INEQUALITIES FOR MEASURES OF SUM SETS

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1. INTRODUCTION

Suppose that E, F are Borel subsets of the middle thirds Cantor set with positive Cantor measure μ, then their sum

$$E + F = \{x + y : x \in E, y \in F\}$$

has positive Lebesgue measure. In fact Bill Moran and I showed in [4] that

$$\lambda(E+F) \geq \mu(E)^{\alpha}\mu(F)^{\alpha}, \quad \alpha = \log3/\log4 \tag{1}$$

(here the sets lie on the circle $\mathbf{T}$ and λ is Haar measure).

The value of α in (1) is best possible and it is precise metric results of this type that will concern us here. (Weaker conclusions of more general applicability are discussed in [3]). Attention will be restricted to sums of two sets - although (1) has recently been extended to the case where μ is replaced by the Bernoulli convolution with constant ratio of dissection m + 1. For the latter we must use m summands and α becomes $\log(m+1)/m\log2$. See [5].

D.M. Oberlin, [10], recently gave a related result,

$$\lambda(E+F) \geq \lambda(E)^{\beta}\mu(F), \quad \beta = 1 - (\log2/\log3), \tag{2}$$

and I have now established the following: for $s,t \geq 1$,

$$\lambda(E+F) \geq \mu(E)^{1/s}\mu(F)^{1/t}, \quad s^{-1}+t^{-1} = \log3/\log2,\ 3(s+t) \leq 8; \qquad (1)'$$

$$\lambda(E+F) \geq \lambda(E)^{1/s}\mu(F)^{1/t}, \quad s^{-1}+(\log2/\log3)t^{-1} = 1; \qquad (2)'$$

$$\lambda(E+F) \geq \nu(E)^{1/s}\pi(F)^{1/t}, \quad s^{-1}+(\log2/\log3)t^{-1} = \log4/\log3; \qquad (3)'$$

where ν is the distribution of the random number in whose base 4 expansion the digits 0,1,2 appear with equal probability and 3 is totally suppressed, and π is the distribution of the random number in whose base 4 expansion the digits 0,1 appear with equal probability and 2,3 are suppressed.

All these results are best possible and are proved by reduction to inequalities for real numbers. In fact, for $0 \le x \le 1$,

$$1 + x + x^2 \ge (1+x^s)^{1/s}(1+x^t)^{1/t}, \quad s^{-1}+t^{-1} = \log 3/\log 2, \quad 3(s+t) \le 8; \qquad (1)''$$

$$1 + x + x^2 \ge (1+x^s+x^{2s})^{1/s}(1+x^t)^{1/t}, \quad s^{-1}+(\log 2/\log 3)t^{-1} = 1; \qquad (2)''$$

$$1 + x + x^2 + x^3 \ge (1+x^s+x^{2s})^{1/s}(1+x^t)^{1/t}, \quad s^{-1}+(\log 2/\log 3)t^{-1} = \log 4/\log 3. \qquad (3)''$$

One has a feeling that a result like (1) ought to be old and well-known. Results like (1)", (2)", (3)" ought perhaps to be old, well-known, and easy! Certainly the proofs (which will appear in [2], [3].) use only elementary calculus, but the results are by no means immediate and appear to fall outside the known bestiary of inequalities (see e.g. [1], [8]).

2. DREAM

The inequalities follow an obvious pattern so, rather than use ad hoc techniques for each one, we might hope to find a universal method. Better still we might even hope to exploit the group structure to prove the like of (1)′,(2)′,(3)′ directly and deduce numerical inequalities such as (1)",(2)",(3)". Such a beautiful dream seems less implausible when we read Oberlin′s paper, [10].

A little background is necessary. The proof of (1) used the approximation of E,F by intervals whose end-points are triadic rationals; the proof for these intervals being achieved by induction. The key inductive step turned out to be the inequality

$$x^{\alpha}y^{\alpha} + \max\{x^{\alpha}(1-y)^{\alpha}, y^{\alpha}(1-x)^{\alpha}\} + (1-x)^{\alpha}(1-y)^{\alpha} \geq 1, \tag{4}$$

for $0 \leq x, y \leq 1$, $\alpha = \log 3/\log 4$. This had been proved by Woodall, [11], following Hall's response, [7], to a problem in combinatorial geometry which Moran and I had posed.

Oberlin makes the beautifully simple observation that (4) is for $\mathbb{Z}(3)$ the natural analogue of the inequality, for $f,g \in C^{+}(\mathbb{T})$,

$$\int f\#g\,d\lambda \geq \left(\int f^{s}d\mu\right)^{1/s}\left(\int g^{s}d\mu\right)^{1/s}, \tag{5}$$

where, as before, λ is Haar measure, μ middle-thirds Cantor measure; where $s = \log 4/\log 3$ and

$$f\#g(t) = \max\{f(t-s)g(s) : s \in \mathbb{T}\}. \tag{6}$$

Now one sees that it is more illuminating to derive (1) from the sharper result (5) because the inductive step corresponds to iteration of (4) which can be viewed as a version of (5) for a finite group.

The dream I mentioned corresponds to a further piece of ingenuity in [10]. This is Oberlin's treatment of the following special case of (2),

$$\lambda(E+K) \geq \lambda(E)^{\beta}, \quad \beta = 1 - (\log 2/\log 3), \tag{7}$$

where K is the entire middle thirds Cantor subset of $\mathbb{T}$. (Actually [10] handles the general case of integer ratio of dissection m. Provided the value of β is modified there is no essential difference in the discussion). Oberlin notes that the natural version of (7) for $\mathbb{Z}(3)$ (which is obviously true) iterates to give a version for the infinite product $\prod^{\infty} \mathbb{Z}(3)$. This last result can be used to force an inequality for $\mathbb{Z}(3)$ which is the natural version of the following:

$$\int f\#\chi_K d\lambda \geq \left(\int f^{1/\beta} d\mu\right)^{\beta}, \quad \beta \text{ as before,} \tag{8}$$

where $f \in C^+(\mathbb{T})$ and χ_K is the indicator function of K. Now the version of (8) for $\mathbb{Z}(3)$ iterates to prove (8) tout court. Thus (7) lifts from $\mathbb{Z}(3)$, where it is obvious, to $\mathbb{T}$ where it is not.

That argument appears to make essential use of the fact that K can be identified in a canonical way with an infinite product, and there is no apparent way to use (2)′ for $\mathbb{Z}(3)$ to generate (2)′ itself. The fact that (1)′ is best possible shows there can be no simple universal method of transfer. The point is that the analogue of (1)′ for $\mathbb{Z}(3)$ holds without the restriction $3(s+t) \leq 8$, and that result cannot transfer to $\mathbb{T}$.

Before we consider what must be done to prove (1)′,(2)′,(3)′, let me describe some more inequalities which were suggested by considerations of this type.

3. DIVERSION

Larry Shepp (who visited UNSW on a Sydney County Council project) and I have shown the following (see [6]):

THEOREM (Brown and Shepp)

(i) *Suppose that* $\infty > p \geq s \geq q \geq 1$ *and let* $t^{-1} = p^{-1} + q^{-1} - s^{-1}$.

If $f, g \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ *are nonnegative*

$$\|(f^p * g^p)^{1/p}\|_q \geq \|f\|_s \|g\|_t \geq \|(f^q * g^q)^{1/q}\|_p.$$

Unless f *or* g *is null, equality holds only when* $p = q = s = t$ *and then we have*

$$\|(f^p * g^p)^{1/p}\|_p = \|f\|_p \|g\|_p$$

for all $f,g \in L^p(\mathbb{R})$.

(ii) *Suppose that* $s^{-1} + t^{-1} = 1$, $s > 1$, $t > 1$. *If* f,g *are continuous with compact support then*

$$\int \sup_y |f(x-y)g(y)|dx \geq \|f\|_s \|g\|_t \geq \sup_x \int |f(x-y)g(y)|dy$$

Equality holds if and only if f *or* g *is null.*

Of course the right hand part of the inequality in (ii) is well-known and the right hand part of the inequality in (i) is the integral version of a famous inequality due to Young, [12]. (See also [8] p.199 et seq.). When the middle terms are deleted then we have a known result about repeated means given by Jessen, [9]. The remaining assertions are, to the best of our knowledge, novel.

Our proof of the theorem depends upon a useful convexity property of the p-norm. In its simplest form this is the following lemma.

LEMMA *Let* $1 \leq s_1 < s_2 < s_3$ *and let* t_1,t_2,t_3 *be conjugate to* s_1,s_2,s_3 *respectively. Then*

$$\|f\|_{s_2} \|g\|_{t_2} \leq \max(\|f\|_{s_1} \|g\|_{t_1}, \|f\|_{s_3} \|g\|_{t_3}).$$

The point of the lemma is that $\log\|f\|_s$ need not be convex as a function of s, so it is not clear how to describe the behaviour of $\log\|f\|_t$ as a function of s. Thus the graph of $\log\|f\|_s + \log\|g\|_t$ need not be cup-shaped as a function of s, but we are able to show that it has no caps.

4. REALITY

It remains to make some general comments about the proofs of (1)′,(2)′,(3)′. We are to be concerned with inequalities of the type

$$\int f \# g d\lambda \geq \|f\|_{s,\mu}\|g\|_{t,v} \quad (f,g \in C^{+}(G)) \tag{9}$$

for suitable choices of λ,μ,v,s,t. (The symbol $\|f\|_{s,\mu}$ denotes the $L^{s}(\mu)$-norm of f).

It turns out that the lemma quoted in the previous section admits strengthening so that it can be brought to bear upon the present situation. In the first place conjugacy can be replaced by an affine relationship of the form,

$$as_i^{-1} + bt_i^{-1} = 1, \text{ a,b fixed, } i = 1,2,3. \tag{10}$$

The equations linking s,t in (1)′,(2)′,(3)′ can all be recast in the form of (10). Secondly the norms of f,g can be taken with respect to arbitrary fixed probabilities μ,v. Accordingly the lemma applies to show that we need only check (1)′,(2)′ and (3)′ in the appropriate limiting cases. (For the examples discussed here we need check (1)′ only when $s = 1.0246...$, $t = 1.6420...$; (2)′ for $s = 1$, $t = 2.7095...$, and $s = \infty$, $t = 1$; and (3)′ only for $s = 1$, $t = 2.4094...$, and $s = 1.5850...$, $t = 1$.)

We proceed to establish the natural versions of (9) for $G = \mathbb{Z}(3)$ or $\mathbb{Z}(4)$ as appropriate, and for the special choice of $f(j) = x^j$, $g(j) = x^j$. This corresponds to proving (1)″,(2)″,(3)″ in the limit cases outlined above, and is an exercise in differential calculus. We extend to general f,g. Next we establish a lifting property for (9). This may be expressed roughly as follows: if we have (9) for λ_1,μ_1,v_1 on H and for λ_2,μ_2,v_2 on G/H, then we have it for $\lambda_1 \otimes \lambda_2$, $\mu_2 \otimes \mu_2$, $v_1 \otimes v_2$ on G. In this way we use an inductive limit such as $\mathbb{T} = \lim \mathbb{Z}(3^n)$ to climb through finite subgroups to establish the full results. (The various details are given in [2], [3]).

All the measures considered here have been Cantor measures with a fixed ratio of dissection. Observe, however, that the methods we have sketched are more flexible. The important point is that s,t should remain fixed in the last part of the argument. Thus we may obtain inequalities for rather general classes of measures, $\mu \sim \bigotimes_{n=1}^{\infty} \mu_n$, μ_n on $\mathbb{Z}(a_n)$, by considering $\underset{\sim}{a}$-adic expansions for $\underset{\sim}{a} = (a_n)$. By the same token it becomes interesting to establish (9) for various choices of λ, μ, ν and $G = \mathbb{Z}(a)$.

REFERENCES

[1] E.F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[2] G. Brown, Some inequalities that arise in measure theory, *J. Austral. Math. Soc. (to appear)*.

[3] G. Brown, Measures of algebraic sums of sets, *preprint*.

[4] G. Brown and W. Moran, Raikov systems and radicals in convolution measure algebras, *J. London Math. Soc.* (2) 28 (1983), 531-542.

[5] G. Brown, M. Keane, W. Moran and C. Pearce, An inequality with applications to Cantor measures, *preprint*.

[6] G. Brown and L. Shepp, A convolution inequality, *preprint*.

[7] R.R. Hall, A problem in combinatorial geometry, *J. London Math. Soc.* (2) 12 (1976), 315-319.

[8] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, London, 1951.

[9] B. Jessen, Om Uligheder imellem Potensmiddelvaerdier, *Mat. Tidsskrift*, B (1931) No.1.

[10] D.M. Oberlin, The size of sum sets, II, *preprint*.

[11] D.R. Woodall, A theorem on cubes, *Mathematika* 24 (1977), 60-62.

[12] W.H. Young, On the determination of the summability of a function by means of its Fourier constants, *Proc. London Math. Soc.* (2) 12 (1913), 71-88.

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