STABLE HARMONIC MAPS WITH VALUES IN $\mathbb{P}^m\mathbb{C}$

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0. INTRODUCTION

Let $f : (M,g) \to (N,h)$ be a stable harmonic map between Kähler manifolds: examples of such maps are holomorphic or antiholomorphic maps and it is important to know when this is the only possible case.

We present here some results on the complex analytic character of stable harmonic maps in the special case where N is the complex projective space $\mathbb{P}^m(\mathbb{C})$ equipped with the Fubini-Study metric and M is low-dimensional.

The results described below start from an investigation of the second variation formula of the energy of f and the information one can get about $\operatorname{Im} f$, using the special form of the curvature tensor of $\mathbb{P}^m(\mathbb{C})$. In the case the differential f_* is generically injective it turns out that f induces, locally on M, an integrable almost complex structure compatible with the metric; i.e. a local complex section of the Twistor Space $Z(M)$ over M. An analysis of the number of such sections allows us to draw conclusions.

These results have been announced in [2]; details and proofs will appear in a subsequent paper.

1. LOCAL CONSEQUENCES OF STABILITY

Let (M,g) be a Riemannian oriented compact manifold and let $N = G/H$ be a compact irreducible Hermitian symmetric space equipped with complex structure tensor $\hat{J}$ and canonical connection; let

$f : M \to N$ be a harmonic map and $E = f^{*}TN$ the induced bundle with induced connection ∇^E. The second variation of the energy of f is a quadratic form on $\mathscr{C}(E)$,

$$Q(v) = \int_M H_p(f)(v)d\mu(p) \quad ,$$

where $H_p(f)(v) = |\nabla^E v|^2(p) - \sum_{i=1}^{q} K_N(f_*(x_i),v,f_*(x_i),v)(f(p))$

K_N is the curvature tensor of N and $\{x_1,\ldots x_q\}$ is an orthonormal basis of T_pM.

Using the special structure of N and in particular of its curvature tensor, we have the following:

LEMMA 1.1

Let $\{v_j\}$ *be an orthonormal basis, with respect to the Killing form, of the algebra of Killing vector fields on* N. *Then for any* $p \in M$,

$$\sum_j H_p(f)(\hat{J}f^{*}v_j) = 0 \quad .$$

LEMMA 1.2

If f *is stable, then*

$$[L,\hat{J}] = 0$$

where L *is the Jacobi operator induced by* f ,

$$L(v) = -\sum_{i=1}^{q} \nabla^E_{x_i} \nabla^E_{x_i} v + \nabla^E_{\Sigma\nabla_{x_i}x_i} v - \sum_{i=1}^{q} K_N(f_*(x_i),v)f_*(x_i)$$

In particular, from **Lemma 1.2** it follows that if v is a Jacobi field along f, then $\hat{J}v$ is a Jacobi field along f.

Now
$$f_* : T_pM \to T_{f(p)}N \qquad \text{induces}$$
$$^tf_* : T_{f(p)}N \to T_pM \; .$$

From now on, let us consider the special case $N = \mathbb{P}^m(\mathbb{C})$. Since the curvature tensor of the complex projective space is

$$K_{\mathbb{P}^m(\mathbb{C})}(X,Y,X,Y) = \frac{1}{4}\left(\|X\|^2\|Y\|^2 - \langle X,Y\rangle - 3\langle X,\hat{J}Y\rangle\right) ,$$

from **Lemma 1.2** we find

LEMMA 1.3

If $f : (M,g) \to (\mathbb{P}^m(\mathbb{C}), \textit{Fubini-Study})$ *is a stable harmonic map, then*

$$[f_*{}^tf_*, \hat{J}] = 0 \quad ,$$

and

COROLLARY 1.4

With the assumptions of **Lemma 1.3** we have :

(a) for every $p \in M$, $\operatorname{Im} f_*(p)$ is $\hat{J}$-invariant

(b) $\operatorname{rank}_{\mathbb{R}} f_*(p) \equiv 0 (\operatorname{mod} 2)$

(c) if f_* is locally injective, then $J = f_*^{-1} \circ \hat{J} \circ f_*$ is a locally integrable almost complex structure on M which is compatible with the metric g (and of course f is $(J,\hat{J})$-holomorphic)

From **Corollary 1.4** (c) and the fact that on a Riemann surface the conformal structure completely determines the complex structure, we obtain :

PROPOSITION 1.5

Stable harmonic maps from a compact Riemann surface to $\mathbb{P}^m(\mathbb{C})$ are holomorphic or antiholomorphic.

2. THE ROLE OF THE TWISTOR SPACE

From now on, suppose that $\dim_{\mathbb{R}} M \leq 2m$ and $\mathrm{rank}_{\mathbb{R}} f_*$ attains its maximum somewhere, so that $q = 2n$.

Let $Z_+(M)$ (resp. $Z_-(M)$) be the Twistor Space over M; i.e. the bundle of all almost complex structures on $T(M)$ which are compatible with the given metric and induce the same (resp. opposite) orientation, equipped with their canonical almost complex structure (cf e.g. [1]). Let M_+ (resp. M_-) be the open subset of M where f_* is injective and where the orientation induced by J agrees (resp. disagrees) with the original orientation of M.

We can restate the results of **Corollary 1.4** in terms of the Twistor structure. In fact the structure J gives rise to a section $\sigma_{\pm}(M)$ of $Z_{\pm}(M)$ over $M_{\pm}$ and we have:

LEMMA 2.1

$\sigma_{\pm}(M)$ is a complex submanifold of $Z_{\pm}(M)$ and the Nijenhuis Tensor of $Z_{\pm}(M)$ vanishes along $\sigma_{\pm}(M)$ i.e. for every $p \in M_+ \cup M_-$

$$\text{(\#)} \qquad \sigma(J)[W,\sigma(J)]\sigma(J) = 0 \quad \text{at } p ,$$

where $W : \Lambda^2(M) \to \Lambda^2(M)$ is the Weyl tensor of M, i.e. the conformal invariant part of the curvature operator of M and σ is the spin representation of $\mathrm{End}(T_xM)$ in $\mathrm{End}(\Lambda^2_x(M))$ given by $\sigma(A) = A \circ I$.

3. THE RESULTS

Assume now :

i) M is a Kähler surface with complex structure $\check{J}$

ii) $M_+ \neq \emptyset$

Then W splits as $W_+ + W_-$ and, over M_+ (#) is equivalent to:

$$(\#\#) \qquad \sigma(J)[W_+,\sigma(J)]\sigma(J) = 0 \ ,$$

and (##) is satisfied only for $J = \pm \check{J}$ unless $W_+ \equiv 0$. Since $W_+ \equiv 0$ if and only if the scalar curvature R of M is identically zero, we obtain :

THEOREM 3.1

Let M be a Kähler surface with $R \equiv 0$: then any stable harmonic map $f : M \to \mathbb{P}^m(\mathbb{C})$ such that $M_+ \neq \emptyset$ is holomorphic or antiholomorphic.

In the case $R \equiv 0$, the conclusions one can draw depend on the explicit knowledge of the global geometry of $Z_+(M)$. In fact we have:

LEMMA 3.2

If $R \equiv 0$ *and* $M_+ \neq \emptyset$, *then* $M_- = \emptyset$ *and the closure of* $\sigma_+(M)$ *in* $Z_+(M)$ *is a complex analytic sub-variety of* $Z_+(M)$.

In particular if M is a minimal Kähler surface, one can prove the only possible cases with $R \equiv 0$ are :

(1) flat metric on M and a finite covering of M is a flat complex torus.

(2) Ric $(M) \equiv 0$ but M is not flat: M is a K3 or Enriques surface with a Calabi-Yau metric.

(3) M is a ruled surface over a compact curve of genus ≥ 2.

In particular for (1) and (2) $Z_+(M)$ admits a 2-real parameter family of analytic cycles which are sections of the Twistor fibration and so correspond to global complex structures on M compatible with metric and orientation, and for (3) the only global analytic cycles are $\pm\check{J}$ Therefore we have :

THEOREM 3.3

Let M be a minimal Kähler surface with $R \equiv 0$ and let $f : M \to \mathbb{P}^m(\mathbb{C})$ be a stable harmonic map such that $M_+ \neq \emptyset$: then

i) if $Ric(M) = 0$, f is holomorphic with respect to one of the complex structures in the Twistor family.

ii) if $Ric(M) \neq 0$, f is holomorphic or antiholomorphic.

REFERENCES

[1] M.F. Atiyah, N. Hitchin, I.M. Singer, *Self-duality in four dimensional Riemannian geometry*, Proc.Roy.Soc.London 1978, A362, 4425-461.

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