SOME BASIC SEQUENCES AND THEIR MOMENT OPERATORS

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1. INTRODUCTION

A well known result in Fourier analysis (see [3, p.107], for example) says that if the Fourier series of a continuous function on the circle group is lacunary, then the series converges uniformly to the function. Equivalently, if $(\alpha(n))$ is a lacunary sequence of positive integers (that is, $\alpha(n+1)\alpha(n)^{-1} \geq \gamma > 1$, for all n and some γ), then the sequence $1, e^{i\alpha(1)t}, e^{-i\alpha(1)t}, e^{i\alpha(2)t}, e^{-i\alpha(2)t}, \ldots$ is basic in $C(0, 2\pi)$.

On the other hand, Gurarii and Macaev ([5]) proved some analogues of this result for power sequences in $C([0,1])$ and $L^p(0,1)$. Letting $1 \leq p < \infty$ and letting $(\alpha(n))$ be a given increasing sequence of positive numbers, they proved that $(\alpha(n))$ is lacunary if and only if $(\alpha(n)^{1/p} t^{\alpha(n)-1/p})$ is basic in $L^p(0,1)$, in which case this basic sequence is equivalent to the standard basis in ℓ^p. They also proved that $(\alpha(n))$ is lacunary if and only if $(t^{\alpha(n)})$ is basic in $C([0,1])$.

In [4], Edwards has considered, in a dual form, a related problem concerning sequences of measures on a compact Hausdorff space K. If (μ_n) is a weak* convergent sequence of measures on K which satsifies a one term recurrence relation, he gives conditions which ensure that $\{(\int_K f d\mu_n) : f \in C(K)\} = c$. This result is closely related to the problem of finding conditions for (μ_n) to be a basic sequence of measures on K.

The present paper presents some analogues of the preceding results which are derived by considering a general problem in Banach spaces. Throughout, X will denote a given Banach space with dual X^*, (b_n) will denote a given sequence of scalars, $\sigma = (v_n)$ will denote a given sequence of vectors in X and $\tau = (x_n)$ will denote the sequence in X

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given by the recurrence relation

$$x_n - b_n x_{n-1} = v_n, \quad \text{for} \quad n \geq 1, \quad \text{where} \quad x_0 = 0. \tag{1.1}$$

The general problem considered is to find conditions which ensure that if σ is basic then τ is basic, and also to find when σ and τ are equivalent bases. If (z_n) is a sequence in X, the moment operator A of (z_n) is defined on X^* by $(Ax^*)(n) = x^*(z_n)$, for $n \in \mathbb{N}$ and $x^* \in X^*$. Whether (z_n) is basic can often be expressed in terms of the range of A ([2,7]). These results are discussed in section 2.

In section 3, basic sequences in a space $L^p(S, \mathcal{S}, \mu)$ are constructed which are of the form $(f|K_n)$, where (K_n) is an increasing sequence of sets in $\mathcal{S}$, f is a given $\mathcal{S}$-measurable function, and $f|K_n$ is the function equal to f on K_n and 0 elsewhere. In section 4, some bases are constructed for some subspaces of $L^p(\mathbb{R})$ consisting of piecewise linear functions. By taking Fourier transforms in some of these results with $p = 2$, conditions are found for weighted sequences of Dirichlet and Fejér kernels in $L^2(\mathbb{R})$ to be basic. The dual versions of these results give statements about the ranges of the various moment operators. For example, the following conditions are equivalent, where $1 \leq p < \infty$, $p^{-1} + q^{-1} = 1$, and $(\alpha(n))$ is an increasing sequence of positive numbers:

$$(\alpha(n)) \quad \text{is lacunary,}$$

$$\left\{ \left(\alpha(n)^{-(1+1/p)} \int_{-\alpha(n)}^{\alpha(n)} (\alpha(n) - |t|) f(t) dt \right) : f \in L^q(\mathbb{R}) \right\} = \ell^q, \quad \text{and}$$

$$\left\{ \left(\alpha(n)^{-3/2} \int_{-\infty}^{\infty} \left(\frac{\sin \alpha(n) t}{t} \right)^2 f(t) dt \right) : f \in L^2(\mathbb{R}) \right\} = \ell^2.$$

Some definitions and notation used throughout the paper now follow. All sequences (z_n) in X or elsewhere are understood to be of the form $(z_n)_{n=1}^{\infty}$, unless indicated otherwise. If (z_n) is a sequence in X, $[z_n : n \in \mathbb{N}]$ denotes the Banach subspace of X generated by $\{z_n : n \in \mathbb{N}\}$. If $\lambda = (z_n)$ is a sequence in X we define

$$A_\lambda = \left\{ d : d \text{ is a scalar sequence and } \sum_{n=1}^{\infty} d_n z_n \quad \text{converges in} \quad X \right\}.$$

Let $S_\lambda : A_\lambda \to X$ be given by $S_\lambda(d) = \sum_{n=1}^{\infty} d_n z_n$. If S_λ is a bijection from A_λ onto $[z_n : n \in \mathbb{N}]$, λ is said to be basic in X and to be a basis for $[z_n : n \in \mathbb{N}]$. If σ and τ are two basic sequences in X, they are said to be equivalent if $A_\sigma = A_\tau$.

A sequence $\lambda = (z_n)$ in X is basic in X and $A_\lambda = \ell^p$ (for some $1 \leq p < \infty$) if and only if there are $A, B > 0$ such that

$$A\|d\|_p \leq \left\| \sum_{n=1}^{\infty} d_n z_n \right\| \leq B\|d\|_p, \quad \text{for all} \quad d \in A_\lambda. \tag{1.2}$$

Also, λ is basic and $A_\lambda = c_0$ if and only if an equality of type (1.2) holds with $p = \infty$ (see [11, p.354-355] or [12, p.30] for these facts). In the case where λ is basic in a Hilbert space, λ is said to be Riesz basic if $A_\lambda = \ell^2$. Standard results on bases may be found in [11] and [12] and used without explicit reference. For convenience rather than necessity, spaces such as $L^p(\mathbb{R})$, ℓ^p will be taken to consist of real valued functions and sequences. The bounded continuous real valued functions on $\mathbb{R}$ are denoted by $C(\mathbb{R})$, and $C_0(\mathbb{R})$ denotes those functions in $C(\mathbb{R})$ vanishing at infinity. The characteristic function of a set A is denoted by $\chi(A)$.

2. GENERAL RESULTS

If the given sequence $\sigma = (v_n)$ in X is basic, there is a sequence (f_n) in X^* which is biorthogonal to σ. That is, $f_n(v_m) = 0$ if $m \neq n$ and $f_n(v_n) = 1$, for all m, n. If (b_n) is a given sequence of scalars we let $x_n - b_n x_{n-1} = v_n$, as in (1.1), and let $h_n = f_n - b_{n+1} f_{n+1}$, for all n.

LEMMA 2.1. *If $\sigma = (v_n)$ is basic in X, then (h_n) is a sequence in X^* which is biorthogonal to (x_n). Also,*

$$\sum_{i=1}^{n} h_i(x) x_i = \sum_{i=1}^{n+1} f_i(x) v_i - f_{n+1}(x) x_{n+1}, \quad \text{for} \quad x \in X, \quad n \in \mathbb{N}.$$

Proof. It is straightforward to prove this from (1.1) and the definition of h_n (see also [4,p.11] and [11,p.29]).

THEOREM 2.2. *Let $\sigma = (v_n)$ be a basis for X, let (b_n) be a sequence of scalars with $b_1 = 0$, let $\tau = (x_n)$ be given by (1.1), and let $1 \leq p < \infty$. Then the following hold.*

(2.1) *If σ is bounded away from 0 and τ is bounded, then τ is a basis for X. If σ is bounded and τ is a basis for X, then τ is bounded.*

(2.2) *If τ is a basis for X which is bounded away from 0, then σ is bounded away from 0.*

(2.3) *If A_σ is ℓ^p or c_0, τ is bounded if and only if τ is a basis for X.*

(2.4) *If $||b||_\infty < 1$ and σ is bounded, then $A_\sigma = \ell^p$ (respectively c_0) if and only if τ is a basis for X and $A_\tau = \ell^p$ (respectively c_0).*

(2.5) *Let $||b||_\infty < 1$, let A_σ be either ℓ^p or c_0 and let $A, B > 0$ be chosen so that (1.2) holds for σ. Then for all $d \in A_\tau$,*

$$A(1 + ||b||_\infty)^{-1}||d||_p \leq \left\|\sum_1^\infty d_n x_n\right\| \leq B\,(1 - ||b||_\infty)^{-1}\,||d||_p,$$

where, if $A_\sigma = c_0$, $||d||_\infty$ is taken in place of $||d||_p$.

Proof. As σ is a basis for X, $x = \sum_{n=1}^{\infty} f_n(x) v_n$, for all $x \in X$. Assume that σ is bounded away from 0. Then $\lim_{n\to\infty} f_n(x) = 0$, for $x \in X$. Hence, if τ is bounded, we deduce from Lemma 2.1 that $x = \sum_{n=1}^{\infty} h_n(x) x_n$, for all $x \in X$, and it follows that τ is a basis for X. This proves half of (2.1).

Now let σ be bounded and τ be a basis for X. Because (h_n) is biorthogonal to τ, there is $K > 0$ so that $||x_n||.||h_n|| \leq K$ for all n. Thus,

$$||x_n|| \leq K||h_n||^{-1} \leq K||v_n||.|h_n(v_n)|^{-1} \leq K||v_n||,$$

so that τ is bounded. This proves the rest of (2.1).

If τ is a basis for X bounded away from 0, choose K as above and observe that, using Lemma 2.1,

$$||v_n|| = ||x_n - b_n x_{n-1}|| \geq |h_n(x_n - b_n x_{n-1})|.||h_n||^{-1} \geq ||h_n||^{-1} \geq K^{-1}||x_n||.$$

Hence σ is bounded away from 0. This proves (2.2).

If A_σ is ℓ^p or c_0, an inequality of type (1.2) holds, so σ is bounded and also bounded away from 0. Hence (2.3) is a consequence of (2.1).

If $||b||_\infty < 1$ and σ is bounded, use (1.1) to obtain

$$\begin{aligned} ||x_n|| &\le ||v_n|| + \sum_{j=1}^{n-1} ||b||_\infty^{n-j} ||v_j||, \\ &\le (1 - ||b||_\infty)^{-1} \sup\{||v_n|| : n \in \mathbb{N}\}. \end{aligned}$$

Hence τ is bounded. Now let A_σ be ℓ^p (respectively, c_0). It follows from (2.3) that τ is a basis for X so that

$$x = \sum_{n=1}^{\infty} (f_n(x) - b_{n+1} f_{n+1}(x))\, x_n = \sum_{n=1}^{\infty} f_n(x) v_n, \qquad \text{for} \quad x \in X. \tag{2.6}$$

Hence, $(S_\tau^{-1} \circ S_\sigma)(d) = (I - SM)(d)$, for $d \in A_\sigma$, where S, M are the operators given by $Sd = (d_{n+1})$, $Md = (b_n d_n)$ and I is the identity. SM maps ℓ^p into ℓ^p (respectively c_0 into c_0) and $||SM|| \le ||b||_\infty < 1$. Hence $I - SM$ is a bounded invertible operator on ℓ^p (respectively c_0) and $A_\tau = (I - SM)A_\sigma = \ell^p$. This proves half of (2.4). For the other half, let τ be a basis with $A_\tau = \ell^p$ (respectively c_0). Then τ is bounded away from 0. By (2.3), σ is bounded away from 0, so $A_\sigma \subseteq c_0$. It is easy to see that $I - SM$ is injective on c_0. Thus, as $\ell^p = A_\tau = (I - SM)A_\sigma$, we deduce that $A_\sigma = \ell^p$ (respectively, c_0). This proves (2.4).

To prove (2.5), observe that $||I - SM|| \le 1 + ||b||_\infty$ and $||(I - SM)^{-1}|| \le (1 - ||b||_\infty)^{-1}$. Then (1.2) and (2.6) give

$$A||d||_p \le \left\| \sum_{n=1}^{\infty} ((I - SM)d)_n\, x_n \right\| \le B||d||_p, \qquad \text{for} \quad d \in A_\sigma.$$

Replacing d by $(I - SM)^{-1}(d)$ now gives (2.5).

COROLLARY 2.3. *Let $\sigma = (v_n)$ be a bounded basis for X which is also bounded away from* 0. *Let (d_n) be a sequence of non-zero scalars, let $y_n = \sum_{j=1}^{n} d_j v_j$ and let $\lambda = (d_n^{-1} y_n)$. Then the following conditions are equivalent:* (i) *λ is basic in X,* (ii) *λ is bounded, and* (iii) *$(d_{n+1}^{-1} y_n)$ is bounded. If there is $\theta < 1$ so that $|d_{j-1} d_j^{-1}| \le \theta$ for all $j \ge 2$, then conditions* (i) *to* (iii) *do hold, and $A_\sigma = \ell^p$ (respectively c_0) if and only if $A_\lambda = \ell^p$ (respectively c_0).*

Proof. If $x_n = d_n^{-1}y_n$, $b_n = d_{n-1}d_n^{-1}$, $b_1 = 0$ then $x_n - b_n x_{n-1} = v_n$, all n. The equivalence of (i), (ii) now follows from (2.1). As $d_n^{-1}y_n - d_n^{-1}y_{n-1} = v_n$ and σ is bounded, (ii) and (iii) are equivalent. If $||b||_\infty < 1$, (x_n) is bounded and the remaining statements follow from (2.3) and (2.4).

REMARK. The equivalence of (i), (ii) and (iii) is known ([11, p.29]) and may be regarded as the special case of (2.1) which arises when it is assumed that in the recurrence relation (1.1), $b_n \neq 0$ for all n.

THEOREM 2.4. *Let X be reflexive, let $\sigma = (v_n)$ be a basis for X with $||v_1|| = 1$ and $||v_n|| \leq 1$ for $n \geq 2$. Let $\sigma' = (||v_n||^{-1}v_n)$ and assume that $A'_\sigma = \ell^p$, for some $1 < p < \infty$. For $n \geq 1$ let $b_n = (1 - ||v_n||^p)^{1/p}$ and let $\tau = (x_n)$ be the sequence in X given by (1.1). Let (f_n), (h_n) be the sequences in X^* which are biorthogonal to σ, τ respectively, as described in Lemma 2.1. Then the following conditions are equivalent.*

(2.7)
$$[h_n : n \in \mathbb{N}] = X^*,$$

(2.8)
$$\prod_{j=r}^{\infty} b_j = 0, \quad \textit{for all} \quad r \in \mathbb{N}, \quad \textit{and}$$

(2.9)
$$\sum_{j=1}^{\infty} ||v_j||^p = \infty.$$

Proof. By reflexivity, (2.7) holds if and only if $x \in X$ and $h_n(x) = 0$ for all n implies $x = 0$. Let $x = \sum_{n=1}^{\infty} d_n ||v_n||^{-1} v_n$, where $d \in \ell^p$, be such that $h_n(x) = 0$ for all n. Then $||v_n||^{-1}d_n = b_{n+1}||v_{n+1}||^{-1}d_{n+1}$ for all n.

If $b_n = 0$ for an infinite number of n, we deduce that $d = 0$. In this case (2.7) to (2.9) hold.

On the other hand suppose that there is q so that $b_q = 0$ and $b_n \neq 0$ for $n > q$. Then $d_j = 0$ for $1 \leq j \leq q-1$ and $d_n = ||v_n||.||v_q||^{-1}(b_n b_{n-1} \ldots b_{q+1})^{-1} d_q$ for $n > q$. Hence

$$\sum_{n=q+1}^{\infty} |d_n|^p = |d_q|^p ||v_q||^{-p} \lim_{n \to \infty} (b_n b_{n-1} \cdots b_{q+1})^{-p}.$$

As $d \in \ell^p$, either $d = 0$ or $\prod_{n=q+1}^{\infty} b_n \neq 0$. Hence (2.8) implies (2.7). The converse argument may be used to show that if (2.8) fails, there is $x \in X$, $x \neq 0$ so that $h_n(x) = 0$ for all n. Hence (2.7) implies (2.8).

Thus, (2.7) and (2.8) are equivalent, and the latter is equivalent to (2.9) by a standard result on infinite products ([9, p.292]).

COROLLARY 2.5. *Let H be a Hilbert space, let (x_n) be a normalized sequence in H, and let (b_n) be a scalar sequence such that $b_1 = 0$ and the projection of x_n into $[x_j : 1 \leq j \leq n-1]$ is equal to $b_n x_{n-1}$ for all $n \geq 2$. Let $v_1 = x_1$ and $v_n = x_n - b_n x_{n-1}$ for $n \geq 2$. Then (x_n) is basic in H if and only if (v_n) is bounded away from 0, in which case (x_n) is Riesz basic. The subspaces $[x_n : n \in \mathbb{N}]$ and $[\|v_n\|^{-2} v_n - \overline{b_{n+1}} \|v_{n+1}\|^{-2} v_{n+1} : n \in \mathbb{N}]$ of H are equal if and only if $\sum_{n=1}^{\infty} \|v_n\|^2 = \infty$.*

Proof. Let $X = [x_n : n \in \mathbb{N}]$. Then (v_n) is an orthogonal basis for H, and

$1 = \|x_n\|^2 = |b_n|^2 + \|v_n\|^2$. Hence (v_n) is bounded away from 0 if and only if $\|b\|_\infty < 1$. The first statement now follows from Theorem 2.2. The rest follows from Theorem 2.4 with $p = 2$.

The following result concerns the relationship between a sequence in X and its associated moment operator. The result is essentially known (see [2], [7, Theorem 1] and [12, p.169], for similar results) and is included for completeness.

THEOREM 2.6. *Let $\sigma = (z_n)$ be a sequence in X, let $M = [z_n : n \in \mathbb{N}]$ and for $x^* \in X^*$ let $Sx^* = (x^*(z_n))$. Then the following hold.*

(2.10) *If $S(X^*)$ is equal to ℓ^r for some $1 \leq r \leq \infty$ (respectively c_0), then S is bounded from X^* onto ℓ^r (respectively, c_0).*

(2.11) *If $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, then $S(X^*) = \ell^q$ (respectively ℓ^1) if and only if σ is a basic sequence in X which is equivalent to the standard basis in ℓ^p (respectively c_0).*

(2.12) *If σ is a basic sequence in X which is equivalent to the standard basis in ℓ^1,* *then* $S(X^*) = \ell^\infty$.

(2.13) *If* M *is complemented in* X *and* π *is a projection from* X *onto* M, *the restriction of* S *to* $\pi^*(M^*)$ *is a bijection onto* $S(X^*)$.

Proof. (2.10) follows from the closed graph theorem.

Now let $1 < p < \infty$ and $S(X^*) = \ell^q$. Define T on M^* by $T\mu = (\mu(z_n))$. Then $T(M^*) = \ell^q$ and T is a bounded bijection from M^* to ℓ^q. Hence T^* is a bounded bijection from ℓ^p to M^{**}. If $d \in \ell^p$ and $\mu \in M^*$ we have $(T^*d)(\mu) = \sum_{n=1}^{\infty} d_n\mu(z_n)$, and it is easy to see that this series converges uniformly on the unit ball in M^*. It follows that $\sum_{n=1}^{\infty} d_n z_n$ converges in M and that $T^*d = \sum_{n=1}^{\infty} d_n z_n$, for $d \in \ell^p$. As T^* is a bounded bijection onto M^{**}, it follows that $M = M^{**}$ and that σ is a basic sequence with $A_\sigma = \ell^p$. When $p = \infty$ and $q = 1$, apply a similar argument to prove that T^* is a bounded bijection from c_0 onto M and that $T^*d = \sum_{n=1}^{\infty} d_n z_n$ for $d \in c_0$ – then σ is basic with $A_\sigma = c_0$. This proves part of (2.11).

Conversely, if σ is basic and $A_\sigma = \ell^p$ for some $1 \leq p < \infty$, let $S_\sigma d = \sum_{n=1}^{\infty} d_n z_n$, for $d \in \ell^p$. Then $S_\sigma^*(X^*) = \ell^q$. As $S = S_\sigma^*$, this proves (2.12) and the rest of (2.11). The proof of (2.13) is straightforward.

3. BASES AND RESTRICTIONS

In this section, $(S, \mathcal{S}, \mu)$ will denote a given measure space, $K = (K_n)$ will denote an increasing sequence of sets in $\mathcal{S}$ such that $\mu(K_{n+1} - K_n) > 0$ for all n, and f will denote a given $\mathcal{S}$-measurable scalar valued function on $\mathcal{S}$. It will be assumed that $1 \leq p \leq \infty$ is given and that, for all n, $f\chi(K_n - K_{n-1})$ is a non-zero element of $L^p(S, \mathcal{S}, \mu)$, where $K_0 = \emptyset$ when $n = 1$. We let $R(f, p, K)$ denote all functions g in $L^p(S, \mathcal{S}, \mu)$ such that $g = 0$ on $S - \bigcup_{n=1}^{\infty} K_n$ and on each set $K_n - K_{n-1}$, the restriction of g is a multiple of the restriction of f. Then $R(f, p, K)$ is a Banach subspace of $L^p(S, \mathcal{S}, \mu)$ and it is clear that $(f\chi(K_n - K_{n-1}))$ is a basis for $R(f, p, K)$. This section is concerned with when $(f\chi(K_n))$ is also a basis for

$R(f,p,K)$. Let, for $n \in \mathbb{N}$,

(3.1) $$f_n = \|f\chi(K_n)\|_p^{-1} f\chi(K_n), \qquad f_0 = 0,$$
$$b_n = \|f\chi(K_n)\|_p^{-1} \|f\chi(K_{n-1})\|_p, \qquad \text{and}$$
$$v_n = \|f\chi(K_n)\|_p^{-1} f\chi(K_n - K_{n-1}).$$

It is immediate from (3.1) that

(3.2) $$f_n - b_n f_{n-1} = v_n, \qquad \text{for} \quad n \in \mathbb{N}.$$

THEOREM 3.1. *Let $\tau = (f_n)$ and consider the following conditions.*

There is $\delta > 0$ such that for all $n \in \mathbb{N}$,

(3.3) $\|f\chi(K_n - K_{n-1})\|_p \geq \delta \|f\chi(K_n)\|_p$,

(3.4) *τ is a basis for $R(f,p,K)$, and*

(3.5) *there is $\gamma > 1$ such that for all $n \in \mathbb{N}$, $\|f\chi(K_n)\|_p \geq \gamma \|f\chi(K_{n-1})\|_p$.*

Then if $1 \leq p \leq \infty$, (3.3) *and* (3.4) *are equivalent. If $1 \leq p < \infty$,* (3.3), (3.4) *and* (3.5) *are equivalent and imply that $A_\tau = \ell^p$. If $p = \infty$ and* (3.5) *holds,* (3.3) *and* (3.4) *also hold and $A_\tau = c_0$.*

Proof. (3.1) shows that $b_1 = 0$ and it follows from (3.2) that Theorem 2.2 applies. Also τ is bounded, by (3.1). Now if (3.3) holds, (v_n) is bounded away from 0 and (3.4) follows from (2.1). Conversely, if (3.4) holds, (3.3) is a consequence of (2.2).

When $1 \leq p < \infty$, it is easy to prove that (3.3) and (3.5) are equivalent. As (3.5) means that $\|b\|_\infty < 1$, it follows from (2.4) that $A_\tau = \ell^p$.

When $p = \infty$, (3.5) implies that $\|f\chi(K_n)\|_\infty = \|f\chi(K_n - K_{n-1})\|_\infty$ so that (3.3) holds. (3.5) also implies that $\|v_n\|_\infty = 1$ and that $\|b\|_\infty < 1$, so that $A_\sigma = c_0$ (where $\sigma = (v_n)$) and $A_\tau = c_0$ by (2.4). This completes the proof.

If $(\alpha(n))$ is a strictly increasing sequence of positive numbers let

(3.6) $$\gamma(\alpha) = \inf\{\alpha(n+1)\alpha(n)^{-1} : n \in \mathbb{N}\} \quad \text{and} \quad \psi(\alpha) = \sup\{\alpha(n+1)\alpha(n)^{-1} : n \in \mathbb{N}\}.$$

We allow the possibility that $\psi(\alpha) = \infty$, in which case $\psi(\alpha)^{-1} = 0$. Clearly, $\gamma(\alpha) \geq 1$.

COROLLARY 3.2. *Let $(\alpha(n))$ be a strictly increasing sequence of positive real numbers and let $1 \leq p < \infty$. Then $\gamma(\alpha) > 1$ if and only if there are $C, D > 0$ such that*

$$C\left(\sum_{n=1}^{r} |d_n|^p\right)^{1/p} \leq \left(\sum_{j=1}^{r} (\alpha(j) - \alpha(j-1)) \left|\sum_{n=j}^{r} \frac{d_n}{\alpha(n)^{1/p}}\right|^p\right)^{1/p} \leq D\left(\sum_{n=1}^{r} |d_n|^p\right)^{1/p},$$

for all scalars $d_1, d_2, \ldots, d_r$ and $r \in \mathbb{N}$. In this case we may take

$$C = \frac{(\gamma(\alpha) - 1)^{1/p}}{\gamma(\alpha)^{1/p} + 1} \qquad \textit{and} \qquad D = \frac{\gamma(\alpha)^{1/p}}{\gamma(\alpha)^{1/p} - 1} \left(1 - \psi(\alpha)^{-1}\right)^{1/p}.$$

Proof. Apply Theorem 3.1 to $L^p(\mathbb{R})$ with $f = 1$ and $K_n = (0, \alpha(n))$. Then $f_n = \alpha(n)^{-1/p}\chi(0, \alpha(n))$ and (3.5) holds if and only if $\gamma(\alpha) > 1$. Now observe that

$$\left\|\sum_{n=1}^{r} d_n f_n\right\|_p = \left(\sum_{j=1}^{r} (\alpha(j) - \alpha(j-1)) \left|\sum_{n=j}^{r} \frac{d_n}{\alpha(n)^{1/p}}\right|^p\right)^{1/p}.$$

Thus, an inequality of the above type is equivalent to saying that $\tau = (f_n)$ is basic in $L^p(\mathbb{R})$ with $A_\tau = \ell^p$ (see (1.2)). The estimates for C, D are consequences of applying (2.5) with $\sigma = (\alpha(n)^{-1/p}\chi((\alpha(n-1), \alpha(n))))$, τ as above and $b_n = \alpha(n-1)^{1/p}\alpha(n)^{-1/p}$. This completes the proof.

PROPOSITION 3.3. *Let $(H, <,>)$ be a Hilbert space, let (e_n) be a Riesz basis for H, let (c_n) be a sequence of scalars and let $(\alpha(n))$ be a strictly increasing sequence of positive integers. Then the following conditions are equivalent.*

(3.7) *There is $\eta > 0$ such that for all $n \in \mathbb{N}$,*

$$\left(\sum_{j=1}^{\alpha(n)} |c_j|^2\right)^{-1} \left(\sum_{j=\alpha(n-1)+1}^{\alpha(n)} |c_j|^2\right) \geq \eta.$$

(3.8) *The sequence $\left(\sum_{j=1}^{\alpha(n)} c_j e_j\right)$ is basic in H.*

(3.9) *If we let*

$$a_{j,k} = \frac{\sum_{r=1}^{\alpha(j)} \sum_{s=1}^{\alpha(k)} c_r \bar{c}_s < e_r, e_s >}{\left(\sum_{r=1}^{\alpha(j)} |c_r|^2\right)^{1/2} \left(\sum_{s=1}^{\alpha(k)} |c_s|^2\right)^{1/2}},$$

then there are $A, B > 0$ such that for all scalar sequences (d_n) of finite support,

$$A\|d\|_2^2 \leq \left|\sum_{j,k=1}^{\infty} a_{j,k} d_j \overline{d}_k\right| \leq B\|d\|_2^2.$$

When the above conditions hold,

$$\left(\left(\sum_{j=1}^{\alpha(n)} |c_j|^2\right)^{-1/2} \sum_{j=1}^{\alpha(n)} c_j e_j\right)$$

is Riesz basic in H.

Proof. Apply Theorem 3.1 to $\ell^2(\mathbb{N})$, with $f = (c_n)$ and $K_n = \{1, 2, \ldots, \alpha(n)\}$. Then (3.7) is equivalent to (3.3) with $p = 2$. Let $Jd = \left(\sum_{j=1}^{\infty} |d_j|^2\right)^{-1/2} \left(\sum_{j=1}^{\infty} d_j e_j\right)$, for $d \in \ell^2$. Then J is an isomorphism from $\ell^2(\mathbb{N})$ onto H such that $J(f\chi(K_n)) = \left(\sum_{j=1}^{\alpha(n)} |c_j|^2\right)^{-1/2} \left(\sum_{j=1}^{\alpha(n)} c_j e_j\right)$. Hence the equivalence of (3.7) and (3.8) is a consequence of the equivalence of (3.3) and (3.4). Condition (3.9) is equivalent to saying that $(J(f\chi(K_n)))$ is Riesz basic in H. This observation and Theorem 3.1 give the remaining conclusions.

REMARKS. 1. An alternative proof of Proposition 3.3 may be based upon Corollary 2.5.

2. If (e_n) is an orthonormal basis for H and $c_n = 1$ for all n, then

$$a_{j,k} = \text{minimum}\left(\alpha(j)^{1/2}\alpha(k)^{-1/2}, \quad \alpha(k)^{1/2}\alpha(j)^{-1/2}\right).$$

In this case the inequality (3.9) is the same as the one in Corollary 3.2 with $p = 2$.

COROLLARY 3.4. *Let $(\alpha(n))$ be an increasing sequence of positive integers. For $n \in \mathbb{N}$, let $D_n(t) = \sin(n + \frac{1}{2})t / \sin \frac{1}{2}t$, for $t \in (0, 2\pi)$. Then $\gamma(\alpha) > 1$ if and only if $(D_{\alpha(n)})$ is basic in $L^2(0, 2\pi)$, in which case $(\alpha(n)^{-1/2} D_{\alpha(n)})$ is Riesz basic. If $f \in L^2(0, 2\pi)$, then $(D_{\alpha(n)} * f)$ is not basic in $L^2(0, 2\pi)$. If $\gamma(\alpha) > 1$, a function $f \in L^2(0, 2\pi)$ has a unique expression in $L^2(0, 2\pi)$ of the form $\sum_{n=1}^{\infty} d_n \alpha(n)^{-1/2} D_{\alpha(n)}$, $d \in \ell^2$, if and only if the Fourier transform of f is constant on the set $\{-\alpha(1), \ldots, \alpha(1)\}$ and also upon each set of the form $\{-\alpha(n), \ldots, -\alpha(n-1) - 1\} \cup \{\alpha(n-1) + 1, \ldots, \alpha(n)\}$, for $n \geq 2$.*

Proof. Apply Proposition 3.3 with $H = L^2(0, 2\pi)$, $c_n = 1$ for all n, $e_1 = 1$ and $e_n(t) = e^{i(n-1)t} + e^{-i(n-1)t}$, for $n \geq 2$. Then (3.7), (3.8) imply that $\gamma(\alpha) > 1$ if and only if $(D_{\alpha(n)})$ is basic in $L^2(\mathbb{R})$. $D_{\alpha(n)} * f$ is the nth partial sum of the Fourier series of f, and $(D_{\alpha(n)} * f)$ is thus not basic by Corollary 2.3. Finally, observe that the Fourier transform of f is constant on $\{-\alpha(1), \ldots, \alpha(1)\}$ and upon each set

$$\{-\alpha(n), \ldots, -\alpha(n-1)-1\} \cup \{\alpha(n-1)+1, \ldots, \alpha(n)\}$$

if and only if $f \in [D_{\alpha(n)} : n \in \mathbb{N}]$. This completes the proof.

REMARKS. A consequence of Corollary 3.6 is that there exist basic sequences $(D_{\alpha(n)})$ in $L^2(0, 2\pi)$ such that for no $f \in L^2(0, 2\pi)$ is $(D_{\alpha(n)} * f)$ basic in $L^2(0, 2\pi)$.

COROLLARY 3.5. *Let $(\alpha(n))$ be an increasing sequence of positive real numbers. For $\beta \in \mathbb{R}$, let $D_\beta^{\mathbb{R}}(t) = \sin \beta t / t$, for $t \in \mathbb{R}$. Then $\gamma(\alpha) > 1$ if and only if $(D_{\alpha(n)}^{\mathbb{R}})$ is basic in $L^2(\mathbb{R})$, in which case $\left(\alpha(n)^{-1/2} D_{\alpha(n)}^{\mathbb{R}}\right)$ is Riesz basic. If $\gamma(\alpha) > 1$, a function $f \in L^2(\mathbb{R})$ has a unique expansion in $L^2(\mathbb{R})$ of the form $\sum_{n=1}^{\infty} d_n \alpha(n)^{-1/2} D_{\alpha(n)}^{\mathbb{R}}$, $d \in \ell^2$, if and only if the Fourier transform of f is constant on each subset of $\mathbb{R}$ of the form $(-\alpha(n), -\alpha(n-1)] \cup [\alpha(n-1), \alpha(n))$.*

Proof. This is similar to Corollary 3.4.

PROPOSITION 3.6. *Let $1 \leq p < \infty$, let $(\alpha(n))$ be a strictly increasing sequence of positive integers, let $a_{ij} = \alpha(i)^{-1/p}$ for $1 \leq j \leq \alpha(i)$, and let $a_{ij} = 0$ if $j > \alpha(i)$. Let A denote the operator obtained by multiplying by (a_{ij}). Then A is a bounded operator from ℓ^q onto ℓ^q (where $p^{-1} + q^{-1} = 1$) if and only if $\gamma(\alpha) > 1$. In this case, the restriction of A to the subspace of ℓ^q consisting of those sequences which are constant on each interval $\{\alpha(n-1)+1, \ldots, \alpha(n)\}$ in $\mathbb{N}$ is a bounded invertible operator on ℓ^q.*

Proof. Let a_n denote the nth row of A. Then by Theorem 3.1, $\sigma = (a_n)$ is basic in ℓ^p if and only if $\gamma(\alpha) > 1$, in which case $A_\sigma = \ell^p$. By (2.10), (2.11) and (2.12), A is bounded from ℓ^q onto ℓ^q. If $1 < p < \infty$ and $A(\ell^q) = \ell^q$, then (2.11) implies σ is basic and thus $\gamma(\alpha) > 1$. If $p = 1$ and $A(\ell^\infty) = \ell^\infty$, we have for $d \in \ell^\infty$,

$$(Ad)(n) - (Ad)(n+1) = \alpha(n+1)^{-1}\left(\alpha(n+1)\alpha(n)^{-1} - 1\right)\left(\sum_{i=1}^{\alpha(n)} d_i\right) - \alpha(n+1)^{-1}\left(\sum_{i=\alpha(n)+1}^{\alpha(n+1)} d_i\right),$$

so that

$$|(Ad)(n) - (Ad)(n+1)| \le \|d\|_\infty 2(1 - \alpha(n)\alpha(n+1)^{-1}).$$

Hence, if $A(\ell^\infty) = \ell^\infty$, $\gamma(\alpha) > 1$.

Now let M_p denote the subspace of ℓ^p consisting of those sequences which are constant on each interval $[\alpha(n-1)+1, \alpha(n)]$. Then if

$$(\pi d)_n = (\alpha(k) - \alpha(k-1))^{-1}\left(\sum_{i=\alpha(k-1)+1}^{\alpha(k)} d_i\right),$$

for $d \in \ell^p$ and $n \in [\alpha(k-1)+1, \alpha(k)]$, then π is a projection from ℓ^p onto M_p and $\pi^*(M_p^*) = M_q$. By (2.13) the restriction of A to M_q is a bounded invertible operator onto ℓ^q, as required.

REMARK. Proposition 3.6 should perhaps be compared with the result ([1] and [6, p.239]) that if $p > 1$, the Cesàro operator is bounded on ℓ^p, and with a recent result ([8]) on the partial invertibility of the Cesàro operator.

PROPOSITION 3.7. *Let $1 \le p < \infty$, let $(\alpha(n))$ be a strictly increasing sequence of positive integers and let*

$$(Af)(n) = \alpha(n)^{-1/p}\int_{-\alpha(n)}^{\alpha(n)} f(t)dt, \qquad \textit{for} \quad n \in \mathbb{N} \quad \textit{and} \quad f \in L^q(\mathbb{R}),$$

where $p^{-1} + q^{-1} = 1$. Then $\gamma(\alpha) > 1$ if and only if A is a bounded operator from $L^q(\mathbb{R})$ onto ℓ^q. In this case the restriction of A to the subspace of $L^q(\mathbb{R})$ consisting of those functions which are constant on each set $[-\alpha(n), -\alpha(n-1)] \cup [\alpha(n-1), \alpha(n)]$ is a bounded invertible operator onto ℓ^q.

Proof. This is similar to Proposition 3.6.

THEOREM 3.8. *Let* $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ *and for* $n \in \mathbb{N}$ *let*

$$w_n = \left(\int_{K_n - K_{n-1}} |f|^p d\mu \right)^{-1/q} \chi(K_n - K_{n-1})(\text{sign } f)|f|^{p-1}, \qquad \textit{and}$$

$$h_n = \left(\int_{K_n - K_{n-1}} |f|^p d\mu \right)^{-1} \chi(K_n - K_{n-1})(\text{sign } f)|f|^{p-1}$$
$$- \left(\int_{K_{n+1} - K_n} |f|^p d\mu \right)^{-1} \chi(K_{n+1} - K_n)(\text{sign } f)|f|^{p-1}.$$

Then $[w_n : n \in \mathbb{N}] = [h_n : n \in \mathbb{N}]$ *in* $L^q(S, \mathcal{S}, \mu)$ *if and only if* $\lim_{n\to\infty} \|f\chi(K_n)\|_p = \infty$.

Proof. Let $X = [v_n : n \in \mathbb{N}]$ in $L^p(S, \mathcal{S}, \mu)$. As the v_n have disjoint supports, $\sigma' = (\|v_n\|_p^{-1} v_n)$ is a basis for X and $A_{\sigma'} = \ell^p$. It is easy to check that $w_n \in L^q(S, \mathcal{S}, \mu)$, that $\|w_n\|_q = 1$ and that $\int_S v_n w_n d\mu = \|v_n\|_p$. It follows that $(\|v_n\|_p^{-1} w_n)$ is a sequence in $L^q(S, \mathcal{S}, \mu)$ which is biorthogonal to (v_n). Also, X^* is isometrically isomorphic to $[w_n : n \in \mathbb{N}]$ in $L^q(S, \mathcal{S}, \mu)$ under T, where $T\lambda = \sum_{n=1}^{\infty} \lambda(v_n)\|v_n\|_p^{-1} w_n$, for $\lambda \in X^*$. From (3.1) it follows that $b_n = (1 - \|v_n\|_p^p)^{1/p}$, and, as X is reflexive and (3.2) holds, we may apply Theorem 2.4. The result now follows from the equivalence of (2.7) and (2.8) by observing that, in the present context, (2.7) means that $[w_n : n \in \mathbb{N}]$ equals $[h_n : n \in \mathbb{N}]$ and (2.8) means that $\lim_{n\to\infty} \|f\chi(K_n)\|_p = \infty$. This completes the proof.

4. BASES IN SPACES OF PIECEWISE LINEAR FUNCTIONS

Let $\alpha = (\alpha(n))$ denote a given strictly increasing sequence of positive numbers and let $\gamma(\alpha)$ be defined as in (3.6). If $1 \le p \le \infty$, $PLC(p, \alpha)$ will denote the piecewise linear, even functions in $L^p(\mathbb{R})$ which are linear on each interval $[\alpha(n-1), \alpha(n))$, continuous on $\bigcup_{n=1}^{\infty} (-\alpha(n), \alpha(n))$, and zero off this union. Let $PLC_0(\infty, \alpha) = PLC(\infty, \alpha) \cap C_0(\mathbb{R})$. Then for $1 \le p \le \infty$, $PLC(p, \alpha)$ is a Banach subspace of $L^p(\mathbb{R})$. Also, $PLC_0(\infty, \alpha)$ is a Banach subspace of $C_0(\mathbb{R})$. Let

$$f(t) = \text{maximum}(0, 1 - |t|), \qquad \text{for} \quad t \in \mathbb{R},$$

and for $n \in \mathbb{N}$ and $t \in \mathbb{R}$ let

(4.1) $$g_n(t) = 2^{-1/p}(p+1)^{1/p}\alpha(n)^{-1/p} f(\alpha(n)^{-1}(t), \quad \text{if} \quad 1 \le p < \infty, \quad \text{and}$$

$$g_n(t) = f(\alpha(n)^{-1}t), \qquad if \quad p = \infty.$$

Then, for $1 \leq p \leq \infty$, $g_n \in PLC(p, \alpha)$ and $\|g_n\|_p = 1$. We also let, for $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

(4.2)
$$\phi_n(t) = \frac{\alpha(n) - |t|}{\alpha(n) - \alpha(n-1)}, \qquad \text{for} \quad \alpha(n-1) \leq |t| \leq \alpha(n),$$
$$\phi_n(t) = 0, \qquad \text{if} \quad |t| < \alpha(n-1) \quad \text{or} \quad |t| > \alpha(n),$$
$$\phi'_n(t) = \frac{|t| - \alpha(n-1)}{\alpha(n) - \alpha(n-1)}, \qquad \text{for} \quad \alpha(n-1) \leq |t| \leq \alpha(n),$$
$$\phi'_n(t) = 0, \qquad \text{if} \quad |t| < \alpha(n-1) \quad \text{or} \quad |t| > \alpha(n), \quad \text{and}$$
$$z_n = 2^{-1/p}(p+1)^{1/p}\alpha(n)^{-1/p}(\phi'_{n-1} + \phi_n).$$

Let $\phi'_0 = 0$ and $\alpha(0) = \alpha(-1) = 0$. Note that g_n, z_n depend upon p. The function z_n is a type of Schauder hat function used in discussing bases of $C([0,1])$ (see [10, section 2.3]). Expressions of the from $a^{1/p}$, $(p+1)^{1/p}$, etc., will be taken to be 1 when $p = \infty$. The main result in this section is the following.

THEOREM 4.1. *Let $1 \leq p < \infty$, let (g_n) be given by* (4.1) *and $\nu = (z_n)$ be given by* (4.2). *Then ν is a basis for $PLC(p, \alpha)$ and $\ell^p \subseteq A_\nu$. Also, if we consider the conditions*

(4.3) $\quad A_\nu = \ell^p$,

(4.4) $\quad \gamma(\alpha) > 1$,

(4.5) $\quad$ *(g_n) is a basis for $PLC(p, \alpha)$, and*

(4.6) $\quad$ *$(\alpha(n)^{-3/2}t^{-2}\sin^2 2^{-1}\alpha(n)t)$ is basic in $L^2(\mathbb{R})$,*

then (4.4), (4.5) *and* (4.6) *are equivalent,* (4.4) *implies* (4.3), *and if $(\alpha(n) - \alpha(n-1))$ is increasing then* (4.3) *and* (4.4) *are equivalent. When conditions* (4.4) *to* (4.6) *hold, (g_n) is equivalent to the standard basis in ℓ^p, and the sequence in* (4.6) *(which is a sequence of weighted Fejér kernels in $L^2(\mathbb{R})$) is Riesz basic in $L^2(\mathbb{R})$.*

The case $p = \infty$ is covered by

THEOREM 4.2. *Let $p = \infty$, let (g_n) be given by* (4.1) *and let $\nu = (z_n)$ be given by* (4.2). *Then ν is a basis for $PLC_0(\infty, \alpha)$ and $A_\nu = c_0$. Also, $\gamma(\alpha) > 1$ if and only if (g_n) is a basis for $PLC_0(\infty, \alpha)$.*

A function $f \in PLC(\infty,\alpha)$ *if and only if there exists a (necessarily unique)* $d \in \ell^\infty$ *so that the series* $\sum_{n=1}^{\infty} d_n z_n$ *converges uniformly to* f *on each compact subset of* $\mathbb{R}$.

The proofs of Theorems 4.1 and 4.2 require some preliminary results and observations. Let $1 \le p \le \infty$ be given. We define

$$r_n(t) = \alpha(n) - \alpha(n-1), \quad \text{for} \quad |t| \le \alpha(n-1),$$

$$r_n(t) = \alpha(n) - |t|, \quad \text{for} \quad \alpha(n-1) \le |t| \le \alpha(n), \quad \text{and}$$

$$r_n(t) = 0, \quad \text{for} \quad |t| > \alpha(n).$$

Also, let

$$w_n = 2^{-1/p}(p+1)^{1/p}\alpha(n)^{-1/p}(\alpha(n)-\alpha(n-1))^{-1}r_n. \tag{4.7}$$

From (4.2) and (4.7) we now have

$$||\phi_n||_p = ||\phi_n'||_p = 2^{1/p}(p+1)^{-1/p}(\alpha(n)-\alpha(n-1))^{1/p}, \tag{4.8}$$

$$||z_n||_p = (1-\alpha(n-2)\alpha(n)^{-1})^{1/p}, \quad \text{and} \tag{4.9}$$

$$||w_n||_p = (1+p\alpha(n-1)\alpha(n)^{-1})^{1/p}. \tag{4.10}$$

The sequences (g_n) and (w_n) also satisfy the following recurrence relations.

$$g_n - \alpha(n-1)^{1+1/p}\alpha(n)^{-(1+1/p)}g_{n-1} = (1-\alpha(n-1)\alpha(n)^{-1})w_n, \quad \text{and} \tag{4.11}$$

$$w_n - \alpha(n-1)^{1/p}\alpha(n)^{-1/p}w_{n-1} = z_n, \quad \text{for all} \quad n \in \mathbb{N}. \tag{4.12}$$

LEMMA 4.3. *Let* $1 \le p < \infty$ *and let* $C_p = \inf\{(1+t)^{-1}(1+t^{p+1}) : 0 \le t \le 1\}$. *Then for all* $a, b \in \mathbb{R}$,

$$C_p^{1/p}2^{1/p}(p+1)^{-1/p}(\alpha(n)-\alpha(n-1))^{1/p} \textit{ maximum } (|a|,|b|)$$
$$\le ||a\phi_n' + b\phi_n||_p \le 2^{1+1/p}(p+1)^{-1/p}(\alpha(n)-\alpha(n-1))^{1/p} \textit{ maximum } (|a|,|b|).$$

Proof. The right hand inequality follows easily from (4.8). For the left hand inequality, note that $||a\phi_n' + b\phi_n||_p$ is symmetric in a, b. If $ab \ge 0$ and $|a| > |b|$,

$$||a\phi_n' + b\phi_n||_p = 2^{1/p}(p+1)^{-1/p}(\alpha(n)-\alpha(n-1))^{1/p}|a|\left(\frac{1-|b/a|^{p+1}}{1-|b/a|}\right)^{1/p},$$
$$\ge 2^{1/p}(p+1)^{-1/p}(\alpha(n)-\alpha(n-1))^{1/p}|a|.$$

If $a = b$, then

$$||a\phi_n' + b\phi_n||_p = 2^{1/p}(\alpha(n) - \alpha(n-1)^{1/p}|a|.$$

If $ab < 0$ and $|a| \geq |b|$, then

$$||a\phi_n' + b\phi_n||_p = 2^{1/p}(p+1)^{-1/p}(\alpha(n) - \alpha(n-1))^{1/p}|a| \left(\frac{1+|b/a|^{p+1}}{1+|b/a|}\right)^{1/p},$$
$$\geq C_p^{1/p} 2^{1/p}(p+1)^{-1/p}(\alpha(n) - \alpha(n-1))^{1/p}|a|.$$

Lemma 4.3 now follows from these observations.

LEMMA 4.4. *Let $1 \leq p < \infty$ and let (d_n) be a sequence of scalars. Then the following conditions are equivalent.*

(4.13) $$\sum_{n=1}^{\infty} d_n z_n \quad \textit{converges in} \quad PLC(p, \alpha),$$

(4.14)
$$2^{-1/p}(p+1)^{1/p}\left(\sum_{n=1}^{\infty}\left(d_{n+1}\alpha(n+1)^{-1/p}\phi_n' + d_n\alpha(n)^{-1/p}\phi_n\right)\right) \quad \textit{converges in} \quad L^p(\mathbb{R}), \quad \textit{and}$$

(4.15) $$\sum_{n=1}^{\infty}(\alpha(n) - \alpha(n-1))\textit{maximum}\left(\frac{|d_n|^p}{\alpha(n)}, \frac{|d_{n+1}|^p}{\alpha(n+1)}\right) < \infty.$$

When these conditions hold, the sums of the series in (4.13), (4.14) *are equal. If $d \in \ell^p$, then $\sum_{n=1}^{\infty} d_n z_n$ converges in $PLC(p, \alpha)$. If $\gamma(\alpha) > 1$, $\sum_{n=1}^{\infty} d_n z_n$ converges in $PLC(p, \alpha)$ if and only if $d \in \ell^p$, and in this case,*

(4.16) $$C_p^{1/p}(1 - \gamma(\alpha)^{-1})^{1/p}||d||_p \leq \left\|\sum_{n=1}^{\infty} d_n z_n\right\|_p \leq 2||d||_p.$$

Proof. First observe that

$$\sum_{j=1}^{n} d_j z_j = 2^{-1/p}(p+1)^{1/p}\left\{\left(\sum_{j=1}^{n-1}\left(d_{j+1}\alpha(j+1)^{-1/p}\phi_j' + d_j\alpha(j)^{-1/p}\phi_j\right)\right) + d_n\alpha(n)^{-1/p}\phi_n\right\}.$$

Now let (4.15) hold. Then by (4.8), $\lim_{n\to\infty} d_n\alpha(n)^{-1/p}\phi_n = 0$. Also, by Lemma 4.3,

$$\sum_{j=1}^{\infty}\left\|d_{j+1}\alpha(j+1)^{-1/p}\phi_j' + d_j\alpha(j)^{-1/p}\phi_j\right\|_p^p$$
$$\leq 2^{p+1}(p+1)^{-1}\left(\sum_{j=1}^{\infty}(\alpha(n) - \alpha(n-1))\max\left(\frac{|d_n|^p}{\alpha(n)}, \frac{|d_{n+1}|^p}{\alpha(n+1)}\right)\right),$$

and we deduce that the series in (4.13) and (4.14) converge and have equal sums.

Now let (4.13) hold. Then $\lim_{n\to\infty} d_n z_n = 0$ and it follows from (4.2) that $\lim_{n\to\infty} d_n \alpha(n)^{-1/p}\phi_n = 0$. From the initial observation in the proof, we now see that (4.14) holds and that the series in (4.13), (4.14) have equal sums.

Let (4.14) hold. Then

$$\sum_{n=1}^{\infty}\left\|\left(d_{n+1}\alpha(n+1)^{-1/p}\phi_n' + d_n\alpha(n)^{-1/p}\phi_n\right)\right\|_p^p < \infty.$$

Applying Lemma 4.3 shows that (4.15) then holds. This proves the equivalence of (4.13) to (4.15).

If $d \in \ell^p$, (4.15) holds and hence (4.13) holds.

Now let $\gamma(\alpha) > 1$. Then if $\sum_{n=1}^{\infty} d_n z_n$ converges, we deduce from Lemma 4.3 and (4.14) that

$$\begin{aligned} C_p 2(p+1)^{-1}\left(1-\gamma(\alpha)^{-1}\right)\|d\|_p^p &\le C_p 2(p+1)^{-1}\left(\sum_{n=1}^{\infty}\left(1-\alpha(n-1)\alpha(n)^{-1}\right)|d_n|^p\right), \\ &\le C_p 2(p+1)^{-1}\left(\sum_{n=1}^{\infty}(\alpha(n)-\alpha(n-1))\max\left(\frac{|d_n|^p}{\alpha(n)}, \frac{|d_{n+1}|^p}{\alpha(n+1)}\right)\right), \\ &\le \left\|\sum_{n=1}^{\infty}\left(d_{n+1}\alpha(n+1)^{-1/p}\phi_n' + d_n\alpha(n)^{-1/p}\phi_n\right)\right\|_p^p, \\ &= 2(p+1)^{-1}\left\|\sum_{n=1}^{\infty} d_n z_n\right\|_p^p. \end{aligned}$$

Hence $d \in \ell^p$ and the left hand side of (4.16) holds.

If $d \in \ell^p$,

$$\begin{aligned} \left\|\sum_{n=1}^{\infty} d_n z_n\right\|_p &\le \left\|\sum_{n=1}^{\infty} d_{2n} z_{2n}\right\|_p + \left\|\sum_{n=1}^{\infty} d_{2n-1} z_{2n-1}\right\|_p \\ &\le 2\|d\|_p, \end{aligned}$$

as $\|z_n\|_p \le 1$ by (4.9), and the z_{2n} have disjoint supports, as do the z_{2n-1}. This proves the right hand side of (4.16).

LEMMA 4.5. *Let $1 \le p < \infty$. Then a function is in $PLC(p, \alpha)$ if and only if it is the sum of a convergent series in $L^p(\mathbb{R})$ which is of the form $\sum_{n=1}^{\infty}(a_n\phi_n + a_{n+1}\phi_n')$.*

Proof. The condition is clearly sufficient. For necessity, observe that if $f \in PLC(p,\alpha)$, there are a_n, b_n so that $f = a_n\phi_n + b_n\phi_n'$ on $[\alpha(n-1), \alpha(n)]$. As f is continuous, we must have $b_n = a_{n+1}$. This completes the proof.

Proof of Theorem 4.1. Let $1 \leq p < \infty$ and let $f \in PLC(p,\alpha)$. By Lemma 4.5 choose a sequence (a_n) so that $f = \sum_{n=1}^{\infty}(a_n\phi_n + a_{n+1}\phi_n')$ in $L^p(\mathbb{R})$. Let $d_n = 2^{1/p}(p+1)^{-1/p}\alpha(n)^{1/p}a_n$. Then by Lemma 4.4, $f = \sum_{n=1}^{\infty} d_n z_n$, where this series converges in $L^p(\mathbb{R})$. Also, if $\sum_{n=1}^{\infty} d_n z_n = 0$, then $d_n z_n + d_{n+1}z_{n+1} = 0$ on $[\alpha(n-1), \alpha(n)]$. As z_n, z_{n+1} are independent on $[\alpha(n-1), \alpha(n)]$ we deduce that $d_n = d_{n+1} = 0$, hence $d = 0$. This proves that $\nu = (z_n)$ is a basis for $PLC(p,\alpha)$. Lemma 4.4 implies that $\ell^p \subseteq A_\nu$.

If $\gamma(\alpha) > 1$, Lemma 4.4 shows that $A_\nu = \ell^p$. If $(\alpha(n) - \alpha(n-1))$ is increasing, then (4.15) is equivalent to having $\sum_{n=1}^{\infty}(1-\alpha(n-1)\alpha(n)^{-1})|d_n|^p < \infty$. Together with Lemma 4.4, this implies that if $(\alpha(n) - \alpha(n-1))$ is increasing, then $\gamma(\alpha) > 1$ if and only if $A_\nu = \ell^p$.

Let $\gamma(\alpha) > 1$. The recurrence relation (4.12) shows that we may apply Theorem 2.2 with $\sigma = (z_n)$, $\tau = (w_n)$ and $b_n = (\alpha(n-1)\alpha(n)^{-1})^{1/p}$. We see from (4.9) that σ is bounded away from 0, and from (4.10) that τ is bounded, so we deduce from (2.1) that $\tau = (w_n)$ is a basis for $PLC(p,\alpha)$. As $||b||_\infty = \gamma(\alpha)^{-1/p} < 1$, (2.4) implies that $A_\tau = \ell^p$.

Now as $\gamma(\alpha) > 1$, $((1-\alpha(n-1)\alpha(n)^{-1})w_n)$ is also a basis for $PLC(p,\alpha)$ which is equivalent to the standard basis for ℓ^p. The recurrence relation (4.11) shows that Theorem 2.2 may be applied again, with $\sigma = ((1-\alpha(n-1)\alpha(n)^{-1})\, w_n)$, $\tau = (g_n)$ and $b_n = (\alpha(n-1)\alpha(n)^{-1})^{1+1/p}$. Then τ is bounded, σ is bounded away from 0 and $||b||_\infty = \gamma(\alpha)^{-(1+1/p)} < 1$. It follows from (2.1) and (2.4) that (g_n) is a basis for $PLC(p,\alpha)$ which is equivalent to the standard basis in ℓ^p. This proves that (4.4) implies (4.5).

Conversely, let (g_n) be a basis for $PLC(p,\alpha)$. As $||g_n||_p = 1$, (2.2) and (4.11) imply that $((1-\alpha(n-1)\alpha(n)^{-1})w_n)$ is bounded away from 0. As (4.10) shows that (w_n) is bounded, we deduce that $\gamma(\alpha) > 1$. Thus, (4.5) implies (4.4).

If $p = 2$, observe that the Fourier transform of g_n in $L^2(\mathbb{R})$ is a multiple, independent of n, of $\alpha(n)^{-3/2}\left((\sin\alpha(n)t/2)/t\right)^2$. The equivalence of (4.5) and (4.6) is thus a consequence of Plancherel's theorem. If (g_n) is basic in $L^2(\mathbb{R})$, we have seen that it is Riesz basic, so in this case Plancherel's theorem also implies that the sequence in (4.6) is Riesz basic in $L^2(\mathbb{R})$. This completes the proof of Theorem 4.1.

REMARK. If we let $\alpha(2n) = 2^n$ and $\alpha(2n+1) = 2^n + 1$, it can be shown that (4.15) holds if and only if $d \in \ell^p$. By Lemma 4.4, $A_\nu = \ell^p$. Thus $\gamma(\alpha) = 1$ but $A_\nu = \ell^p$, so (4.3) does not, in general, imply (4.4).

COROLLARY 4.6. *If* $m, n \in \mathbb{N}$ *let*

$$a_{m,n}(\alpha) = \left(\frac{\alpha(m)}{\alpha(n)}\right)^{1/2}\left(3 - \frac{\alpha(m)}{\alpha(n)}\right), \quad \textit{if} \quad m \le n, \quad \textit{and}$$
$$a_{m,n}(\alpha) = \left(\frac{\alpha(n)}{\alpha(m)}\right)^{1/2}\left(3 - \frac{\alpha(n)}{\alpha(m)}\right), \quad \text{if} \quad n \le m.$$

Then $\gamma(\alpha) > 1$ *if and only if there are* $A, B > 0$ *such that for all scalar sequences* (d_n) *of finite support,*

$$A\|d\|_2^2 \le \sum_{m,n=1}^{\infty} d_m d_n a_{m,n}(\alpha) \le B\|d\|_2^2.$$

Proof. Let $p = 2$. Then $(a_{m,n}(\alpha))_{m,n=1}^{\infty}$ is the Gram matrix of (g_n), except for a constant factor. The inequality is thus equivalent to saying that (g_n) is Riesz basic in $L^2(\mathbb{R})$ (see [12, p.32]). The result now follows from Theorem 4.1.

COROLLARY 4.7. *Let* $\gamma(\alpha) > 1$. *Then a function* $h \in L^2(\mathbb{R})$ *has an expansion as a convergent series in* $L^2(\mathbb{R})$ *of the form* $\sum_{n=1}^{\infty} \alpha(n)^{-3/2} d_n t^{-2} \sin^2 2^{-1}\alpha(n)t$, *for* $d \in \ell^2$, *if and only if the Fourier transform of* h *is in* $PLC(2, \alpha)$.

Proof. Observe that the Fourier transform $\hat{h}$ of h is in $PLC(2, \alpha)$ if and only if $h \in [\hat{g}_n : n \in \mathbb{N}]$, where g_n is given by (4.1) with $p = 2$. Now apply Theorem 4.1.

Proof of Theorem 4.2. Let $p = \infty$. Note that $\|z_n\|_\infty = 1$ and that z_n is supported by $[\alpha(n-2), \alpha(n)]$. Hence $\sum_{n=1}^{\infty} d_n z_n$ converges in $PLC_0(\infty, \alpha)$ if and only if $d \in c_0$. It also follows that $f \in PLC(\infty, \alpha)$ if and only if there is $d \in \ell^\infty$ so that $\sum_{n=1}^{\infty} d_n z_n$ converges uniformly to f on compact subsets of $\mathbb{R}$. It is easy to prove that $\nu = (z_n)$ is a basis for $PLC_0(\infty, \alpha)$ by analogy with the case $1 \le p < \infty$ in Theorem 4.1.

If $\gamma(\alpha) > 1$, we apply (2.1) of Theorem 2.2 twice, using the recurrence relations (4.11) and (4.12) with $p = \infty$. This is similar to the case $1 \le p < \infty$ in Theorem 4.1, and we deduce in a similar way that (g_n) is a basis for $PLC_0(\infty, \alpha)$.

Conversely, if (g_n) is a basis for $PLC_0(\infty, \alpha)$, then $(\|g_{n+1} - g_n\|_\infty)$ is bounded away from 0. As

$$\|g_{n+1} - g_n\|_\infty = |g_{n+1}(\alpha(n))| = (1 - \alpha(n)\alpha(n+1)^{-1}),$$

we deduce that $\gamma(\alpha) > 1$. This proves Theorem 4.2.

PROPOSITION 4.8. *If $\gamma(\alpha) > 1$, there is a projection π_1 from $C_0(\mathbb{R})$ onto $PLC_0(\infty, \alpha)$ such that $\pi_1^*(PLC_0(\infty, \alpha)^*) = PLC(1, \alpha)$.*

If $1 \le p < \infty$, $p^{-1} + q^{-1} = 1$ and $\gamma(\alpha) > 1$, there is a projection π_2 from $L^p(\mathbb{R})$ onto $PLC(p, \alpha)$ such that $\pi_2^*(PLC(p, \alpha)^*) = PLC(q, \alpha)$.

Proof. Let $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ and $\gamma(\alpha) > 1$. We let

$$z'_n = 2^{-1/q}(q+1)^{1/q}\alpha(n)^{-1/q}(\phi'_{n-1} + \phi_n).$$

By (4.9), $\|z'_n\|_q \le 1$. Also, z'_n is supported by $F_n \cup -F_n$, where $F_n = [\alpha(n-2), \alpha(n)]$. Hence, for $f \in L^p(R)$,

$$\left|\int_{\mathbb{R}} f(t) z'_n(t)\,dt\right| \le \|f\chi(F_n \cup -F_n)\|_p, \quad \text{for} \quad n \in \mathbb{N}. \tag{4.17}$$

Now let $A_1 = 0$,

$$A_n = \int_{\mathbb{R}} z_n(t) z'_{n-1}(t)\,dt = \frac{(p+1)^{1/p}(q+1)^{1/q}}{6}\left(\frac{\alpha(n-1)}{\alpha(n)}\right)^{1/p}\left(1 - \frac{\alpha(n-2)}{\alpha(n-1)}\right), \quad \text{for} \quad n \ge 2,$$

$$B_n = \int_{\mathbb{R}} z_n(t) z'_n(t)\,dt = \frac{(p+1)^{1/p}(q+1)^{1/q}}{3}\left(1 - \frac{\alpha(n-2)}{\alpha(n)}\right), \quad \text{for} \quad n \ge 1, \quad \text{and}$$

$$C_n = \int_{\mathbb{R}} z_n(t) z'_{n+1}(t)\,dt = \frac{(p+1)^{1/p}(q+1)^{1/q}}{6}\left(\frac{\alpha(n)}{\alpha(n+1)}\right)^{1/q}\left(1 - \frac{\alpha(n-1)}{\alpha(n)}\right), \quad \text{for} \quad n \ge 1.$$

As $\gamma(\alpha) > 1$, (B_n^{-1}) is bounded. If $f \in L^p(\mathbb{R})$, we now let

$$\pi(f) = \sum_{n=1}^{\infty} B_n^{-1} \left(\int_{\mathbb{R}} f(t) z_n'(t) dt \right) z_n.$$

From (4.16) and (4.17) we see that the series of πf converges in $L^p(\mathbb{R})$ and that

$$\|\pi(f)\|_p \leq 2^{1+1/p} \|(B_n^{-1})\|_\infty \|f\|_p.$$

Hence π is bounded from $L^p(\mathbb{R})$ into $PLC(p, \alpha)$. We will now show that π is invertible on $PLC(p, \alpha)$. If $f \in PLC(p, \alpha)$, as (z_n) is a basis for $PLC(p, \alpha)$ by Theorem 4.1, there is $d \in \ell^p$ so that $f = \sum_{n=1}^{\infty} d_n z_n$. Then

$$\begin{aligned} (4.18) \qquad \pi(f) &= \sum_{n=1}^{\infty} \left(A_{n+1} B_n^{-1} d_{n+1} + d_n + B_n^{-1} C_{n-1} d_{n-1} \right) z_n, \quad \text{where,} \quad d_0 = 0, \\ &= \sum_{n=1}^{\infty} \left((I + S) d \right)_n z_n, \end{aligned}$$

where I is the identity operator on ℓ^p, and

$$(Sd)_n = A_{n+1} B_n^{-1} d_{n+1} + B_n^{-1} C_{n-1} d_{n-1}, \quad \text{for} \quad d \in \ell^p.$$

Now,

$$\begin{aligned} A_{n+1} B_n^{-1} &= 2^{-1} \alpha(n)^{1/p} \alpha(n+1)^{-1/p} \left(\alpha(n) - \alpha(n-1) \right) \left(\alpha(n) - \alpha(n-2) \right)^{-1}, \\ &\leq 2^{-1} \gamma(\alpha)^{-1/p}, \qquad \text{and} \\ B_n^{-1} C_{n-1} &= 2^{-1} \alpha(n-1)^{1/q} \alpha(n)^{-1/q} \left(1 - \alpha(n-2) \alpha(n-1)^{-1} \right) \left(1 - \alpha(n-2) \alpha(n)^{-1} \right)^{-1}, \\ &\leq 2^{-1} \gamma(\alpha)^{-1/q}. \end{aligned}$$

Hence S is bounded on ℓ^p and $\|S\| \leq 2^{-1} \left(\gamma(\alpha)^{-1/p} + \gamma(\alpha)^{-1/q} \right) < 1$, so $I + S$ is invertible on ℓ^p. By (4.16), $PLC(p, \alpha)$ is isomorphic to ℓ^p, and we deduce from (4.18) that π is invertible on $PLC(p, \alpha)$. Denote this inverse by λ and let $\pi_2 = \lambda \circ \pi$. Then π_2 is a projection from $L^p(\mathbb{R})$ onto $PLC(p, \alpha)$.

Now by Theorem 4.1, $\nu = (z_n)$ is a basis for $PLC(p, \alpha)$ and $A_\nu = \ell^p$. Then (2.11) shows that $\{(\mu(z_n) : \mu \in PLC(p, \alpha)^*\} = \ell^q$. Hence if $\mu \in PLC(p, \alpha)^*$, $(B_n^{-1} \mu(z_n)) \in \ell^q$ and the series $\sum_{n=1}^{\infty} B_n^{-1}\mu(z_n)z'_n$ converges in $PLC(q,\alpha)$. It is easy to prove that $\pi^*(\mu) = \sum_{n=1}^{\infty} B_n^{-1}\mu(z_n)z'_n$, for $\mu \in PLC(p,\alpha)^*$, and it follows that $\pi^*(PLC(p,\alpha)^*) = PLC(q,\alpha)$ (here we have used the fact that (B_n) is bounded above and below and that (z'_n) is a basis for $PLC(q,\alpha)$ equivalent to the standard basis in ℓ^q). Finally, as λ is invertible on $PLC(p,\alpha)$,

$$\begin{aligned} \pi_2^*\left(PLC(p,\alpha)^*\right) &= \pi^*\left(\lambda^*\left(PLC(p,\alpha)^*\right)\right), \\ &= \pi^*\left(PLC(p,\alpha)^*\right), \\ &= PLC(q,\alpha), \qquad \text{from above.} \end{aligned}$$

This proves the proposition for $1 < p < \infty$.

When $p = 1$ and $q = \infty$, the proof proceeds on the lines above, except that when we have $\pi^*(\mu) = \sum_{n=1}^{\infty} B_n^{-1}\mu(z_n)z'_n$, this series is taken as converging uniformly on compact sets, rather than in the $L^\infty(\mathbb{R})$ norm.

When $p = \infty$ and $q = 1$, the proof is again similar to the preceding. Instead, π is defined on $C_0(\mathbb{R})$, ℓ^p is replaced by c_0, and Theorem 4.2 is used in place of Theorem 4.1.

REMARKS. 1. If one only wishes to show that $PLC_0(\infty,\alpha)$ is complemented in $C_0(\mathbb{R})$ a simpler proof than the one above may be found in [10, p.27] – this proof does not require $\gamma(\alpha) > 1$, but it does not give the identity $\pi_1^*\left(PLC_0(\infty,\alpha)^*\right) = PLC(1,\alpha)$.

2. Let $PL(p,\alpha)$ denote those (not necessarily continuous) functions in $L^p(\mathbb{R})$ which are even and linear on each interval $[\alpha(n-1),\alpha(n)]$. Then it can be proved that for $1 \le p < \infty$, $PL(p,\alpha)$ is complemented in $L^p(\mathbb{R})$ under a projection π such that $\pi^*(PL(p,\alpha)^*) = PL(q,\alpha)$. This is true without restriction on $\gamma(\alpha)$. Thus, it is not clear whether the role played in Proposition 4.8 by the condition $\gamma(\alpha) > 1$ is essential, although $\gamma(\alpha) > 1$ *is* essential for the next result.

PROPOSITION 4.9. *Let* $1 \le p < \infty$ *and* $p^{-1} + q^{-1} = 1$. *For* $g \in L^q(\mathbb{R})$ *and* $n \in \mathbb{N}$ *let*

$$(Ag)(n) = \alpha(n)^{-(1+1/p)} \int_{-\alpha(n)}^{\alpha(n)} (\alpha(n) - |t|)\, g(t)\,dt.$$

Then $\gamma(\alpha) > 1$ *if and only if* $A(L^q(\mathbb{R})) = \ell^q$. *In this case, the restriction of* A *to the subspace* $PLC(q,\alpha)$ *of* $L^q(\mathbb{R})$ *is a bounded invertible operator onto* ℓ^q.

Proof. By Theorem 4.1, $\gamma(\alpha) > 1$ is equivalent to saying that (g_n) is a basis for $PLC(p, \alpha)$ which is equivalent to the standard basis for ℓ^p. When $1 < p < \infty$, we deduce from (2.11) that this is equivalent to $A(L^q(\mathbb{R})) = \ell^q$. When $p = 1$ and $q = \infty$, $\gamma(\alpha) > 1$ implies that $A(L^\infty(\mathbb{R})) = \ell^\infty$ is a consequence of (2.12). Conversely, if $\gamma(\alpha) = 1$ and $g \in L^\infty(\mathbb{R})$ let $a_n = \alpha(n)^{-2} \int_{-\alpha(n)}^{\alpha(n)} (\alpha(n) - |t|)\, g(t) dt$. Then it can be shown that $\liminf_{n \to \infty} |a_{n+1} - a_n| = 0$ (compare with the corresponding part of the proof of Proposition 3.6). Hence, if $\gamma(\alpha) = 1$, $A(L^\infty(\mathbb{R})) \subset \ell^\infty$ and $A(L^\infty(\mathbb{R})) \neq \ell^\infty$. The final statement in the proposition comes from (2.13) and Proposition 4.8.

There are also discrete versions of the preceding results, some of which are presented.

THEOREM 4.10. *Let $1 \leq p \leq \infty$ be given, and let $(\alpha(n))$ be an increasing sequence of positive integers. Let $h_n \in \ell^p(\mathbb{Z})$ be given by*

$$h_n(j) = \alpha(n)^{-(1+1/p)}(\alpha(n) - |j|), \quad \textit{for} \quad |j| \leq \alpha(n), \quad \textit{and}$$

$$h_n(j) = 0, \quad \textit{for} \quad |j| > \alpha(n).$$

Then $\gamma(\alpha) > 1$ if and only if (h_n) is basic in $\ell^p(\mathbb{Z})$. If $\gamma(\alpha) > 1$ and $1 \leq p < \infty$, (h_n) is equivalent to the standard basis in ℓ^p. Also, $\gamma(\alpha) > 1$ if and only if the sequence $(\alpha(n)^{-3/2} \sin^2(\alpha(n)t/2) \sin^{-2} t/2)$ is basic in $L^2([0, 2\pi])$, in which case it is Riesz basic.

Proof. Let $PLC(p)$ denote the closed subspace of $L^p(\mathbb{R})$ consisting of the even, continuous functions which are linear on $[n-1, n]$ for $n \in \mathbb{N}$. If $f \in PLC(p)$, let $(Tf)(n) = f(n)$, for $n \in \mathbb{Z}$. It follows from Lemma 4.3 that T is an isomorphism from $PLC(p)$ into $\ell^p(\mathbb{Z})$. Also, $T(g_n) = 2^{-1/p}(p+1)^{1/p} h_n$, for all n. The statements concerning (h_n) are thus a consequence of the equivalence of (4.4) and (4.5), and Theorems 4.1 and 4.2. When $p = 2$ the Fourier transform of $\alpha(n)^{3/2} h_n$ is the Fejér kernel $\sin^2(\alpha(n)t/2) \sin^{-2} t/2$. The remainder of Theorem 4.10 now follows from Plancherel's theorem.

COROLLARY 4.11. *Let* $\gamma(\alpha) > 1$. *Then a function* $h \in L^2([0, 2\pi])$ *has an expansion as a convergent series in* $L^2([0, 2\pi])$ *of the form*

$$\sum_{n=1}^{\infty} \alpha(n)^{-3/2} d_n \sin^2(\alpha(n)t/2) \sin^{-2} t/2, \qquad \textit{for} \quad d \in \ell^2,$$

if and only if the Fourier transform of h *is the restriction to* $\mathbb{Z}$ *of some function in* $PLC(2, \alpha)$.

Proof. This is analogous to the proof of Corollary 4.7.

COROLLARY 4.12. *Let* $1 \leq p < \infty$ *and* $p^{-1} + q^{-1} = 1$. *Let* $a_{ij} = \alpha(i)^{-(1+1/p)}(\alpha(i) - j + 1)$, *for* $1 \leq j \leq \alpha(i)$, *and* $a_{ij} = 0$, *for* $j > \alpha(i)$. *Let* A *denote the operator obtained by multiplying by the matrix* (a_{ij}). *Then* A *is a bounded operator from* ℓ^q *onto* ℓ^q *if and only if* $\gamma(\alpha) > 1$.

Proof. This is similar to the proof of Proposition 4.9.

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