REGULARITY AND SINGULARITY FOR ENERGY MINIMIZING MAPS

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1. INTRODUCTION

We will consider the occurrence of singularities in a class of boundary-value mapping problems. Suppose M is an m dimensional smooth compact Riemannian manifold with boundary and N is a smooth compact Riemannian manifold without boundary. Via an isometric embedding, we view N as a Riemannian submanifold of $\mathbb{R}^k$. We will consider the following type of problem:

Given a smooth $\varphi : \partial M \to N$, *find a least energy* $u : M \to N$ *with* $u \mid \partial M = \varphi$.

While various general energy functionals may be treated, we will mainly discuss, for $1 < p < \infty$, the ordinary *p-energy*

$$\int_M |\nabla u|^p \, dM \quad .$$

Here, the most important case is $p = 2$ where critical points are *harmonic maps.* In local coordinates $x_1, x_2, \cdots, x_m$ on M, the expression $|\nabla u|^p$ should be interpreted as $[\Sigma_{\alpha,\beta} \Sigma_{i,j} (\partial u^i/\partial x_\alpha) g^{\alpha,\beta} (\partial u^j/\partial x_\beta)]^{p/2}$ and the volume element dM as $(\det g)^{1/2} dx$ where $g = g_{\alpha,\beta} = [g^{\alpha,\beta}]^{-1}$ is the matrix representing the metric of M in these coordinates. Since only the topology and geometry of N will be relevant for our discussion of regularity and singularity, we will, for simplicity of notations, assume that M *is an open subset of* $\mathbb{R}^m$ *with the standard Euclidean metric.*

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2. SINGULARITIES BY TOPOLOGICAL OBSTRUCTION.

Topological obstructions may be relevant for the existence or regularity of least energy maps. Perhaps the simplest example concerns the case of maps from the unit ball $\mathbb{B}^m$ in $\mathbb{R}^m$ to the sphere $\mathbb{S}^{m-1} = \partial\mathbb{B}^m$. Here an elementary topological condition is

(*) *there exists a continuous map* $u : \overline{\mathbb{B}}^m \to \mathbb{S}^{m-1}$
if and only if $u \mid \partial\mathbb{B}^m$ *has degree* 0 .

For $p \geq m$, (*) *implies that there exists no function* $u : \overline{\mathbb{B}}^m \to \mathbb{S}^{m-1}$ *of finite p-energy whose trace* $u \mid \partial\mathbb{B}^m$ *has nonzero degree* .

In fact, for $p > m$, a finite energy u would be essentially continuous by Sobolev embedding while for $p = m$, one could suitably approximate u by a continuous map to contradict (*). Thus for $p \geq m$, the least energy boundary-value problem may be meaningless.

On the other hand, if $1 < p < m$, then there is such a finite p-energy extension given, for example, by the homogeneous-degree-0 extension, $w(x) = \varphi(x/|x|)$. In fact, using spherical polar coordinates,

$$\int_{\mathbb{B}^m} |\nabla w|^p dx = \int_{\mathbb{S}^{m-1}} \int_0^1 r^{-p} |\nabla_{tan}\varphi|^p r^{m-1} \, dr \, d\theta = (m-p)^{-1} \int_{\mathbb{S}^{m-1}} |\nabla_{tan}\varphi|^p d\theta < \infty .$$

In this case, the boundary-value problem is meaningful, but (*) implies that *a solution will necessarily be discontinuous on* $\overline{\mathbb{B}}^m$ *if the given* φ *has nonzero degree* (e.g. φ = identity).

In general, if there exists some finite p-energy extension of φ , then one easily obtains the *existence* of a solution of the least p-energy boundary value problem by direct methods using the weak compactness, lower semi-continuity, and trace theory in the space $L^{1,p}(M,\mathbb{R}^k)$ [KJF] of functions of finite p-energy. Of relevance here is the fact that any sequence weakly convergent in $L^{1,p}$ is strongly convergent in $L^{0,p} = L^p$. In particular, a weakly convergent sequence in

$$L^{1,p}(M,N) = L^{1,p}(M,\mathbb{R}^k) \cap \{u : u(x) \in N \text{ a.e. in } M\}$$

has limit in $L^{1,p}(M,N)$, and we may minimize in the class

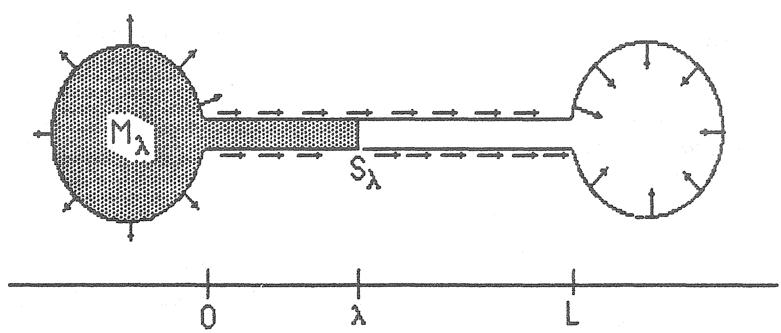
$$\mathcal{U}_\varphi = L^{1,p}(M,N) \cap \{u : u \,|\, \partial M = \varphi \text{ (in the sense of } L^{1,p} \text{ traces)}\}.$$

3. SINGULARITIES WITHOUT TOPOLOGICAL OBSTRUCTION.

In [HL$_1$] is a specific example of a smooth map $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ of degree 0 for which there is a definite *gap* between the two numbers

$$\inf\{\int_{\mathbb{B}^m} |\nabla u|^2 dx \ : u \in \mathcal{U}_\varphi\} < \inf\{\int_{\mathbb{B}^m} |\nabla v|^2 dx \ : \ v \in \mathcal{U}_\varphi \cap C^0(\overline{\mathbb{B}}^m, \mathbb{S}^{m-1})\}.$$

In particular the energy minimizer must have a singularity even though there is no topological restriction in the sense that there does exist some continuous finite-energy extension of φ. The gap may be made arbitrarily large by suitable choice of φ. To explain the idea of this construction we will describe an analogous problem where $1 < p < 2 = m$, $N = \mathbb{S}^1$, and M is a region in the plane shaped like a barbell with a thin handle of length L. The boundary data φ is given by a unit vectorfield on ∂M as shown.



As a map from ∂M to $\mathbb{S}^1$, φ has degree 0. By considering a comparison function that is approximately constant on the handle and homogeneous of degree 0 on each end, we readily check that

$$\inf\{\int_{\mathbb{B}^m} |\nabla u|^p dx \ : u \in \mathcal{U}_\varphi\} < 4\pi(2-p)^{-1} + 1 .$$

On the other hand, suppose that $v \in \mathcal{U}_\varphi \cap \mathcal{C}^0(\overline{\mathbb{B}}^m, \mathbb{S}^{m-1})$. Then, for all $0 < \lambda < L$, $v \mid \partial M_\lambda$ has degree 0 where M_λ is the subregion of M indicated above. Then the restriction of v to the vertical slice S_λ must almost cover $\mathbb{S}^1$ because $v \mid \partial M_\lambda \sim S_\lambda = \varphi \mid \partial M_\lambda \sim S_\lambda$ almost covers $\mathbb{S}^1$. This, along with Hölder's inequality, gives a lower bound for the energy of the slice:

$$\int_{S_\lambda} |\nabla v|^p \, dy \geq \int_{S_\lambda} |\partial v/\partial y|^p \, dy \geq c\Big[\int_{S_\lambda} |\partial v/\partial y| \, dy \Big]^p \geq \tilde{c} .$$

By Fubini's theorem, $\int_M |\nabla v|^p \, dx \geq \tilde{c} L \to \infty$ as $L \to \infty$.

4. PARTIAL REGULARITY THEORY.

Having seen that singularities in solutions are often unavoidable, we now discuss estimations on the their size. First we give a brief summary of what is known.

In case $p = 2$, and there is some restriction on N (e.g. negatively curved or lying in a coordinate neighborhood) there are several interesting early works, e.g. [ES], [Ha], [HKW]. For discussion of these and many other works, we refer to the excellent surveys of S. Hildebrandt [Hi] and J. Jost [J_1], [J_2]. In case $p > 2$ and N lies in a coordinate neighborhood, the work of N. Fusco and J. Hutchinson [FH] and of M. Giaquinta and G. Modica [GM] implies, among other things, the partial $\mathcal{C}^{1,\alpha}$ regularity of an energy minimizer, for some $0 < \alpha < 1$.

With no restriction on N and $p = 2$, the fundamental work of R. Schoen and K. Uhlenbeck [SU_1] (See also [GG], [SU_2], [JM], [SU_3]) showed that the interior singular set of an energy minimizer has Hausdorff dimension at most $m-3$. The study of liquid crystals leads to consideration of a more general energy functional with quadratic growth for mappings from 3 dimensional spatial domains to $\mathbb{S}^2$. For these minimizers, R. Hardt, D. Kinderlehrer, and F. H. Lin [HKL_1] showed that the singular set had 1 dimensional Hausdorff measure zero. For $p > 2$ and more general energy functionals with p-power growth, S. Luckhaus [L_1], [L_2] established the $\mathcal{C}^{1,\alpha}$ regularity of minimizers away from a singular set of dimension $m-p$. In the independent work [HL_2], p-energy minimizers, for any $1 < p < \infty$, were shown to be $C^{1,\alpha}$ regular away from an interior singular set of dimension at most $m-[p]-1$. For more general functionals with p-power growth, arguments in [HKL_2] lead to the (not necessarily optimal) dimension estimate $m-p-\varepsilon$ for some positive ε.

Next we will sketch some of the arguments used in [HKL_1], [HL_2], and [HKL_2]. The partial regularity proof goes in two steps:

Step I. Partial Hölder continuity
Step II. Locally Hölder continuity implies higher regularity .

For Step II, one may localize in M to reduce to the case that N is a graph, $\{(y, f(y)) : y \in \Omega\}$ for some open $\Omega \subset \mathbb{R}^n$ and smooth $f : \Omega \to \mathbb{R}^{k-n}$. Then we may write a minimizer u as $u(x) = (\tilde{u}(x), f(\tilde{u}(x)))$, hence, $\nabla u = (\nabla\tilde{u}, \nabla f \circ \nabla\tilde{u})$, and examine the functional minimized by $\tilde{u}$. For $p = 2$, $\tilde{u}$ satisfies an elliptic system of diagonal type to which standard regularity theory [G, VII, 3] applies. For $p \neq 2$, more argument is required and the highest regularity that can be expected is $C^{1,\alpha}$ for some $0 < \alpha < 1$ (although there may be a partial higher regularity result). For $p > 2$, the corresponding problem with N replaced by $\mathbb{R}^n$ was first treated in the work of K. Uhlenbeck [U]. To obtain Step II in [HL_2] for all p with $1 < p < 2$, we combine arguments of E. DiBenedetto [D] and P. Tolksdorff [T].

For Step I an important notion is that of the *normalized energy*

$$\mathbb{E}_{r,a}(u) = r^{p-m}\int_{\mathbb{B}_r(a)} |\nabla u|^p\,dx \quad \text{for } \mathbb{B}_r(a) \subset \mathbb{R}^m \text{ and } u \in L^{1,p}(\mathbb{B}_r(a), N)\,.$$

Note the appropriateness of the factor r^{p-m}; for a homogeneous degree 0 function u, $\mathbb{E}_{r,0}(u)$ is independent of r. The use of normalized energy in [HL_2], as previoiusly in [SU_1] and [HKL_1], is motivated by

MORREY'S LEMMA.[M, 3.5.2] *If* $c > 0$, $0 < \alpha < 1$, *and* $\mathbb{E}_{r,a}(u) \leq c\,r^{p\alpha}$ *for all balls* $\mathbb{B}_r(a) \subset \mathbb{B} = \mathbb{B}_1(0)$, *then* $u\,|\,\mathbb{B}_{1/2} \in C^{0,\alpha}$.

While we wish to show such uniform power decay, we can at least get the normalized energy arbitrarily small by taking a sufficiently small ball centered about most points in the domain by the following:

DENSITY LEMMA. *For any function* $u \in L^{1,p}(M,N)$, *the energy density* $\Theta(a) = \limsup_{r\downarrow 0} \mathbb{E}_{r,a}(u)$ *equals* 0 *at all points* $a \in M \sim S_u$ *for some set* S_u *having* $m-p$ *dimensional Hausdorff measure zero.*

This follows, as in [SU_1, 2.7], from an elementary covering argument. In case u is a p-energy minimizer, Hölder continuity off of S_u now follows by iteration and scaling from the:

REGULARITY LEMMA. *There exists a positive number* $\theta < 1$ *so that if* $u \in L^{1,p}(\mathbb{B},N)$ *is a p-energy minimizer with* $\mathbb{E}_{1,0}(u) < \theta$, *then* $\mathbb{E}_{\theta,0}(u) < \frac{1}{2}\mathbb{E}_{1,0}(u)$.

This may be proven by arguing by contradiction or "blowing-up". Here, as in similar situations in geometric measure theory and elliptic systems, a key problem is controlling the "blow-up sequence" whose convergence is initially only known to be weak in a Sobolev norm. The extra ingredient is a "Caccioppoli-type" inequality which in this context follows from the:

COMPARISON LEMMA. *There exists positive constants* c, ε *so that for any* $0 < \lambda < \infty$ *and any* $\varphi : \partial\mathbb{B} \to N$ *with* $\int_{\partial\mathbb{B}} |\nabla_{\tan}\varphi| dS < \varepsilon$, *there exists a function* $w : \mathbb{B} \to N$ *with* $w \,|\, \partial\mathbb{B} = \varphi$ *and*

$$\int_{\mathbb{B}} |\nabla w|^p dx \le \lambda \int_{\partial\mathbb{B}} |\nabla_{\tan}\varphi|^p dS + c\lambda^{-p} \int_{\partial\mathbb{B}} |\varphi|^p dS .$$

This was first proven for $p=2$ in [SU_1, 4.3] and generalized for $p \neq 2$ in [HL_2] and [L_2] .

All of the discussion so far carries over to more general functionals having p-power growth. For a minimizer u of the ordinary p-energy, one has the additional

MONOTONICITY PROPERTY. $\mathbb{E}_{r,a}(u) \le \mathbb{E}_{s,a}(u)$ *whenever* $0 < r < s < \operatorname{dist}(a, \partial M)$.

Using this along with a discussion of homogeneous minimizers and an induction argument of H. Federer [F], one may, as in [SU_1], show that the singular set S_u has not only $m-p$ dimensional measure 0 , but even Hausdorff dimension $m-[p]-1$. Here $S_u = \emptyset$ in case $m < [p]+1$ and is finite in case $m = [p]+1$. Using the examples of singularities discussed in §2-3, one can easily show that the dimension estimate is optimal.

5. DENSITY BOUNDS.

In [HL_2] it was observed that

if N *is simply* [p]-1 *connected* (i.e. $\Pi_0(N)=\Pi_1(N)=\cdots=\Pi_{[p]-1}(N)=0$), *then one may delete the smallness hypothesis* $\int_{\partial\mathbb{B}}|\nabla_{\tan}\varphi|\,dS<\varepsilon$ *from the Comparison Lemma.*

The proof of this is different from the inductive construction of [SU_1, 4.3] and involves choosing projections from a generically-situated complex in the manner of [FF] or [W]. The idea is easily described in case $N=\mathbb{S}^n$. One first verifies that the $\mathbb{R}^{n+1}$-valued harmonic extension h of φ satisfies the inequality in the conclusion of the Comparison Lemma. Of course, h probably does not have image in $\mathbb{S}^n$. To correct this we take $w=\Pi_a\circ h$ where $\Pi_a:\mathbb{R}^{n+1}\sim\{a\}\to\mathbb{S}^n$ is an appropriate retraction and $a\in\mathbb{B}_{1/2}$. To see that a suitable a can be found, one uses Fubini's theorem to obtain an estimate

$$\int_{\mathbb{B}_{1/2}}\int_{\mathbb{B}}|\nabla(\Pi_a\circ h)(x)|^p\,dx\,da\ \le\ c\int_{\mathbb{B}}|\nabla h(x)|^p\,dx\ .$$

Next we examine some easy consequences of the new Comparison Lemma. We assume $u\in L^{1,p}(M,N)$ is a p-energy minimizer and $\mathbb{B}_r(a)\subset M$.

(1) $\mathbb{E}_{r/2,a}(u)\le\lambda\mathbb{E}_{r,a}(u)+C\lambda^{-p}$ *where* C *depends only on* $m, N,$ *and* p.

(2) $\mathbb{E}_{r/2,a}(u)\ \le\ (1+C)[\mathbb{E}_{r,a}(u)]^{p/1+p}$.

(3) $\Theta(a)\ \le\ (1+C)^{1+p}\,2^{m-p}$.

(4) $\nabla u\in L^q_{loc}$ *for some* $q>p$ *and* $\dim(S_u)<m-p$.

To verify (1) we use Fubini's Theorem to choose s with $\frac{1}{2}r<s<r$ so that

$$\int_{\partial\mathbb{B}_s(a)} |\nabla u|^p \, dS \le 2r^{-1} \int_{\mathbb{B}_r(a)} |\nabla u|^p \, dx \ ,$$

apply the new Comparison Lemma with $\varphi(x) = u(sx)$, and note that $u \mid \mathbb{B}_s$ is minimizing and that N is bounded. Inequality (2) follows from (1) by taking $\lambda = [\mathbb{E}_{r,a}(u)]^{-1/1+p}$. To obtain (3) we iterate (2) to see that

$$\mathbb{E}_{r/2^j,a}(u) \le (1+C)^{1+\cdots+\beta^j} [\mathbb{E}_{r,a}(u)]^{\beta^j} \quad \text{where} \quad \beta = p/1+p \ ,$$

and then let $j \to \infty$. For (4) one may combine (2) with an appropriate "boundary-Sobolev" inequality to get a reverse Hölder-type inequality and then argue in a manner similar to [G, §5] (See [HKL_2] for details).

6. NATURE OF THE SINGULAR SET.

The best results on the behavior near singularities are known for ordinary energy $\int_M |\nabla u|^2 dx$ when $m = \dim M = 3$. Here by [SU_1] and the study of asymptotics by L. Simon in [S_1], there exists, for any singularity a of an energy minimizer u , a smooth harmonic map $v : \mathbb{S}^2 \to N$ such that

$$\sup_{x \in \mathbb{B}_r(a)} \left| u(x) - v\big((x-a)/|x-a|\big) \right| \to 0 \quad \text{as} \quad r \to 0 \ .$$

If, moreover $N = \mathbb{S}^2$, then R. Gulliver and B. White [GW] have verified that an easier asymptotics result [S_2] is applicable so that this convergence is as fast as a positive power of r . They also find a real analytic N and a minimizer $u : \mathbb{B}^3 \to N$ for which positive power decay to the tangent map fails.

Harmonic maps from $\mathbb{S}^2$ to $\mathbb{S}^2$ are classified by rational functions of z or $\bar{z}$ [J_2, 1.5]. By the general bound (3) above, a homogeneous degree 0 extension of such a map will not be minimizing if the degree of this map is too large. Recently H. Brezis, J.M. Coron, and E. Lieb [BCL] have shown that any such homogeneous minimizer is, in fact, of degree 1 and is precisely in the form $g(x/|x|)$ for some rotation $g \in \mathbb{O}(3)$.

In general, there are many open problems on the nature of the singular set of an energy minimizing map. One would like to extend some of the above results to handle $m > 3$ or $p \neq 2$ or more general energy functionals.

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