JACKSON'S THEOREM FOR COMPACT CONNECTED LIE GROUPS

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This is an announcement of results which will appear in detail in the *J. Approx. Theory*.

Let E be a Banach space of periodic functions on $\mathbb{R}$, let $f \in E$ and let $n \geq 1$ be an integer. A basic problem in approximation theory is to estimate the quantity

$$\mathcal{E}_n(f) = \inf_t \{\|f - t\|_E\},$$

the infimum being taken over all trigonometric polynomials t of degree at most n. Jackson's Theorem is the fundamental "direct theorem" here; it asserts that if the r-th derivative $f^{(r)}$ exists in E (in the appropriate sense) and if E is suitable, then $\mathcal{E}_n(f) \leq C_r n^{-r} \omega_1(n^{-1}, f^{(r)}) = o(n^{-r})$ (see [6]). More precise versions of Jackson's Theorem provide estimates $\mathcal{E}_n(f) \leq C_r \omega_r(n^{-1}, f)$ for any $f \in E$, where $\omega_r(t, f)$ is the r-th modulus of continuity of f.

Jackson's Theorem extends in a straightforward way to periodic functions of k variables (i.e. functions on the group $\mathbf{T}^k$), and it is natural to ask whether it also applies to functions on nonabelian groups. We can prove that Jackson's Theorem is true for any compact connected Lie group:

THEOREM *Let $G \neq \{1\}$ be any compact connected Lie group. Let E denote one of the spaces $C(G)$ or $L^p(G)$, $1 \leq p < \infty$, and let $r \geq 1$ be an integer. Then there is a constant C_r and for each integer $n \geq 1$ there is a central trigonometric polynomial K_n of degree $\leq n$ such that*

$$\|f - K_n * f\|_E \leq C_r \omega_r(\frac{1}{n}, f)$$

for each $f \in E$.

Here a *central trigonometric polynomial of degree* $\leq n$ is a linear combination of the characters χ_γ, where $\gamma \in \bar{K} \cap I^*$ and $\|\gamma\| \leq n$ (The dual object $\hat{G}$ of G may be identified with a semilattice $\bar{K} \cap I^*$ as in [1, p. 242], and $\|.\|$ is a norm obtained from an inner product on $\mathfrak{g}$ which is invariant under the adjoint action of G on $\mathfrak{g}$.) Let $f \in E$, where $E = C(G)$ or $L^p(G)$, $1 \le p < \infty$. The r-th modulus of continuity $\omega_r(t, f)$ of f is defined as follows: For any integer $r \ge 1$ and for $t > 0$, let

$$\omega_r(t,f) = \sup\{\|\Delta^r_{\exp X} f\|_E : X \in \mathfrak{g} \text{ and } \|X\| \le t\}.$$

Here

$$(\Delta^r_h f)(x) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(h^{-j}x)$$

for $x, h \in G$.

Johnen [5] proved this theorem in the special case $r = 2$, but our method is quite different from his. The kernels K_n are related to the $\tilde{\Phi}_n$ of [3], but even more to those used in [6] and [7] in proving the $\mathbf{T}^k$ case.

As an application of our theorem, we use the sharp estimates for the Lebesgue constants recently obtained by Giulini and Travaglini [4] to give "best possible" criteria for the norm convergence of Fourier series of functions on G. Let $E = C(G)$ or $L^1(G)$. For $f \in E$ and $n \ge 1$, $s_n f = \sum_{\gamma \in C_n} d_\gamma \chi_\gamma * f$ is called the n-th *spherical* [resp. *polyhedral*] partial sum of the Fourier series $\sum_{\gamma \in \bar{K} \cap I^*} d_\gamma \chi_\gamma * f$ of f if $C_n = \{\gamma \in \bar{K} \cap I^* : \|\gamma + \varrho\| \le n\}$ [resp. $C_n = \{\gamma \in \bar{K} \cap I^* : \gamma \le n\omega\}$, where $\omega \in K \cap I^*$ is fixed]. Giulini and Travaglini [4] showed that the *Lebesgue constants* $\sup\{\|s_n f\|_E : \|f\|_E \le 1\} = \|\sum_{\gamma \in C_n} d_\gamma \chi_\gamma\|_1$ for spherical partial sums satisfy

$$c_1 n^{(d-1)/2} \le \| \sum_{\gamma \in C_n} d_\gamma \chi_\gamma \|_1 \le c_2 n^{(d-1)/2}$$

for $d = \dim G$ and for suitable constants $c_1, c_2 > 0$, while for polyhedral sums similar inequalities hold, but with $(d-1)/2$ replaced by $|R_+|$. We can now state a refinement of the Proposition in [4].

PROPOSITION *Let G be a semisimple compact connected Lie group and let $E = C(G)$ or $L^1(G)$.*

(a) *If $f \in E$ and $\omega_r(t, f) = o(t^{(d-1)/2})$ as $t \to 0$ for some integer $r \ge (d-1)/2$, then the spherical partial sums $s_n f$ converge to f in E.*

(b) *There exists $F \in E$ such that $\omega_r(t, F) = O(t^{(d-1)/2})$ as $t \to 0$ but for which $s_n F$ does not converge to F in E. In fact, if $0 \le s < (d-1)/2$ is an integer, we may choose $F \in E^{(s)}$ with $\omega_{r-s}(t, F^{(s)}) = O(t^{(d-1)/2-s})$ for all $r \ge (d-1)/2$.*

The corresponding result holds for polyhedral partial sums with $(d-1)/2$ replaced by $|R_+|$ throughout.

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