SOLVABILITY OF DIFFERENTIAL OPERATORS II: SEMISIMPLE LIE GROUPS

F.D. BATTESTI and *A.H. DOOLEY*

1. INTRODUCTION

Let G be a noncompact, connected, semisimple Lie group with finite centre and P a differential operator on G. P is <u>left invariant</u> if for all $g \in G$ and for all $f \in C^\infty(G)$, $PL_g f = L_g Pf$, where L_g denotes left translation. Similarly one defines <u>right invariance</u> and <u>bi-invariance.</u>

Elements X of the Lie algebra $\underline{\underline{g}}$ act on $C^\infty(G)$ by

$$(Xf)(g) = \frac{d}{dt}\Big|_{t=0} f(g \exp tX)$$

These define left invariant operators and in fact every left invariant operator is obtained by the extension of this map to the universal enveloping algebra $U(\underline{\underline{g}}_C)$. The bi-invariant operators correspond to the centre $Z(\underline{\underline{g}}_C)$ of $U(\underline{\underline{g}}_C)$.

DEFINITION A differential operator P has <u>fundamental solution</u> E if E is a distribution (in $\mathfrak{D}'$) such that $PE = \delta_e$.

We say that P is <u>locally solvable</u> if $PC^\infty(V) \supseteq C_c^\infty(V)$ for some neighbourhood V of the identity, and P is <u>globally solvable</u> if $PC^\infty(G) = C^\infty(G)$.

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2. HISTORY

Some of the major results concerning these concepts are as follows.

THEOREM (Helgason [6]) *(i) We have local solvability for all* $P \in Z(\mathfrak{g})$.

(ii) For all G*-invariant operators on* G/K, $PC^{\infty}(G/K) = C^{\infty}(G/K)$, *where* K *denotes a maximal compact subgroup of* G.

EXAMPLE (Cerezo and Rouvière [4]) If G is complex, the imaginary part of the complex Casimir operator belongs to $Z(\mathfrak{g})$ but is not globally solvable.

(In fact, this operator reduces to the well-known example of H. Levy of an operator without solution.)

This example notwithstanding, there are a number of results concerning the solvability of the real Casimir operator C.

THEOREM (Rauch and Wigner [9]) C *is always globally solvable.*

THEOREM (Benabdallah-Rouvière, Johnson) *(i) If* G *has no discrete series, then* C *has a central fundamental solution.*

(ii) If G *has a discrete series, then* C *has a fundamental solution on* G.

The paper [3] actually constructs a fundamental solution; Benabdallah [2] has shown that on $SL(2,\mathbb{R})$, C can have no central fundamental solution.

THEOREM (Rouvière [10]) *The equality* $C\mathcal{D}' = \mathcal{D}'$ *holds if* $G = SL(2,\mathbb{R})$ *or* $SL(2,\mathbb{C})$ *but fails if real rank* $G > 1$.

As to other invariant operators, we have

THEOREM (Cerezo-Rouvière [4]) *If* G *is complex and* p *is a polynomial then* $p(C)$ *has a fundamental solution and is globally solvable.*

Finally, one result on general operators. Recall the Harish-Chandra isomorphism for a Cartan subgroup H of G. This provides a map γ_H : $Z(\underline{\underline{g}}_C) \to S_W(\underline{\underline{h}}_C)$ (where the image is the Weyl group invariant elements of the symmetric algebra of $\underline{\underline{h}}_C$.)

THEOREM (Benabdallah-Rouvière [3]). *Suppose* $P \in Z(\underline{\underline{g}})$ *and suppose there is a* θ*-stable Cartan subgroup* H *of* G *such that* $\gamma_H(P)$ *has a fundamental solution on* H. *Then* P *has a central fundamental solution on* G, *and* $PC^\infty(G) = C^\infty(G)$.

This result allows one to give a sufficient condition for solvability in terms of the Cerezo and Rouvière necessary and sufficient condition for solvability on H. However, considering the example of the Casimir on $SL(2,\mathbb{R})$, the existence of a central fundamental solution seems a rather special property.

In conclusion, we should like to mention some recent work of Rouvière [11] and G. Lion [8] where conditions for existence of solutions on symmetric spaces are considered in a rather general setting.

3. GROUP CONTRACTIONS

Our program for studying solvability of differential operators on G is based on the use of group contractions to transfer the operators to the Cartan motion group $V \rtimes K$ associated to G. Then the results of [1] can be applied. This general approach has something in common with §6 of [11], where the passage between the symmetric spaces G/K and $(V \rtimes K)/K$ is considered.

Choosing a maximal compact subgroup K of G one may write $\underline{\underline{g}} = \underline{\underline{k}} + \underline{\underline{p}}$, where $\underline{\underline{p}}$ is a vector space complement for $\underline{\underline{k}}$. Letting V be $\underline{\underline{p}}$ considered as a vector group, we take the semidirect product

$V \rtimes K$ relative to the adjoint representation. This is the associated Cartan motion group. (If $G = SL(2,\mathbb{R})$ then $V \rtimes K$ is the Euclidean motion group $M(2)$.)

The contraction maps $\pi_\lambda : V \times K \to G : (v,k) \to \exp\frac{v}{\lambda}.k$ are approximate homomorphisms in the sense that

$$\pi_\lambda^{-1}(\pi_\lambda(x)\pi_\lambda(y)) \to xy \qquad \text{as} \qquad \lambda \to \infty.$$

These maps have been used, for example in [5] to transfer harmonic analysis from G to $V \rtimes K$. We shall use them to transfer differential operators.

DEFINITION Let P be an element of $\underset{=}{U}(g)$. Define P_λ, a linear operator on $C^\infty(V \rtimes K)$, by

$$(P_\lambda f) = P(f \circ \pi_\lambda^{-1}) \circ \pi_\lambda.$$

We have

PROPOSITION 1 *(i) Suppose* P_λ *has a fundamental solution* E_λ *and define* $F_\lambda \in \mathcal{D}'(G)$ *by*

$$\langle F_\lambda, f\rangle = \langle E_\lambda, f \circ \pi_\lambda\rangle$$

Then F_λ *is a fundamental solution for* P.

(ii) Suppose P *has a fundamental solution* F. *Then* $E_\lambda \in \mathcal{D}'(V \rtimes K)$, *defined by*

$$\langle E_\lambda, g\rangle = \langle F, g \circ \pi_\lambda^{-1}\rangle$$

is a fundamental solution for P_λ.

PROOF One calculates that for $f \in \mathcal{D}(G)$

$$\langle PF_\lambda, f\rangle = \langle F_\lambda, {}^tPf\rangle = \langle E_\lambda, ({}^tPf) \circ \pi_\lambda\rangle.$$

$$\text{Now} \quad ({}^tPf) \circ \pi_\lambda = ({}^tP)_\lambda (f \circ \pi_\lambda) = {}^t(P_\lambda)(f \circ \pi_\lambda).$$

Thus $$\langle PF_\lambda, f\rangle = \langle P_\lambda E_\lambda, f \circ \pi_\lambda\rangle.$$

If E_λ is a fundamental solution for P_λ, then the right hand side is $f(\pi_\lambda(e)) = f(e)$, and so F_λ is a fundamental solution for P. If, on the other hand, F is fundamental solution for P then the same calculation gives, for $g \in \mathcal{D}(V \rtimes K)$,

$$g(e) = \langle PF, g \circ \pi_\lambda^{-1} \rangle = \langle P_\lambda E_\lambda, g \rangle .$$

The hard work in the proof involves showing in (i) that $F_\lambda \in \mathcal{D}'(G)$ and in (ii), that $E_\lambda \in \mathcal{D}'(V \rtimes K)$. ▯

This proposition allows us to pass between G and $V \rtimes K$ for each fixed λ. However, it is not very satisfactory since even if P is a bi-invariant operator on G, P_λ will usually not even be left invariant on $V \rtimes K$. Thus, there is no good method for finding a fundamental solution for P_λ. In fact, using the computer package MACSYMA, an explicit expression has been calculated for C_λ, where C is the Casimir operator on $SL(2,\mathbb{R})$. This expression currently runs to around sixty pages of computer printout and is apparently not left invariant. The problem occurs with differentiations in the V direction. For K-directions, things work out nicely.

LEMMA 2 *Suppose that* P *is left (resp. right)* K*-invariant on* G. *Then* P_λ *is left (resp. right)* K*-invariant on* $V \rtimes K$.

PROOF Let $k_0 \in K$. Then

$$k_0\pi_\lambda(v,k) = k_0 \exp\frac{v}{\lambda}.k = \exp\frac{k_0.v}{\lambda}.k_0k = \pi_\lambda((0,k_0)(v,k))$$

and

$$\pi_\lambda(v,k).k_0 = \pi_\lambda((v,k)(0,k_0)).$$

These facts, combined with the definition of P_λ, prove the lemma.▯

The fact that π_λ is not a homomorphism is, of course, the reason for this failure to preserve invariance of the operator. Nevertheless, π_λ is an approximate homomorphism and, relying on this fact, one may expand P_λ in a power series in λ. It is of interest to identify the dominant term in such an expansion. The derivative of the map π_λ is the map

$$\phi_\lambda : V \oplus \underline{\underline{k}} \to \underline{\underline{g}} : Y + X \to \frac{1}{\lambda}Y + X,\ Y \in V,\ X \in \underline{\underline{k}}.$$

This map extends to a linear map, also denoted ϕ_λ: $U(V\oplus\underline{\underline{k}}) \to U(\underline{\underline{g}})$. Note that ϕ_λ is not a Lie algebra homomorphism.

LEMMA 3 $P_\lambda = \phi_\lambda^{-1}(P)$ + *lower order terms.*

PROOF If $X \in \underline{\underline{k}}$, the previous lemma shows that $X_\lambda = X$. On the other hand, if $Y \in V$,

$$(Y_\lambda f)(v,k) = \frac{d}{dt}\Big|_{t=0} f(\pi_\lambda^{-1}(\exp\tfrac{v}{\lambda}.k.\exp tY))$$

A rather messy calculation based on the Campbell-Baker-Hausdorff formula shows that this may be expanded as

$$\lambda(Yf + O(\tfrac{1}{\lambda})) = \phi_\lambda^{-1}(Y) + \text{lower order terms.}$$

The general case now follows. ▯

In general, the terms of this series are not left invariant, even if $P \in Z(\underline{\underline{g}})$, although they do share the K-invariance properties of P. The leading term of the series, which is part of $\phi_\lambda^{-1}(P)$ is an exception.

LEMMA 4 *(i) Suppose* $P \in U(\underline{\underline{g}})$. *Then* $\phi_\lambda^{-1}(P) \in U(V \oplus \underline{\underline{k}})$

(ii) Suppose $P \in Z(\underline{\underline{g}})$ *Then the leading term of* P_λ *belongs to* $Z(V \oplus \underline{\underline{k}})$.

PROOF (i) is clear.

(ii) results from the fact that π_λ is an approximate homomorphism. Thus for $x \in V \rtimes K$ and left invariant P,

$$L_x(P_\lambda f)(y) = (P_\lambda f)(xy) = P(f \circ \pi_\lambda^{-1})(\pi_\lambda(xy))$$
$$= P(f \circ \pi_\lambda^{-1})(\pi_\lambda(x)\pi_\lambda(y)) + \text{lower terms}$$

$$= L_{\pi_\lambda(x)}\, P(f \circ \pi_\lambda^{-1})(\pi_\lambda(y)) + \text{lower terms}$$

$$= P\, L_{\pi_\lambda(x)}\, (f \circ \pi_\lambda^{-1}))(\pi_\lambda(y)) + \text{lower terms.}$$

Now by the mean value theorem,

$$L_{\pi_\lambda(x)}(f \circ \pi_\lambda^{-1})(\pi_\lambda(z)) = f(\pi_\lambda^{-1}(\pi_\lambda(x)\pi_\lambda(z)))$$

$$= f(xz) + \text{lower terms.}$$

Thus $L_{\pi_\lambda(x)}(f \circ \pi_\lambda^{-1}) = (L_x f) \circ \pi_\lambda^{-1}$ + lower terms.

From this it follows that $L_x(P_\lambda f) = P_\lambda(L_x f)$ + lower terms.

This proves that the leading term of P_λ is left invariant. Similarly, if P is right invariant, so is its leading term. ▯

Consider the example of the Casimir operator on $SL(2,\mathbb{R})$, $C = X^2 + Y^2 - T^2$. This is bi-invariant. The operator $\phi_\lambda^{-1}(C)$ on $M(2)$ is $\lambda^2(X^2+Y^2) - T^2$. The leading term, $\lambda^2(X^2+Y^2)$ is a bi-invariant operator on $M(2)$.

The results of [1] allow us to find a solution for any bi-invariant operator on $V \rtimes K$, and we are currently working on ways of finding fundamental solutions for other operators.

However, in the case where fundamental solutions for $\phi_\lambda^{-1}(P)$ do exist (for example, if $\phi_\lambda^{-1}(P)$ is bi-invariant), we have the following results.

PROPOSITION 5 *Suppose that* $\phi_\lambda^{-1}(P)$ *has a fundamental solution* $\tilde{F}_\lambda \in \mathcal{D}'(V \rtimes K)$.

The formula $\langle \tilde{E}_\lambda, f\rangle = \langle \tilde{F}_\lambda, f \circ \pi_\lambda\rangle$ *defines a distribution* $\tilde{E}_\lambda \in \mathcal{D}'(G)$.

(The proof of this proposition is similar to that of Proposition 1).

PROPOSITION 6 *Suppose that there is a sequence* $\lambda_j \to \infty$ *so that*

(i) For all j, *a fundamental solution* $\tilde{F}_{\lambda_j} \in \mathcal{D}'(V \rtimes K)$ *exists for* $\phi_{\lambda_j}^{-1}(P)$.

(ii) The sequence $(\tilde{E}_{\lambda_j})_j$ *defined in Proposition 5 converges in* $\mathcal{D}'(G)$ *to a distribution* E.

Then E *is a fundamental solution for* P.

PROOF Omitting details of the estimates, the proof boils down to the following simple calculation.

$$\begin{aligned}
\langle PE,f\rangle = \langle E,{}^tPf\rangle &= \lim_{j\to\infty} \langle \tilde{E}_{\lambda_j},{}^tPf\rangle \\
&= \lim_{j\to\infty} \langle \tilde{F}_{\lambda_j},({}^tPf)\circ \pi_{\lambda_j}\rangle \\
&= \lim_{j\to\infty} \langle \tilde{F}_{\lambda_j},{}^tP_{\lambda_j}(f\circ \pi_{\lambda_j})\rangle \\
&= \lim_{j\to\infty} \langle P_{\lambda_j}\tilde{F}_{\lambda_j},f\circ \pi_{\lambda_j}\rangle \\
&= \lim_{j\to\infty} (\langle \phi_\lambda^{-1}(P)\tilde{F}_{\lambda_j},f\circ \pi_{\lambda_j}\rangle + \text{lower terms in } \lambda_j) \\
&= \lim_{j\to\infty} (f(\pi_{\lambda_j}(e)) + O(\tfrac{1}{\lambda_j})) \\
&= f(e)
\end{aligned}$$ ▯

4. INVARIANT OPERATORS

In this section, we show how to find a fundamental solution for operators P on G such that $\phi_\lambda^{-1}(P)$ is bi-invariant. In this case, a solution $\tilde{F}_\lambda$ may be explicitly written down for $\phi_\lambda^{-1}(P)$, using essentially Hörmander's formula. Let $\underline{\underline{g}} = \underline{\underline{k}} + V$ be the Cartan decomposition of $\underline{\underline{g}}$, let $X_1,\dots,X_n$ be an orthonormal basis in V. Let P be a polynomial in $X_1^2 + \dots + X_n^2$. Then $\phi_\lambda^{-1}(P)$ is $P^\lambda(X_1,\dots,X_n) = P(\lambda^2(X_1^2+\dots+X_n^2))$, considered as an operator on $V \rtimes K$.

A fundamental solution $\tilde{F}_\lambda$ is then given by

$$\langle \tilde{F}_\lambda,\phi\rangle = \frac{1}{(2\pi)^{\dim V}} \int_V d\xi \int_{V^C} \frac{(\tilde{\phi})^\wedge(-\xi-\zeta)}{P^\lambda(\xi+\zeta)}\, \Psi(P_\xi^\lambda,\zeta)d\lambda(\zeta)$$

Here, $\tilde{\phi}(v) = \int_K \phi(v,k)dk$ for $\phi \in \mathcal{D}(V\rtimes K)$ and $\hat{}$ denotes the Euclidean Fourier transform in V. The Hörmander function $\Psi(P_\xi^\lambda,\cdot-\xi)$ is chosen to be analytic, to have integral one, and to vanish when P^{λ}_{ξ} vanishes, c.f. [12], lemma 7.3.12. In our case, we may also assume that Ψ is invariant under the action of K on V.

Choose a maximal abelian subalgebra $\underline{\underline{a}}$ of V.

The measure in V may be decomposed as

$$dv = \pi_{\alpha\varepsilon P_+} \alpha(H)\, dH\, dk$$

where $v = k.H$, $H \varepsilon \underline{\underline{a}}$. After several changes of variables, and using the K-invariance of $\phi^{-1}_{\lambda}(P)$, we see that

$$\langle\tilde{F}_{\lambda},\phi\rangle = \frac{1}{(2\pi)^{\dim V}} \int_V d\xi \int_{\underline{\underline{a}}+C} \frac{(\tilde{\tilde{\phi}})\hat{}(H)}{P^{\lambda}(H)} \Psi(P^{\lambda}_{\xi},H-\xi) \pi_{\alpha\varepsilon P_+} \alpha(H)dH$$

where $\tilde{\tilde{\phi}}(H) = \int_k \tilde{\phi}(k.H)dk$.

We thus have, for $\beta \,\varepsilon \mathcal{D}(G)$,

$$\langle\tilde{E}_{\lambda},\beta\rangle = \langle\tilde{F}_{\lambda},\ \beta \circ \pi_{\lambda}\rangle$$

An easy calculation shows that

$$(\beta \circ \pi_{\lambda})\hat{} = \lambda^{\dim V}\, \tilde{\tilde{\beta}}(\lambda H),$$

where we note that $\tilde{\tilde{\beta}}(g) = \int_K \int_K \beta(k,gk_2)dkdk_2$ depends only on the "A" in the $KA^{+}K$ decomposition of G, and by abuse of notation we have written $\tilde{\tilde{\beta}}(H) = \tilde{\tilde{\beta}}(\exp H)$ for $H \varepsilon \underline{\underline{a}}^{+}$.

Applying this to Hörmander's formula, we find that

$$\langle\tilde{E}_{\lambda},\beta\rangle = \frac{1}{(2\pi)^{\dim V}} \int_V d\xi \int_{\underline{\underline{a}}+C} \lambda^{\dim V} \frac{(\tilde{\tilde{\beta}})\hat{}(H)}{P(\lambda H)} \Psi(P^{\lambda}_{\xi},H-\xi) \pi_{\alpha\varepsilon P_+} \alpha(H)dH$$

$$= \frac{1}{(2\pi)^{\dim V}} \int_V d\xi \int_{\underline{\underline{a}}+C} \lambda^{\dim \underline{\underline{a}}} \frac{(\tilde{\tilde{\beta}})\hat{}(H)}{P(H)} \Psi(P^{\lambda}_{\xi},\frac{H}{\lambda}-\xi) \pi_{\alpha\varepsilon P_+} \alpha(H)dH$$

A careful choice of the function Ψ enables one to see when this limit exists; hence we can discuss solvability of these non bi-invariant operators.

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School of Mathematics
The University of New South Wales
P.O. Box 1, Kensington NSW 2033.