FOURIER TRANSFORM ASSOCIATED WITH HOLOMORPHIC DISCRETE SERIES

Takeshi Kawazoe

1. INTRODUCTION

The Plancherel formula on semisimple Lie groups G implies that each L^2 function f on G has a decomposition: $f = f_P + {}^\circ f$, where f_P consists of wave packets and $^\circ f$ the discrete part of f, that is, a linear combination of the matrix coefficients of the discrete series of G. We assume that $\Omega=G/K$, K is a maximal compact subgroup of G, is one of classical bounded symmetric domains. Then we shall give a characterization of $^\circ L^p(G)$ $(1\leqq p\leqq 2)$, the discrete part of L^p functions on G, by using the Fourier transform associated with the holomorphic discrete series realized on a Bergman space on Ω. This characterization is related to the theory of the weighted Bergman spaces on Ω and the fractional derivatives of holomorphic functions on Ω.

In this introduction we shall treat the case of Ω= the open unit disk; in the rest of two sections we shall state the generalization on bounded symmetric domains of classical type.

First we shall recall the Fourier transform on the open unit disk $D=\{ z \in C ; |z| < 1 \}$. For $\lambda \in \mathbf{R}$ and $b \in \partial D$, the boundary of D, it is given by

$$\hat{f}(\lambda,b) = \int_D f(z)\left(\frac{1-|z|^2}{|z-b|^2}\right)^{\frac{1}{2}(-i\lambda+1)} dz.$$

As well known, we can identify D with the symmetric space G/K, where G= SU(1,1) and K=SO(2). By this identification G acts on D transitively and a function f(z) on D corresponds to the function $\tilde{f}(g)$ on G given by $\tilde{f}(g)= f(g\cdot 0)$, where $0 \in D$ and "$\cdot$" means the action of G. Then we can rewrite the above integral as follows.

$$\hat{f}(\lambda,b) = \int_D \tilde{f}(g) V_{0,\frac{1}{2}(-i\lambda+1)}(g^{-1}) 1(b)\, dg,$$

where $(V_{j,s}, L^2(\partial D))$ ($j=0$, $\frac{1}{2}$ and $s \in C$) is the principal series representation of G (cf. [Su], p.212) and 1(b) is the constant function in $L^2(\partial D)$ taking value 1. Then this transform has the same properties as the Euclidean Fourier transform; if we regard it as the transform of functions on G by the identification, we see that the image of $L^2(G)$ is exactly given by $L^2(R^+ \times \partial D,\ \lambda \mathrm{th}\frac{1}{2}\lambda d\lambda db)$, however, the one for $L^p(G)$ $(1 \leq p < 2)$ is not clear.

Now we shall recall that G has other irreducible representations, namely, the so called holomorphic discrete series : $(T_n, A_{2,n-1}(D))$ ($n \in \frac{1}{2}\mathbf{Z}$ and $n \geq 1$), where $A_{2,n-1}(D)$ is the weighted Bergman space which will be define below. Then we reach the following problem.

Problem *Let F_n be the Fourier transform associated with the discrete series T_n defined by*

$$F_n(f)(z) = \int_G f(g) T_n(g^{-1}) 1(z)\, dg.$$

Then what is the image $F_n(L^p(G))$ of $L^p(G)$?

Before stating the answer we shall give the definition of the weighted Bergman space on D and recall the fractional derivatives of holomorphic functions on D. For $0<p<\infty$ and $r \in \mathbf{R}$ we put

$$A_{p,r}(D) = \{F : D \to C \ ;\ \text{(i)}\ F \text{ is holomorphic on } D \quad \text{(ii)}\ \|F\|_{p,r}^p = \int_D |F(z)|^p (1-|z|^2)^{2r}\, dz < \infty\}.$$

Since it consists of holomorphic functions on D, $A_{p,r}(D) = \{0\}$ if $r \leq -\frac{1}{2}$.

Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be a holomorphic function on D and $\alpha \geqq 0$. Then the fractional derivative $F^{[\alpha]}$ of F of order α is defined by

$$F^{[\alpha]}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)} a_n z^n.$$

Then Duren, Romberg and Shields obtained the following,

Theorem ([DRS]) *If $r > -\frac{1}{2}$, $\alpha \geqq 0$ and $F \in A_{1,r}(D)$, then $F^{[\alpha]} \in A_{1,r+\frac{1}{2}\alpha}(D)$.*

This theorem gives a relation between weighted Bergman spaces and fractional derivatives. Moreover, the answer of the problem gives a group theoretical interpretation of this theorem.

Theorem (Answer) *Let $n \in \frac{1}{2}Z$, $n \geqq 1$ and $1 \leqq p \leqq 2$.*

If $(n,p) \neq (1,1)$, then $F_n(L^p(G)) = A_{p,\frac{1}{2}np-1}(D)$.

If $(n,p) = (1,1)$, then $F_1(L^1(G)) = H_0^1(D)$.

Here the space $H_0^1(D)$ is defined as follows. First we shall fix a positive number α. Then

$$H_0^1(D) = \{F : D \to C \ ; \ \text{(i)} \ F \text{ is holomorphic on } D \quad \text{(ii)} \ F^{[\alpha]} \in A_{1,\frac{1}{2}(\alpha-1)}(D) \ \}.$$

The theorem of [DRS] asserts that the second condition (ii) does not depend on α, that is, the definition of $H_0^1(D)$ is independent of $\alpha > 0$; for example, we can replace (ii) by (ii)' $\frac{\partial F}{\partial z} \in A_{1,0}(D)$. On the other hand the answer-the above theorem-also asserts the independence, because the left hand side $F_1(L^1(G))$ does not depend on $\alpha > 0$. In this sense the answer gives a group theoretical interpretation of the theorem of [DRS]. This is important in generalizing the classical theorem to other bounded symmetric domains.

Next we shall give a sketch of the proof. The proof is based on some properties of the matrix coefficients of the discrete series T_n. We put

$$e_n^m(z) = B(m+1,2n-1)^{\frac{1}{2}} z^m \quad (m=0,1,2, \ldots).$$

Then $\{ e_n^m(z) \; ; \; m \in \mathbf{N} \}$ is a complete orthonormal system of $A_{2,n-1}(D)$ and the normalized matrix coefficients of T_n are given by

$$f_{pq}^n(g) = [T_n(g)e_n^q, e_n^p]_{n-1} / \| [T_n(\cdot)e_n^q, e_n^p]_{n-1} \|_2,$$

where $[\ ,\ ]_{n-1}$ is the inner product in $A_{2,n-1}(D)$ and $\| \cdot \|_p$ the L^p norm on G. Then these matrix coefficients satisfy the following properties.

Facts

(1) $f_{pq}^n * f_{p'q'}^{n'} = c_n \delta_{nn'} \delta_{qp'} f_{pq'}^n$

(2) If $n > 1$, then $f_{pq}^n \in L^1(G)$.
If $n = 1$, then $f_{pq}^1 \notin L^1(G)$, but $\in L^q(G)$ for all $q > 1$.

(3) We put $f_m^n = f_{m0}^n$ and $\psi_n = f_{00}^n$. Then $F_n(f_m^n)(z) = c_n e_n^m(z)$.

(4) $F_n(f)(z) = F_n(f * \psi_n)(z)$.

(5) $\| F_n(f) \|_{p, \frac{1}{2}np-1} = c_n \| f * \psi_n \|_p$ (if finite).

(1) and (2) are easy consequence from the explicit forms of the matrix coefficients which are given by hypergeometric functions (cf. [Sa]); (3) and (4) follow from (1) and the relation: $T_n(g^{-1})1(z) = \sum_{m=0}^{\infty} f_m^n(g) e_n^m(z)$; (5) follows from (4) and the integral formula on G corresponding to the Cartan decomposition of G (cf. [Su], p.252).

If the norm in (5) is finite, that is, the convolution operator: $f \to f*\psi_n$ is L^p-bounded on G, then the desired characterization of $F_n(L^p(G))$ is given by the weighted Bergman space $A_{p,\frac{1}{2}np-1}(D)$. Actually, except for case of $(n,p)=(1,1)$, we have the following,

Proposition *Let $n \in \frac{1}{2}Z$, $n \geqq 1$, $1 \leqq p \leqq 2$ and $(n,p) \neq (1,1)$. Then*

$$\| f * \psi_n \|_p \leqq c_n \| f \|_p \quad \text{for } f \in L^p(G).$$

When $n > 1$, $\psi_n \in L^1(G)$ by (2), and thus, the inequality holds as in the Euclidean case. When $n=1$, $\psi_1 \in L^q(G)$ for all $q > 1$. Then the desired one is nothing but the Kunze-Stein phenomenon on G (cf. [Cow]).

Therefore, if $(n,p) \neq (1,1)$, we easily conclude that $F_n(L^p(G)) = A_{p,\frac{1}{2}np-1}(D)$. Next we shall consider the case of $(n,p)=(1,1)$. In this case the above proposition does not hold, and the above characterization of $F_n(L^p(G))$ is nonsense, because $A_{1,-\frac{1}{2}}(D)$ consists of only 0 function. Therefore, we must think out another approach. The basic idea is the following,

Lemma *For each f_m^1 there exists an L^1 function $[f_m^1]$ on G such that*

$$[f_m^1] * \psi_1 = f_m^1.$$

Actually, for a fixed positive α- this α corresponds to the order of the fractional derivative - the desired $[f_m^1]$ is given by

$$[f_m^1](g) = c_m^\alpha |\psi_1(g)|^\alpha f_m^1(g), \text{ where}$$

$$c_m^\alpha = \frac{\Gamma(m+2+\alpha)}{\Gamma(m+2)} = \left(\int_G |\psi_1(g)|^\alpha |f_m^1(g)|^2 dg \right)^{-1}.$$

Since $|\psi_1(g)|^\alpha$ ($\alpha > 0$) decays fast, $[f_m^1]$ belongs to $L^1(G)$, and the orthogonal relation of the matrix coefficients over K deduces that $[f_m^1]*\psi_1=f_m^1$.

Now we shall prove that $F_1(L^1(G))=H_0^1(D)$. We have to show that (into): $(F_1(f))^{[\alpha]} \in A_{1,\frac{1}{2}(\alpha-1)}(D)$ for all $f \in L^1(G)$; (onto): for each $F \in H_0^1(D)$ there exists an L^1 function h on G such that $F_1(h)=F$.

(into) follows from a direct calculation.

(onto) is proved by using the Lemma. For $F(z)=\sum_{m=0}^{\infty} a_m e_1^m(z)$ in $H_0^1(D)$ we put $f(g)=\sum_{m=0}^{\infty} a_m f_m^1(g)$. Then (3) means that $F_1(f)=F$, however, this f does not belong to $L^1(G)$ (see (2)). So we need a modification of f. In fact we put $h(g)=\sum_{m=0}^{\infty} a_m [f_m^1](g)$ (see Lemma). Then we see that $F_1(h)=F_1(h*\psi_1)=F_1(f)=F$ by (4) and Lemma; $\|h\|_1=\|z^{-1}(zF)^{[\alpha]}\|_{1,\frac{1}{2}(\alpha-1)}$ by the integral formula on G. Since F belongs to $H_0^1(D)$, we easily see that the last norm is finite and thus h belongs to $L^1(G)$. Therefore, h satisfies the desired conditions.

This completes the proof of the answer to the problem in the case of $\Omega=D$.

Remark. $H_0^1(D)$ is contained in $H^1(D)$, the classical Hardy space on D; it is not equal to, but dense in $H^1(D)$ (see [K]).

2. NOTATION

Let G be a simple matrix group and K a maximal compact subgroup of G. We suppose G/K is Hermitian. Let $\underline{g}$ and $\underline{k}$ be the Lie algebras of G and K, and $\underline{h}$ a maximal abelian subalgebra of $\underline{k}$. We denote the complexification of an algebra $\underline{a}$ by $\underline{a}_C$. Let Σ be the set of non zero roots for the pair $(\underline{g}_C, \underline{h}_C)$ equipped with an order in which every non compact positive root is larger than every compact root. Let $\underline{n}_C^{\pm}$ be the sum of the $\pm$

root spaces (± refers to "positive and negative"), $\underline{p}^{\pm}$ the sum of the non compact ± root spaces and $\underline{b} = \underline{h}_C + \underline{n}_C^-$. Let G_C be an analytic subgroup of the matrices with the Lie algebra $\underline{g}_C$, and $N_C^{\pm}$, $P_C^{\pm}$, B, K_C, T_C the subgroups of G_C corresponding to $\underline{n}_C^{\pm}$, $\underline{p}_C^{\pm}$, $\underline{b}$, $\underline{k}_C$, $\underline{h}_C$ respectively. Then it is well known that BG is open in G_C and there exists a bounded open set Ω in P^+ such that $BG = P^-K_CG = P^-K_C\Omega$. Moreover G acts transitively on Ω as an holomorphic automorphism on Ω and $G \cap P^-K_C = K$ is the subgroup fixing 1 in Ω. This gives the identification: $\Omega = K \backslash G$.

Let $\Lambda \in (\underline{h}_C)^*$, the set of complex linear functionals on $\underline{h}_C$, and suppose that Λ is an integral form on $\underline{h}_C$, dominant with respect to $\underline{k}$. Let $(\tau_\Lambda, V_\Lambda)$ be an irreducible finite dimensional representation of K with highest weight Λ. Let ϕ_Λ be the normalized highest weight vector in V_Λ and χ_Λ the associated character. For a complex valued function f on G_C we define

$$E_\Lambda(f)(x) = \int_K \chi_\Lambda(k^{-1}) f(kx)\, dk \qquad (x \in G_C).$$

Moreover, we shall define a function on $P^-K_CP^+$ which plays an important role in the following arguments as follows.

$$\psi_\Lambda(x) = (\tau_\Lambda(\mu(x))\phi_\Lambda, \phi_\Lambda)_{V_\Lambda} \qquad (x \in P^-K_CP^+),$$

where $\mu(x)$ refers to the K_C-component of x in $P^-K_CP^+$.

Now we shall define the holomorphic discrete series of G. Let Λ be as above and suppose that $\langle\Lambda+\rho,\alpha\rangle < 0$ for every non compact positive root α, where 2ρ is the sum of all positive roots. We define

$$H^p_\Lambda = \{f:BG \to \mathbf{C}\ ;\ \text{(i)}\quad f \text{ is holomorphic on } BG$$
$$\text{(ii)}\quad f(bx)=\xi_\Lambda(b)f(x) \text{ for } b \in B,\ x \in BG$$
$$\text{(iii)}\quad \|f\|^p_p = \int_G |f(g)|^p\, dg < \infty\ \}$$

and $$U_\Lambda(g)f(x) = f(xg) \quad (f \in H^p_\Lambda,\ g \in G \text{ and } x \in BG),$$

where $\xi_\Lambda(nh)=\exp\Lambda(\log(h))$ for $nh \in B=N^-_C T_C$. Then $H^2_\Lambda \neq \{0\}$, since $H^2_\Lambda \ni \psi_\Lambda \neq 0$; (U_Λ, H^2_Λ) is a continuous irreducible unitary representation of G so called the holomorphic discrete series of G. As a representation of K, H^2_Λ is decomposed into irreducible components denoted by V_{Λ_i} $(i \in \mathbf{N})$. Then we choose a complete orthonormal system ϕ^i_j $(1 \leq j \leq \dim V_{\Lambda_i})$ of H^2_Λ such that $\phi^i_j \in V_{\Lambda_i}$; we may assume that $\Lambda_1=\Lambda$ and $\phi^1_1=\|\psi_\Lambda\|^{-1}_2 \psi_\Lambda$. We denote the matrix coefficients of U_Λ as follows.

$$f^{ii'}_{jj'}(x) = (U_\Lambda(x)\phi^{i'}_{j'}, \phi^i_j)_{H^2_\Lambda} \quad (x \in G).$$

Clearly, $\psi_\Lambda = f^{11}_{11}$, and for simplicity we put $f^i_j = f^{1i}_{1j}$.

In what follows we assume that

Assumption. Ω is one of the classical bounded symmetric domains listed below and $\dim \tau_\Lambda = 1$.

Type	G	K	Ω
I	$SU(m,n)$	$S(U(m)\times U(n))$	$\{z \in M_{mn}(\mathbf{C})\ ;\ I_m - z\bar{z}' > 0\}$
II	$Sp(n,\mathbf{R})$	$U(n)$	$\{z \in M_{nn}(\mathbf{C})\ ;\ I_n - z\bar{z} > 0,\ z=z'\}$
III	$SO^*(2n)$	$Sp(n) \cap SO(2n)$	$\{z \in M_{nn}(\mathbf{C})\ ;\ I_n + z\bar{z} > 0,\ z=-z'\}$
IV	$SO(n,2)$	$SO(n)\times SO(2)$	$\{z \in \mathbf{C}^n\ ;\ \lvert zz'\rvert^2+1-2z\bar{z}' > 0,\ \lvert zz'\rvert < 1\}$.

3. MAIN RESULT

First we shall define weighted Bergman spaces on Ω. Let $w(z)$ be a positive function on Ω. Then the w-weighted L^p $(0<p<\infty)$ Bergman space on Ω is defined as follows.

$$H^p_w(\Omega) = \{F:\Omega \to \mathbf{C} \ ; \ \text{(i)} \ \ F \text{ is holomorphic on } \Omega \quad \text{(ii)} \ \ \|F\|^p_{p,w} = \int_\Omega |F(z)|^p w(z)\, dz < \infty \},$$

where dz is a Euclidean measure on Ω. Let $B(z,\bar{\zeta})$ be the Bergman kernel on Ω (cf. [Hu]). Then a G-invariant measure on Ω is given by $B(z,\bar{z})dz$. For simplicity we put

$$w^\Lambda_{p,\alpha}(z) = |\psi_\Lambda(x)|^{(1+\alpha)p} B(z,\bar{z}) \qquad (z=1\cdot x)$$

and $$H^p_\Lambda(\Omega) = H^p_{w^\Lambda_{p,0}}(\Omega).$$

Next we shall consider a realization of U_Λ on $H^2_\Lambda(\Omega)$. For a complex valued function f on BG satisfying $E_\Lambda f=f$ we define

$$I_\Lambda(f)(z) = \psi_\Lambda(g)^{-1} f(g) \qquad (z=1\cdot g),$$

and for a complex valued function F on Ω we put

$$T_\Lambda(g)F(z) = \psi_\Lambda(x)^{-1}\psi_\Lambda(xg)F(z\cdot g) \qquad (g \in G,\ z=1\cdot x).$$

Under the assumption that $\dim \tau_\Lambda = 1$ these definitions are well-defined and moreover, we see that

Lemma *Let Λ be as in §2. Then (U_Λ, H^2_Λ) and $(T_\Lambda, H^2_\Lambda(\Omega))$ are unitary equivalent and I_Λ is the norm-preserving intertwining operator of H^2_Λ onto $H^2_\Lambda(\Omega)$.*

In particular, if we put $\psi^i_j = I_\Lambda(\phi^i_j)$, $\{\psi^i_j\}$ is a complete orthonormal system of $H^2_\Lambda(\Omega)$; we note that $\psi^1_1 = \|\psi_\Lambda\|^{-1}_2$, a constant function on Ω.

Now we shall define the Fourier transform associated with the holomorphic discrete series T_Λ as follows. For a complex valued function f in $L^p(G)$ $(1 \leqq p \leqq 2)$ we put

$$F_\Lambda(f)(z) = \int_G f(g) T_\Lambda(g^{-1}) 1(z)\, dg$$

$$= I_\Lambda(\psi_\Lambda * f)(z) \qquad (z \in \Omega).$$

Before stating the main theorem we shall give two more definitions (the definition of $H^p_{\Lambda,\alpha}(\Omega)$, which appears in the theorem, will be given after the statement). Let Λ be a dominant integral form on $\underline{h}_{\underline{C}}$ and $1 \leqq p \leqq 2$.

Definition. α_Λ is the least number satisfying

$$|\psi_\Lambda|^{1+\alpha} \in L^1(G) \text{ for all } \alpha > \alpha_\Lambda.$$

Definition. We say (Λ, p) is regular if $\psi_\Lambda \in L^p(G)$.

Then the main theorem can be stated as follows.

Theorem *Let Λ be the parameter of the discrete series of G and $1 \leqq p \leqq 2$.*
If (Λ, p) is regular, then $F_\Lambda(L^p(G)) = H^p_\Lambda(\Omega)$.
If (Λ, p) is not regular, then $F_\Lambda(L^p(G)) = H^p_{\Lambda,\alpha}(\Omega)$ for $\alpha > \alpha_\Lambda$.

In the rest we shall give the definition of $H^p_{\Lambda,\alpha}(\Omega)$. First we shall define the fractional derivative $F^{[\alpha]}$ of a holomprphic function F on Ω as follows. For $\alpha > \alpha_\Lambda - 1$ and $F(z) = \sum_{\ell,m} a_{\ell m}\psi^\ell_m(z)$ we put

$$F^{[\alpha]}(z) = \sum_{\ell,m} c^\ell_\alpha a_{\ell m}\psi^\ell_m(z),$$

where $$c^\ell_\alpha = \| \psi_\Lambda \|^2_2 \left(\int_G |f^\ell_m(x)|^2 |\psi_\Lambda(x)|^\alpha dx \right)^{-1}.$$

We easily see that this last integral does not depend on m. Then the space $H^p_{\Lambda,\alpha}(\Omega)$ is given by

$$H^p_{\Lambda,\alpha}(\Omega) = \{F:\Omega \to C \ ; \ \text{(i)} \ \ F \text{ is holomorphic on } \Omega$$
$$\text{(ii)} \ \ F^{[\alpha]} \text{ belongs to } H^{p}{}_{w_{p,\alpha}}^{\Lambda}(\Omega) \ \}.$$

Obviously, $H^p_{\Lambda,0}(\Omega) = H^p_\Lambda(\Omega)$.

The proof of Theorem is carried out as in the case of Ω=D stated in the first section, and the generalization of the theorem of [DRS] also holds on Ω. To the detail see [K2].

REFERENCES

[Cow] M. Cowling, *The Kunze-Stein Phenomenon*, Ann. of Math., 107 (1978), 209-234.

[DRS] P.L. Duren, B.W. Romberg and A.L. Schields, *Linear functionals on H^p spaces with $0<p<1$*, Reine Angew. Math., 238 (1969), 32-60.

[Hu] L.K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, Amer. Math. Soc., Providence, Rhode Island, (1963).

[K] T. Kawazoe, *Discrete part of L^p functions on SU(1,1)*, Keio Univ. Research Report, 10 (1986).

[K2] T. Kawazoe, *Fourier transform associated with holomorphic discrete series and a characterization of the discrete part of L^p functions*, preprint.

[Sa] P.J. Sally, *Analytic continuation of irreducible unitary representations of the universal covering group of SL(2,R)*, Memoirs of A.M.S., 69 (1967).

[Su] M. Sugiura, *Unitary Representations and Harmonic Analysis*, Wiley, New york, (1975).

Takeshi Kawazoe
Department of Mathematics
Faculty of Science and Technology
Keio University
3-14-1, Hiyoshi, Kohokuku
Yokohama 223
Japan