THE FIRST BOUNDARY VALUE PROBLEM FOR SOLUTIONS OF DEGENERATE QUASILINEAR PARABOLIC EQUATIONS

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The study of a class of degenerate quasilinear parabolic equations such as $u_t = (u^m)_{xx} (m>1)$; $u_t = [a(u)]_{xx} + [b(u)]_x$, $a'(0) = 0$; $u_t = \Delta(u^m)(m>1)$; $u_t = \Delta a(u) + \Sigma b_i(u)u_{x_i}$, $a'(0) = 0$ etc. comes from liquid flow through porous medium. Attention is restricted to the nonnegative solutions because of their meaning in this context. A special feature of the above equations is, that at the point (x^0, t^0) where $u(x^0, t^0) > 0$, the equations are nondegenerate. Degeneracy occurs only at the point (x^0, t^0) where $u(x^0, t^0) = 0$. Because of the degeneracy of the equation, the classical solutions do not exist, so that we must consider the generalized solutions.

The following results were obtained in the study of the Cauchy problem and the first boundary value problem of the above equations, for the one space variable case.

Oleinik, Kalasinkov and Czou [1] treated the Cauchy problem and first boundary value problem of $u_t = [\varphi(x,u)]_{xx}$ where $\varphi(x,0) = \varphi_u'(x,0) = 0$.

Gilding and Petelier [2] studied the Cauchy problem of $u_t = (u^m)_{xx} + (u^n)_x$ $(m>1$, m , n , are constants). Under the restriction $n \le \frac{1}{2}(m+1)$, the generalized solutions were proved to be unique, while for the existence of generalized solution this restriction was not needed.

Gilding [3] extended the result of [2] to the equation $u_t = (a(u)u_x)_x + b(u)u_x$ where $a(0) = 0$ and $a(u) > 0$ when $u > 0$, and studied the existence and uniqueness of generalized solutions for the Cauchy problem and the first boundary value problem. A similar restriction $b^2 \leq Ka$ (K is a constant) is needed in the proof of uniqueness of solutions.

There have been many attemmpts to remove the restriction $b^2 \leq Ka$ on the uniqueness of solutions (see [4] and its references). At last, a result of Chen Yazhe (preprint) nearly removes this serious restriction.

In higher space dimensions, Sabinina [5] solved the Cauchy problem of $u_t = \Delta\Phi(u)$ where $\Phi'(0) = 0$.

In 1981, the author studied the existence and uniqueness of solutions for the first boundary value problem of $u_t = \operatorname{div}(a(u)\operatorname{grad}u) + \Sigma\, b_i(u)u_{x_i}$ where $a(0) = 0$ and $a(u) > 0$ when $u > 0$. For the uniqueness of solutions there were some restrictions on $b_i(u)$ related to $a(u)$ (preprint).

Chen Yazhe [6] solved the first boundary value problem of $u_t = \Sigma(a_{ij}(x,t,u)u_{x_j})_{x_i} + \Sigma\, b_i(x,t,u)u_{x_i} + c(x,t,u)u$, where the smallest eigenvalue λ of the nonnegative symmetric matrix $(a_{ij}(x,t,u))$ satisfies $\lambda|_{u=0} = 0$ and $\Sigma\, b_i^2 \leq K\lambda$ (K is a constant). It should be noticed that the restriction $\sum b_i^2 \le K\lambda$ is needed in [6] for the proof of the existence of the solution. This is unlike the one space dimensional case, where $b^2 \le Ka$ is needed only for the proof of uniqueness.

We have succeeded in removing the restriction $\Sigma b_i^2 \le K\lambda$ and prove the existence and uniqueness of generalized solutions of the first boundary value problem and the Cauchy problem [7].

Let Ω be a bounded domain in $\mathbb{R}^n$, $T > 0$, $Q = \Omega\times(0,T)$. Consider the generalized solutions of the first boundary value problem

$$(1)\ \begin{cases} u_t = \Sigma(a_{ij}(x,t,v)_{x_j})_{x_i} + \Sigma b_i(x,t,v)u_{x_i} + c(x,t,v)u & (x,t) \in Q \\ u\big|_{t=0} = u_0(x) \ , \quad x \in \Omega & \\ u\big|_{\partial\Omega\times(0,T)} = \psi(s,t), \quad s \in \partial\Omega \ , \quad 0 \le t \le T & \end{cases}$$

where the initial and boundary data and the coefficients of (1) satisfy the following conditions:

(1) $a_{ij}(x,t,r)$, $b_i(x,t,r)$, $c(x,t,r)$, $\Sigma\frac{\partial}{\partial x_j}a_{ij}(x,t,r)$, $\Sigma\frac{\partial}{\partial x_i}b_i(x,t,r) \in C(\bar{Q}\times\mathbb{R})$, $1 \le i,j \le m$.

(2) $\frac{1}{\Lambda}\nu(|r|)|\xi|^2 \le \Sigma a_{ij}(x,t,r)\xi_i\xi_j \le \Lambda\ \nu(|r|)|\xi|^2$, $\forall\ \xi \in \mathbb{R}^n$, $\forall(x,t,r) \in \bar{Q}\times\mathbb{R}$, where Λ is a positive constant, $\nu(r) \in C[0,\infty]$, $\nu(0) = 0$, $\nu(r) > 0$ when $r > 0$. There exist constants $\delta > 0$, $m > 1$ such that

$$1 \le r\nu(r)/\int_0^r \nu(s)ds \le m \ , \quad r \in (0,\delta] \ .$$

(3) $|b_i(x,t,r)| \left|\Sigma \frac{\partial b_i(x,t,r)}{\partial x_i}\right| , |c(x,t,r)| \le \Lambda \quad (1 \le i \le n)$

$\forall (x,t,r) \in \bar{Q} \times \mathbb{R}$.

(4) There exist constants $a_0 > 0$, $\theta_0 \in (0,1)$ such that

$$\text{mes}\ \{K(\rho) \cap \Omega\} \le (1-\theta_0)\ \text{mes}\ K(\rho) \quad (\rho \le a_0)$$

where $K(\rho)$ is the ball with center on $\partial\Omega$.

(5) $\partial\Omega \in C^{1+\beta_0} (\beta_0 > 0)$, $u_0(x) \ge 0$, $\psi(s,t) \ge 0$, $(s \in \partial\Omega$, $0 \le t \le T)$, satisfy the compatibility condition $u_0(s) = \psi(s,0)$; $u_0(x)$ and $\psi(s,t)$ satisfy Hölder conditions.
A function $u(x,t)$ satisfying $u(x,t) \in C(\bar{Q})$ and $u(x,t) \ge 0$ is called a generalized solution of (1) when it satisfies the initial and boundary conditions in the ordinary sense and satisfies the equation in the generalized sense as follows.

For all $\varphi(x,t) \in C^{2,1}(Q) \cap C^1(\bar{Q})$, $\varphi|_{t=T} = \varphi|_{\partial\Omega\times[0,T]} = 0$,
$\int_Q \{u\varphi_t + \Sigma A_{ij}(x,t,u)\varphi_{x_i x_j} - \Sigma[A_i(x,t,u) + B_i(x,t,u)]\ \varphi_{x_i} + [c(x,t,u) + B(x,t,u)]\varphi\}\ dxdt + \int_\Omega u_0(x)\varphi(x,0)dx - \int_{\partial\Omega\times[0,T]} \Sigma A_{ij}(s,t,\psi(s,t)) \frac{\partial\varphi(s,t)}{\partial N}\cos(N,s_i)\cos(N,s_j)dsdt = 0$, where N is the outer normal of $\partial\Omega$ and

$$A_{ij}(x,t\ r) = \int_0^r a_{ij}(x,t,s)ds\ , \qquad A_j(x,t,r) = \int_0^r \Sigma \frac{\partial a_{ij}}{\partial x_i}(x,t,s)ds\ ,$$

$$1 \le i,j \le n\ .$$

$$B_i(x,t,r) = \int_0^r b_i(x,t,s)ds\ , \qquad B(x,t,r) = \int_0^r \Sigma \frac{\partial b_i}{\partial x_i}(x,t,s)ds\ .$$

THEOREM Under the above conditions the solution of the generalized solution exists. Moreover u satisfies a Hölder condition in $\bar{Q}$. If we further assume $\partial\Omega \in C^2$, then the generalized solutions are unique.

Sketch of the proof:

STEP 1 Prove the classical solution of (1) satisfies a Hölder estimate when $u_0(x) > 0$, $\psi(s,t) > 0$, and the exponent and coefficient of Hölder estimate are independent of the positive lower bound of u in $\bar{Q}$.

In order to obtain this result we first make a change of variables such that $-\Lambda \le c(x,t,u) \le 0$ and then let $w = \int_0^u r(s)ds$ or $u = \Phi(w)$. Multiplying (1) by $\zeta^2(x)\ (w-k)^+$ and $\zeta^2(x)(w-k)^-$ where $\zeta(x)$ is a cutoff function and k is a constant, we have

$$\frac{\partial}{\partial t}\left[e^{-\gamma\Phi'(k)t}\int_{A_{k,\rho}(t)} \zeta^2 \chi_k(w-k)dx + \frac{1}{2M\Lambda}\ e^{-\gamma\Phi'(k)t}\int_{A_{k,\rho}(t)} \zeta^2|\nabla w|^2dx\right.$$

(2)

$$\le \gamma e^{-\gamma\Phi'(k)t}\int_{A_{k,\rho}(t)} |\nabla\zeta|^2(w-k)^2dx$$

where γ is a constant, $A_{k,\rho}(t) = \{x\in K_\rho \cap \Omega | w(x,t)>k\}$, $\chi_k(s) = \int_0^s \Phi'(\tau+k)\tau d\tau$.

$$\frac{\partial}{\partial t}\left[e^{-\gamma\Phi'(k)t}\int_{B_{k,\rho}(t)} \zeta^2\tilde{\chi}(k-w)dx + \frac{1}{2M\Lambda} e^{-\gamma\Phi'(k)t}\int_{B_{k,\rho}(t)} \zeta^2|\nabla w|^2dx\right.$$

(3)

$$\leq \gamma e^{-\gamma\Phi'(k)t}\left[\int_{B_{k,\rho}(t)} |\nabla\zeta|^2(k-w)^2dx + \operatorname{mes}B_{k,\rho}(t)\right]$$

where $B_{k,\rho}(t) = \{x\in K_\rho \cap \Omega | w(x,t)<k\}$, $\tilde{\chi}_k(s) = \int_0^s \Phi'(k-\tau)\tau d\tau$. A function $w(x,t)$ satisfying (2), (3) shall be said to be of generalized $\mathscr{B}_2(Q,M,m,\gamma)$ class. In the ordinary $\mathscr{B}_2$ class of Ladyženskaya and Ural'ceva the function $\chi_k(w-k)$ and $\tilde{\chi}_k(k-w)$ are replaced by $(w-k)^2$, and there is no factor $e^{-\gamma\Phi'(k)t}$. Its Hölder estimate has been established by Ladyženskaya and Ural'ceva already [8]. The generalized $\mathscr{B}_2$ class has properties similar to the ordinary $\mathscr{B}_2$ class and, after some special treatment, the Hölder estimate independent of the lower bound of u in $\bar{Q}$ is ultimately obtained.

STEP 2 The existence of generalized solution

Approximate $u_0(x)$ and $\psi(s,t)$ by $u_0^{(k)}(x) \geq k^{-1}$, $\psi^{(k)}(s,t) \geq k^{-1}$ $(k = 1,2,\dots)$ respectively so that the equation (1) becomes nondegenerate. Then perform a smoothing approximation of coefficients and initial boundary data by $a_{ij}^{(\epsilon)}$, $b_i^{(\epsilon)}$, $c^{(\epsilon)}$, $u_0^{(k,\epsilon)}$, $\psi^{(k,\epsilon)}$, $\nu^{(\epsilon)}(r)$. The approximation of $\nu(r)$ by $\nu^{(\epsilon)}(r)$ must be done so that $1 \leq r\, \nu^{(\epsilon)}(r)/\int_0^r \nu^{(\epsilon)}(s)ds$ holds. The classical solution $u^{(k,\epsilon)}(x,t)$ exists by a well known theorem. Taking a partial sequence, we then have $u^{(k,\epsilon)}(x,t)(\epsilon\to 0) \to u^{(k)}(x,t)(k\to\infty) \to u(x,t)$; i.e., we obtain a generalized solution of (1) with $u(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q})$.

STEP 3 Uniqueness of generalized solutions

If there is another generalized solution $\tilde{u}(x,t) \in C(\bar{Q})$, $\tilde{u}(x,t) \geq 0$ in $\bar{Q}$. Then the difference $u^{(k)}(x,t) - \tilde{u}(x,t)$ satisfies the following integral relation:

$$\int_Q [u^{(k)} - \tilde{u}][\varphi_t + \Sigma a_{ij}^{(k)}(x,t)\varphi_{x_i x_j} + \Sigma b_i^{(k)}(x,t)\varphi_{x_i} + c^{(k)}(x,t)\varphi]dxdt$$

$$+\int_\Omega [u_0^{(k)}(x)-u_0(x)](x,0)dx-\int_{\partial\Omega\times[0,T]} [\psi^{(k)}-\psi]\Sigma a_{ij}^{(k)}(s,t)\frac{\partial\varphi}{\partial N}\cos(N,s_j)dsdt = 0$$

where

$$a_{ij}^{(k)} = \int_0^1 a_{ij}(x,t,\theta u^{(k)}+(1-\theta)\tilde{u})d\theta \ , \ b_i^{(k)} = \ldots, c^{(k)} = \ldots .$$

Then approximate $\tilde{u}$ by $\tilde{u}_\epsilon$ so that $\tilde{u} \leq \tilde{u}_\epsilon \leq \tilde{u}+\epsilon$. With $\tilde{u}$ replaced by $\tilde{u}_\epsilon$, we obtain the approximate coefficients

$$a_{ij}^{(k,\epsilon)} = \int_0^1 a_{ij}(x,t,\theta u^{(k)}+(1-\theta)\tilde{u}_\epsilon)d\theta \ , \ b_i^{(k,\epsilon)} = \ldots, \ c^{(k,\epsilon)} = \ldots .$$

For any $U(x,t) \in C(\bar{Q})$, solve the boundary value problem

$$\begin{cases} \varphi_t+\Sigma a_{ij}^{(k,\epsilon)}\varphi_{x_ix_j}+\Sigma b_i^{(k,\epsilon)}\varphi_{x_i}+c^{(k,\epsilon)}\varphi = U(x,t) , \\ \varphi|_{t=T}=\varphi|_{\partial\Omega\times[0,T]} = 0 . \end{cases}$$

Denote the solution of the above problem by $\varphi^{(k,\epsilon)}$. We prove that $\varphi^{(k,\epsilon)}$ satisfies the following three inequalities

$$|\varphi^{(k,\epsilon)}| \le e^{2\Lambda T}\sup_Q|U(x,t)| \tag{4}$$

$$\left|\frac{\partial\varphi^{(k,\epsilon)}}{\partial N}\right|_{\partial\Omega\times[0,T]}v(\tfrac{1}{k}) \le K \tag{5}$$

$$\int_Q\{\Sigma[\varphi_{x_ix_j}^{(k,\epsilon)}]^2 + \Sigma[\varphi_{x_i}^{(k,\epsilon)}]^2\}dxdt \le \tilde{K}(k) \tag{6}$$

where K , $\tilde{K}(k)$ are constants. Hence

$$\begin{aligned} \left|\int_Q(u-\tilde{u})Udxdt\right| &\le \left|\int_Q(u-u^{(k)})Udxdt\right| + \left|\int_Q(u^k-\tilde{u})[\varphi_t+\Sigma a_{ij}^{(k,\epsilon)}\varphi_{x_ix_j}^{(k,\epsilon)}\right. \\ &\quad \left.+ \Sigma b_i^{(k,\epsilon)}\varphi_{x_i}^{(k,\epsilon)} + C^{(k,\epsilon)}\varphi^{(k,\epsilon)}]dxdt\right| \\ &\le \left|\int_Q(u-u^{(k)})Udxdt\right| + \left|\int_Q(u^{(k)}-\tilde{u})\ \{\Sigma[\Sigma a_{ij}^{(k,\epsilon)}\right. \\ &\quad - a_{ij}^{(k)}]\varphi_{x_ix_j}^{(k,\epsilon)} + \Sigma[b_i^{(k,\epsilon)} - b_i^{(k)}]\varphi_{x_i}^{(k,\epsilon)} \\ &\quad \left.+ [c^{(k,\epsilon)}-c^{(k)}]\varphi^{(k,\epsilon)}\}dxdt\right| + \left|\int_\Omega[u_0^{(k)}(x) - u_0(x)]\right. \\ &\quad \left.\varphi^{(k,\epsilon)}(x,0)dx\right| + \left|\int_{\partial\Omega X[0,T]}[\psi^{(k)}-\psi]\Sigma a_{ij}^{(k)}\frac{\partial\varphi^{(k,\epsilon)}}{\partial N}\right. \\ &\quad \left.\cos(N,s_i)\cos(N,s_j)dsdt\right| \longrightarrow 0 \end{aligned}$$

when $\epsilon \to 0$, $k \to \infty$ by applying (4), (5) and (6) . Hence

$$\int_Q (u-\tilde{u})U dx dt = 0 , \quad u = \tilde{u}$$

i.e. the uniqueness of the generalized solutions follow.

OPEN PROBLEMS The properties of the generalized solution in the higher space dimensional case have had very little study.

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