet's integral.

We assume that M has finitely many branch points, $m_1, \ldots, m_k$, all of which occur in the interior. Let $z_i = \pi(m_i)$, let z_0 be any point in $\partial \mathbf{B}$ and let Γ be a smooth curve in $\mathbf{B}$ passing through $z_0, z_1, \ldots, z_k$ such that $\mathbf{B} \backslash \Gamma$ is simply connected. It then follows that $\pi^{-1}(\mathbf{B} \backslash \Gamma) = \bigcup_{i=1}^Q M_i$, where each M_i is a graph of f_i over $\mathbf{B} \backslash \Gamma$, i.e. $M_i = \{(z, f_i(z)) \mid z \in \mathbf{B} \backslash \Gamma\}$. We define $\text{Dir}(M) = \sum_{i=1}^Q \int_{\mathbf{B}} | \nabla f_i |^2 \, dx \, dy$. M is said to be Dirichlet minimizing if $\text{Dir}(M) \leq$

Dir $(\tilde{M})$ for all disks $\tilde{M}$ which are branched covers of B of degree Q and which have the same boundary values as M. ($\tilde{M}$ has the same boundary values if, after a possible re-ordering of $\tilde{f}_1, \ldots, \tilde{f}_Q$, $\tilde{f}_i(z) = f_i(z)$ for all $z \in \partial \mathrm{B} \backslash \{z_0\}$ and for all $i \in \{1, \ldots, Q\}$.) In particular, if M is Dirichlet minimizing, each f_i is harmonic. The converse need not be true (except for $Q = 1$, of course). The preceding discussion is a very special case of the general theory developed in [A] of multi-valued functions minimizing Dirichlet's integral.

Theorem 3 follows from Theorem 4 and Lemma 5 below.

THEOREM 4 *Let* $M = \{(z^Q, az^\mu + \overline{a}\overline{z}^\mu) \mid z \in \mathrm{B}\}$. *If M is Dirichlet minimizing and μ is a positive integer not divisible by $Q > 1$, then $a \cdot a = 0$ where, if v, $w \in \mathrm{C}^m$, then $v \cdot w = \sum_{i=1}^m v_i w_i$.*

The proof of Theorem 4 runs as follows:

Let $\psi(z; t) = \dfrac{z^Q + tz^{Q-1}}{1 + \overline{t}z}$, $t \in \mathrm{C}, \mid t \mid < 1$. Define $g_t : \partial \mathrm{B} \to \mathrm{R}^m$ by $g_t(e^{i\theta}) = a[\psi(e^{i\theta}; t)]^{\mu/Q} + \overline{a}[\psi(e^{-i\theta}; t)]^{\mu/Q}$ where the branch of $(\psi \mid_{\partial \mathrm{B}})^{1/Q}$ which is chosen is the one which is smooth in t and for which $[\psi(e^{i\theta}; 0)]^{1/Q} = e^{i\theta}$. Let h_t be the harmonic extension of g_t to B and set $M_t = \{(\psi(z; t), h_t(z)) \mid z \in \mathrm{B}\}$. Then $M_0 = M$ and $\partial M_t = \partial M$ for all complex t with $\mid t \mid < 1$. By expanding $\mathrm{Dir}(M_t)$ as a power series in t we show that $\mathrm{Dir}(M_t) < \mathrm{Dir}(M)$ for some t sufficiently close to zero unless Q divides μ or $a \cdot a = 0$.

A similar idea has been previously employed by M. Beeson [B] who gave an analytic proof (i.e. no cut and paste) of Theorem 1 of Osserman.

LEMMA 5 *Let $F : \mathrm{B} \to N$ satisfy the hypothesis of Theorem 3. Then, the Taylor expansion of f has a first term $p(z)$ of degree not divisible by Q. Moreover $\{(z^Q, p(z)) \mid z \in \mathrm{B}\}$ is Dirichlet minimizing.*

<u>Sketch Proof:</u> Let ς be the primitive Q^{th} root of unity and let $\phi(z) = f(z) - f(\varsigma z)$. As in Lemma 8.1 on page 301 of [G], ϕ satisfies an equation of the form $a^{ij} \phi_{ij}^\alpha + a_\beta^{i\alpha} \phi_i^\beta + a_\beta^\alpha \phi^\beta = 0$ with $a^{ij} \in C^1$, $a^{ij}(0) = \delta_{ij}$ and $a_\beta^{i\alpha}$ and a_β^α continuous. Now ϕ is not identically zero because f has

a true branch point at 0 and therefore, a theorem of Hartman and Wintner [H-W] generalized to systems of the type satisfied by ϕ then asserts that, there exists a positive integer μ such that

$$(*)\ldots lim_{z\to 0}\, z^{1-\mu}\frac{\partial\phi}{\partial z} \text{ exists and is non-zero.}$$

The existence of p now follows. Note that $(*)$ already implies that p is harmonic and that it is of degree μ.

Let $M = \{(z^Q, p(z)) \mid z\,\epsilon B\}$. If M is not Dirichlet minimizing we may find a disk $\widetilde{M} \subset B \times R^{n-2}$ such that M and $\widetilde{M}$ have the same boundary values and Dir $(M) <$ Dir $(\widetilde{M})$. We modify the notation used in the first paragraph of this section as follows: $m_1,\ldots,m_k$ are the branch points of $\widetilde{M}$; Γ not only passes through $z_0, z_1,\ldots, z_Q$ but through 0 as well; $f_i(z)$ is now replaced by $p_i(z)$ where $p_i(z) = p(\eta_i(z))$ and $\eta_1(z),\ldots,\eta_Q(z)$ are the Q different branches of $z^{1/Q}$ on $B\backslash\Gamma$. Finally, we let $\tilde{f}_i(z)$ be the function which has the same values as $p_i(z)$ on $\partial B\backslash\{z_0\}$.

Given $\lambda\,\epsilon\,(0,1]$ and $z\,\epsilon B\backslash\Gamma$ with $\mid z \mid\leq \lambda$, let $\hat{f}_i(z) = f(\eta_i(z)) - p_i(z) + \lambda^\mu \tilde{f}_i(z/\lambda)$. Let $S_\lambda = \bigcup_{i=1}^{Q}\{(z,\hat{f}_i(z)) \mid z\,\epsilon B\backslash\Gamma \text{ and } \mid z\mid\leq \lambda\}$ and let $A_\lambda = \{z\,\epsilon B \mid \lambda \leq\mid z\mid\leq 1\}$. We now define $\sum_\lambda \subset R^n$ to be the disk $\overline{S}_\lambda \bigcup_{C_\lambda} F(A_\lambda)$ where $C_\lambda = F(\{z\,\epsilon B \mid \mid z\mid = \lambda\})$ and $\overline{S}_\lambda =$ closure of S_λ. For λ sufficiently small, one then finds that Area $(\sum_\lambda) <$ Area $(F(B))$, thereby contradicting the least area property of F.

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