# STRONG ERGODICITY AND QUOTIENTS OF EQUIVALENCE RELATIONS

*Klaus Schmidt*

## 1. STRONG ERGODICITY

Throughout this note $(X,\mathcal{S},\mu)$ will be a standard, nonatomic probability space. Let $G$ be a countable group, and let $(g,x) \to T_g x$ be a nonsingular, ergodic action of $G$ on $(X,\mathcal{S},\mu)$. A sequence $(B_n) \subset \mathcal{S}$ is asymptotically invariant (a.i.) under the action $T$ of $G$ if $\lim_n \mu(B_n \Delta T_g B_n) = 0$ for every $g \in G$, and $(B_n)$ is trivial if $\lim_n \mu(B_n).(1-\mu(B_n)) = 0$. The action of $G$ on $(X,\mathcal{S},\mu)$ is strongly ergodic if every a.i. sequence is trivial.

The full group $[T]$ of the action $T$ of $G$ on $(X,\mathcal{S},\mu)$ is the group of all nonsingular automorphisms $V$ of $(X,\mathcal{S},\mu)$ such that $Vx \in T_G x = \{T_g x : g \in G\}$ for $\mu$-a.e. $x \in X$. The following assertion is elementary and implies that strong ergodicity is a property of the full group $[T]$ or, equivalently, a property of the equivalence relation of $T$ (cf. section 2).

1.1 PROPOSITION [6] *Let* $(B_n)$ *be an a.i. sequence for* $T$. *Then* $\lim_n \mu(B_n \Delta VB_n) = 0$ *for every* $V \in [T]$.

1.2 EXAMPLE [6] Let $V$ be a measure preserving, ergodic automorphism of a probability space $(X,\mathcal{S},\mu)$. Rokhlin's lemma implies that there exists, for every $n \ge 1$, a set $C_n \in \mathcal{S}$ such that $\mu(C_n) = \frac{1}{2n}$ and $C_n \cap V^k C_n = \phi$ for $1 \le k \le 2n-2$. Put $B_n = \bigcup_{k=0}^{n-1} V^k C_n$ and observe that

$\mu(B_n) = \frac{1}{2}$ and $\mu(B_n \Delta V^k B_n) \le \frac{k}{n}$. In particular, the sequence $(B_n)$ is a.i., and obviously nontrivial.

There exists an analogous version of Rokhlin's lemma for nonsingular, ergodic automorphisms of $(X,\mathcal{S},\mu)$, and one can use it to obtain the following result.

1.3 PROPOSITION [6] *Let* T *be a nonsingular, ergodic action of* $\mathbb{Z}$ *on* $(X,\mathcal{S},\mu)$. *The* T *is not strongly ergodic.*

The theorem of Connes-Feldman-Weiss [1] implies that, if G is a countable amenable group and T a nonsingular, ergodic action of G on $(X,\mathcal{S},\mu)$, then T is <u>approximately finite</u>, i.e. there exists a single automorphism V of $(X,\mathcal{S},\mu)$ such that

$$T_G x = \{V^k x : k \in \mathbb{Z}\} \ \mu\text{-a.e.},$$

i.e. that $[T] = [V]$, where $[V]$ is the full group of the $\mathbb{Z}$-action $(k,x) \to V^k x$ on $(X,\mathcal{S},\mu)$. In particular, T is not strongly ergodic by propositions 1.1 and 1.3. In fact the following is true:

1.4 THEOREM [7] *A countable group* G *is amenable if and only if no nonsingular (or no measure preserving) ergodic action* T *of* G *on* $(X,\mathcal{S},\mu)$ *is strongly ergodic.*

If a group G is not amenable, it must therefore have strongly ergodic actions. There are even groups with the property that all their measure preserving, ergodic action on $(X,\mathcal{S},\mu)$ are strongly ergodic. A (countable) group G has Kazhdan's <u>property T</u> if the following is true for every unitary representation U of G on a separable Hilbert space H: if there exists a sequence of unit vectors $(v_n) \subset H$ with $\lim_n \|U_g v_n - v_n\| = 0$ for every $g \in G$ then there also exists a unit vector $v \in H$ with $U_g v = v$ for every $g \in G$.

1.5 THEOREM [2,7] *A countable group* $G$ *has property* $T$ *if and only if every measure preserving, ergodic action of* $G$ *on* $(X,\mathcal{S},\mu)$ *is strongly ergodic.*

The groups $SL(n,\mathbb{Z})$, $n \geq 3$, have property $T$, but $SL(2,\mathbb{Z})$ and the free groups $F_n$, $n \geq 2$ are neither amenable, nor do they have property $T$. These groups will thus have both strongly ergodic and not strongly ergodic actions.

1.6 EXAMPLE [6,7] Let $G = SL(2,\mathbb{Z})$, $X = \mathbb{R}^2/\mathbb{Z}^2$, and let $G$ act on $X$ by linear automorphisms. This action is ergodic with respect to the Lebesgue measure $\mu$ on $X$, and by looking at the dual action of $G$ on $\mathbb{Z}^2$ one can check that it is strongly ergodic [6]. Consider the cocycle $a : G \times X \to \mathbb{Z}$ defined by

$$a\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \cdot\right) \equiv 0$$

and

$$a\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, x\right) = \begin{cases} -1 & \text{for } x = (s,t) \text{ with } 0 \leq s,t \leq \frac{1}{2} \text{ and } \frac{1}{2} \leq s,t < 1 \\ 1 & \text{otherwise} \end{cases}$$

Then the action $S$ of $G$ on the infinite measure space $X \times \mathbb{Z}$, given by $S_g(x,n) = (gx, n+a(g,x))$, $g \in G$, $n \in \mathbb{Z}$, $x \in X$, is ergodic (cf. [6]). Now consider the action $T$ of $G$ on $Y = \mathbb{R}^3/\mathbb{Z}^3 = X \times \mathbb{R}/\mathbb{Z}$ defined by

$$T_g(x,t) = (gx,\ t+\alpha a(g,x) \pmod 1),$$

where $x \in X$ and $t \in \mathbb{R}/\mathbb{Z}$, and where $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Since the $\mathbb{Z}$-action $(n,t) \to t + n\alpha \pmod 1$ on $\mathbb{R}/\mathbb{Z}$ is not strongly ergodic, there exists a sequence $(D_n)$ of Borel sets in $\mathbb{R}/\mathbb{Z}$ with $\lambda(D_n) = \frac{1}{2}$ for all $n$ and $\lambda(D_n \Delta \alpha + D_n) \to 0$, where $\lambda$ denotes Lebesgue measure on $\mathbb{R}/\mathbb{Z}$. Put $B_n = X \times D_n$ and observe that $(B_n)$ is a.i. under $T$, and that $(B_n)$

is nontrivial. Hence $T$ is ergodic, but not strongly ergodic with respect to $\mu \times \lambda$. Furthermore the map $\phi : X \times \mathbb{R}/\mathbb{Z}$ with $\phi(x,t) = t$ is measure preserving, and

(1.1) $$\phi(T_G y) = S_{\mathbb{Z}} \phi(y)$$

for $\mu \times \lambda$ - a.e. $y \in Y = X \times \mathbb{R}/\mathbb{Z}$, where $S_{\mathbb{Z}}$ denotes the $\mathbb{Z}$-action $(n,t) \to t + n\alpha \pmod 1$ on $\mathbb{R}/\mathbb{Z}$.

1.7 EXAMPLE [6] Let $G$ be a countable group. Then $G$ has a nonsingular, ergodic action $T$ on $(X,\mathcal{S},\mu)$ with the property that the action

$$S_g(x,t) = (T_g x, t + c(g,x))$$

of $G$ on $X \times \mathbb{R}$ is ergodic, where $c(g,x) = \log \frac{d\mu T_g}{d\mu}(x)$. Hence the action $T'$ of $G$ on $X \times \mathbb{R}^2/\mathbb{Z}^2$, given by

$$T'_g(x,(s,t)) = (T_g x,\ s+c(g,x) \pmod 1,\ t+\alpha c(g,x) \pmod 1),$$

where $(s,t) \in \mathbb{R}^2/\mathbb{Z}^2$ and $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, is ergodic. A slight refinement of proposition 1.3 and the argument in example 1.6 shows that $T'$ is not strongly ergodic. We conclude the following proposition.

1.8 PROPOSITION [6] *Let* $G$ *be a countable group. Then* $G$ *has a nonsingular, ergodic action on* $(X,\mathcal{S},\mu)$ *which is not strongly ergodic.*

## 2. APPROXIMATELY FINITE QUOTIENTS OF EQUIVALENCE RELATIONS

Let $T$ be a nonsingular, ergodic action of a countable group $G$ on $(X,\mathcal{S},\mu)$. Then the set

(2.1) $$R = R_T = \{(x,T_g x) : x \in X,\ g \in G\} \subset X \times X$$

is a Borel equivalence relation (i.e. a Borel set and an equivalence relation). For every $x \in X$ we denote by

(2.2) $$R(x) = \{x' \in X : (x,x') \in R\}$$

the equivalence class of $x$, and we write

(2.3) $$R(A) = \bigcup_{x \in A} R(x)$$

for the saturation of a set $A \in \mathcal{S}$. Then $R(A) \in \mathcal{S}$ (cf. [3]), and

(2.4) $$\mu(R(A)) = 0 \text{ if and only if } \mu(A) = 0.$$

In general, a Borel equivalence relation $R \subset X \times X$ is said to be <u>discrete</u> if $R(x)$ is countable for every $x \in X$. If $R$ is discrete then $R(A) \in \mathcal{S}$ for every $A \in \mathcal{S}$ (cf. [3]), and $R$ is called <u>nonsingular</u> if it satisfies (2.4). From now on the term <u>equivalence relation</u> will always denote a discrete, nonsingular, Borel equivalence relation. An equivalence relation $R$ on $(X,\mathcal{S},\mu)$ is <u>ergodic</u> if $\mu(R(A)) \in \{0,1\}$ for every $A \in \mathcal{S}$.

If $R$ is an equivalence relation on $(X,\mathcal{S},\mu)$ there exists a nonsingular action $T$ of a countable group $G$ on $(X,\mathcal{S},\mu)$ such that $R = R_T$ (cf. (2.1) and [3]). This allows us to define the <u>Radon-Nikodym derivative</u> of $R$ by setting $\frac{d\mu(x)}{d\mu(x')} = \frac{d\mu T_g}{d\mu}(x')$ whenever $(x,x') \in R$, $g \in G$, and $T_g x = x$. The relation $R$ <u>preserves $\mu$</u> if $T_G$ preserves $\mu$, and $R$ is ergodic if and only if $T_G$ is ergodic. We write $[R]$ for the <u>full group</u> of $R$, i.e. for the group of nonsingular automorphisms $V$ of $(X,\mathcal{S},\mu)$ with $(Vx,x) \in R$ for $\mu$-a.e. $x \in X$, and note that $[R_T] = [T]$ whenever $T$ is a nonsingular action of a countable group $G$ on $(X,\mathcal{S},\mu)$. Finally we call an equivalence relation $R$ on $(X,\mathcal{S},\mu)$ <u>approximately finite</u> if there exists a nonsingular $\mathbb{Z}$-action $S$ on $(X,\mathcal{S},\mu)$ with $[R] = [S]$.

Proposition 1.1 shows that strong ergodicity is a well defined concept for equivalence relations, and equation (1.1) can be expressed by saying that

$$\phi^{(2)}(R_T) \subset \mathbb{R}^2/\mathbb{Z}^2$$

is approximately finite, where $\phi^{(2)} = \phi \times \phi$. This is a special case of the following general assertion.

2.1 THEOREM [5] *Let* R *be an ergodic equivalence relation on* $(X, \mathcal{S}, \mu)$. *The following statements are equivalent.*

(1) R *is not strongly ergodic;*

(2) *there exists an approximately finite equivalence relation* $R'$ *on a standard, nonatomic probability space* $(Y, \mathcal{T}, \nu)$ *and a nonsingular map* $\phi : X \to Y$ *such that* $\phi^{(2)}(R) = R'$, *where* $\phi^{(2)} = \phi \times \phi$.

The equivalence relation $R'$ is a 'quotient relation' of R the following sense:

2.2 DEFINITION Let R and $R'$ be ergodic equivalence relations on standard, nonatomic probability spaces $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$, and let $\phi : X \to Y$ be a nonsingular map such that $\phi(R(x)) = R'(\phi(x))$ for $\mu$-a.e. $x \in X$. Then $R'$ is said to be a <u>quotient</u> <u>relation</u> of R, and the subrelation

$$R^{\phi} = \{(x, x') \in R : \phi(x) = \phi(x')\} \subset R$$

is called the <u>kernel</u> of the quotient map $\phi^{(2)} : R \to R'$. The quotient $R'$ of R and the quotient map $\phi^{(2)}$ will both be called <u>proper</u> if

$$\mathcal{S}^{R^{\phi}} = \{B \in \mathcal{S} : VB = B \text{ for every } V \in [R^{\phi}]\}$$

$$= \phi^{-1}(\mathcal{T}) \pmod{\mu} \tag{2.5}$$

2.3 THEOREM [5] *Let* R *be an ergodic equivalence relation on* $(X, \mathcal{S}, \mu)$ *which is neither strongly ergodic nor amenable. Then the map*

*$\phi : X \to Y$ in theorem 2.1 (2) is uncountable-to-one, and $\mu$-a.e. equivalence class of $R^{\phi}$ is infinite. Furthermore $R'$ can be chosen to be a proper quotient of $R$.*

## 3. SOME EXAMPLES OF QUOTIENT RELATIONS AND THEIR INFORMATION COCYCLES

3.1 EXAMPLE Consider the action $T$ of $G = SL(2,\mathbb{Z})$ on $Y = \mathbb{R}^3/\mathbb{Z}^3$ defined in example 1.6, and denote by $T'$ the action of $G$ on $X = \mathbb{R}^2/\mathbb{Z}^2$ by linear automorphisms. The first coordinate projection $\psi : Y = X \times \mathbb{R}/\mathbb{Z} \to X$ satisfies that

$$\psi.T_g = T'_g.\psi$$

for every $g \in G$. Hence $\psi(R_T(x)) = \psi(T_G x) = T'_G\psi(x) = R_{T'}(\psi(x))$ $\mu \times \lambda$-a.e., and $\psi$ is uncountable-to-one. However,

$$R^{\psi} = \{(y,y) : y \in Y\},$$

so that $R_{T'}$ is not a proper quotient of $R_T$.

3.2 EXAMPLE Let $X = \mathbb{Z}_4^{\mathbb{N}}$, $Y = \mathbb{Z}_2^{\mathbb{N}}$, when $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z} = \{0,1,\ldots,k-1\}$, and let $\mathcal{S}$ and $\mathcal{T}$ denote the Borel fields in $X$ and $Y$, respectively. We write $\mu$ and $\nu$ for the Haar measures on $X$ and $Y$ and define a measure preserving map $\phi : X \to Y$ by setting $\phi(x)_n = 2x_n$ (mod 2), $n \geq 0$, for every $x = (x_0,x_1,\ldots) \in X$. Put

$$R = \{(x,x') \in X \times X : x_i \neq x'_i \text{ for only finitely many } i \geq 0\} \tag{3.1}$$

$$R' = \{(y,y') \in Y \times Y : y_i \neq y_i \text{ for only finitely many } i \geq 0\}, \tag{3.2}$$

and note that $R$ and $R'$ are measure preserving, ergodic relations on $(X,\mathcal{S},\mu)$ and $(Y,\mathcal{T},\nu)$, respectively, and that $\phi^{(2)}(R) = R'$. It is easy to see that $R'$ is a proper quotient of $R$.

A proper quotient of a finite measure preserving, ergodic equivalence relation need not be measure preserving, as the following examples show.

3.3 EXAMPLE Let $X = \mathbb{Z}_3^{\mathbb{N}}$, $Y = \mathbb{Z}_2^{\mathbb{N}}$ and denote by $\mathcal{S}$ and $\mathcal{T}$ the Borel fields on X and Y, and by $\mu$ the Haar measure on X. Define $\phi : X \to Y$ by

$$\phi(x)_n = \begin{cases} 0 & \text{if } x_n \in \{0,1\} \\ 1 & \text{if } x_n = 2 \end{cases}, \quad n \geq 0,$$

for every $x = (x_0, x_1, \ldots) \in X$ and put $\nu = \mu\phi^{-1}$. Then $\nu = \prod_{k\geq 0} \sigma_k$, where $\sigma_k(0) = \frac{2}{3}$ and $\sigma_k(1) = \frac{1}{3}$. If R and R′ are the equivalence relations defined exactly as in (3.1) and (3.2), then $\phi^{(2)}(R) = R'$, and R′ is a proper quotient of R. Note that R is measure preserving, but R′ has no σ-finite invariant measure $\nu' \sim \nu$.

3.4 EXAMPLE Let $X = \mathbb{Z}_2^{\mathbb{Z}}$, $Y = \mathbb{Z}_2^{\mathbb{N}}$ and denote by $\mathcal{S}$ and $\mathcal{T}$ the Borel fields of X and Y. Let $\mu$ and $\nu$ be the Haar measures on X and Y, and define a measure preserving map $\phi : X \to Y$ by setting $\phi(x)_n = x_n$, $n \geq 0$, for every $x = (x_k) \in X$. Let

R = {(x,x′) ∈ X × X : there exist integers $N \geq 0$, $k \in \mathbb{Z}$, with

(3.3) $\qquad x_n = x'_{n+k}$ for all $|n| \geq N\}$,

and put

R′ = {(y,y′) ∈ Y × Y : there exist integers $N \geq 0$ and $k \in \mathbb{Z}$, with

(3.4) $\qquad N + k \geq 0$ and $x_n = x'_{n+k}$ for all $n > N\}$.

Then R and R′ are ergodic equivalence relations, R is measure preserving, and R′ has no σ-finite, invariant measure $\nu' \sim \nu$.

3.5 DEFINITION Let $R$ be an ergodic equivalence relation on $(X,\mathcal{S},\mu)$, and let $(Y,\mathcal{T},\nu)$ be a standard probability space, $R'$ an equivalence relation on $(Y,\mathcal{T},\nu)$, and $\phi : X \to Y$ a measure preserving map with $\phi(R(x)) = R'(\phi(x))$ $\mu$-a.e. (here we are not assuming $(Y,\mathcal{T},\nu)$ to be nonatomic, although this is the most interesting case). For every $(x,x') \in R$, put

$$J(x,x') = \log \frac{d\nu(\phi(x))}{d\nu(\phi(x'))}$$

(note that $\nu = \mu\phi^{-1}$). Then $J$ is a cocycle, i.e.

$$J(x,x') + J(x',x'') = J(x,x'')$$

for $(x,x')$ and $(x',x'') \in R$, and $J = J_{R,R'}$ is called the <u>information cocycle</u> of the pair $(R,R')$ (cf. [8]).

3.6 EXAMPLES (1) In example 3.2, $J_{R,R'} \equiv 0$.

(2) In example 3.3, $J_{R,R'}(x,x') = 2^{-k}$, where

$k = \#\{n \geq 0 : x_n \neq 2 \text{ and } x'_n = 2\} - \#\{n \geq 0 : x_n = 2 \text{ and } x'_n \neq 2\}$

(3) In example 3.4, $J_{R,R'}(x,x') = 2^{k}$, where $k \in \mathbb{Z}$ is chosen as in (3.3).

If the equivalence relation $R$ is measure preserving the information cocycle can be useful in determining the size of the <u>normalizer</u>

$$N_R(R^\phi) = \{V \in [R] : V[R^\phi]V^{-1} = [R^\phi]\}$$

of $R^\phi$ in $R$.

3.7 THEOREM [4] *In the notation of definition 2.2, let* $R'$ *be a proper quotient of* $R$. *Then*

$$[N_R(R^\phi)] = [R_0],$$

*where*

$$R_0 = \{(x,x') \in R : J_{R,R'}(x,x') = 0\}$$

*and where* $[N_R(R^\phi)]$ *is the full group of* $N_R(R^\phi)$ *(although* $N_R(R^\phi)$ *is uncountable, its orbits are countable, and hence the full group is well defined).*

## 4. PRODUCTS OF EQUIVALENCE RELATIONS

Theorem 2.1 raises the problem whether the approximately finite equivalence relation $R'$ can be written as a direct summand of $R$, i.e. whether there exists an equivalence relation $R''$ on a standard probability space $(Y', \mathcal{T}', \nu')$ and an isomorphism $\psi : X \to Y \times Y'$ such that $\psi^{(2)}(R) = R' \times R''$. Since $R'$ is approximately finite and hence isomorphic to $R' \times R'$ on $Y \times Y$, we see that $R'$ is a summand of $R$ if and only if there exists an isomorphism $\psi' : X \to X \times Y$ such that $\psi'^{(2)}(R) = R \times R'$.

4.1 DEFINITION [5] An ergodic equivalence relation $R$ on $(X, \mathcal{S}, \mu)$ is <u>stable</u> if there exists a nonatomic standard probability space $(Y, \mathcal{T}, \nu)$, a measure preserving, ergodic, approximately finite equivalence relation $R'$ on $(Y, \mathcal{S}, \nu)$, and an isomorphism $\psi : X \to X \times Y$ such that $\psi^{(2)}(R) = R \times R'$.

4.2 DEFINITION [5] Let $R$ be an ergodic equivalence relation on $(X, \mathcal{S}, \mu)$. A sequence $(V_n) \subset [R]$ is called <u>asymptotically central</u> (a.c.) if

$$\lim_n \mu(V_n B \Delta B) = 0 \quad \text{for every } B \in \mathcal{S},$$

$$\lim_n \mu(\{x : V_n W x \neq W V_n x\}) = 0 \quad \text{for every } W \in [R],$$

and

$$\lim_n \frac{d\mu V_n}{d\mu} = 1 \quad \text{in measure.}$$

An a.c. sequence $(V_n)$ is <u>trivial</u> if

$$\lim_n \mu(V_n B_n \Delta B_n) = 0$$

for every a.i. sequence $(B_n, n \geq 1)$ in $\mathcal{S}$.

4.3 THEOREM [5] *Let* R *be an ergodic equivalence relation on* $(X, \mathcal{S}, \mu)$. *Then* R *is stable if and only if* [R] *contains a nontrivial a.c. sequence.*

Stability is a much stronger condition than the existence of nontrivial a.i. sequences, as the following proposition shows. We begin with a definition. A countable group G is <u>inner amenable</u> if there exists a sequence of unit vector $(v_n) \subset \ell^2(G)$ with

$$\lim_n v_n = 0 \quad \text{in the weak topology}$$

and

$$\lim_n \|Ad_g v_n - v_n\| = 0 \quad \text{for every } g \in G ,$$

when Ad denotes the adjoint representation $(Ad_g v)(h) = v(g^{-1}hg)$ of G on $\ell^2(G)$.

4.4 PROPOSITION [5] *Let* G *be a countable group which is not inner amenable, and let* T *be a measure preserving, free, ergodic action of* G *on* $(X, \mathcal{S}, \mu)$. *Then* $R_T$ *is not stable.*

4.6 PROBLEM R. Zimmer [9] has given examples of measure preserving, ergodic equivalence relations which are not isomorphic to any product

$R_1 \times R_2$, where $R_1$ and $R_2$ and both ergodic. Are there any examples of ergodic equivalence relations without (nontrivial) proper quotients?

REFERENCES

[1] A. Connes, J. Feldman and B. Weiss, An amenable equivalence relation is generated by a single transformation, *Ergod. Th. & Dynam. Sys.* 1 (1981), 431-450.

[2] A. Connes and B. Weiss, Property T and almost invariant sequences, *Israel Math. J.* 37 (1980), 209-210.

[3] J. Feldman and C.C. Moore, Ergodic equivalence relations, cohomology and von Neumann algebras I. *Trans. Amer. Math. Soc.* 234 (1977), 289-324.

[4] J. Feldman, C. Sutherland, and R.J. Zimmer (in preparation).

[5] V.F.R. Jones and K. Schmidt, Asymptotically invariant sequences and approximate finiteness, *Amer. J. Math.* 109 (1987), 91-114.

[6] K. Schmidt, Asymptotically invariant sequences and an action of $SL(2,\mathbb{Z})$ on the 2-sphere, *Israel J. Math.* 37 (1980), 193-208.

[7] K. Schmidt, Amenability, Kazhdan's property T, strong ergodicity, and invariant means for ergodic group actions, *Ergod. Th. & Dynam. Sys.* 1 (1981), 223-236.

[8] K. Schmidt, Some solved and unsolved problems concerning orbit equivalence of countable group actions, In: *Ergodic Theory and related topics II,* Ed. H. Michel, Teubner, Leipzig 1987.

[9] R.J. Zimmer, Ergodic actions of semisimple groups and product relations, *Annals of Mathematics* 118 (1983), 9-19.

Mathematics Institute
University of Warwick
Coventry CV4 7AL
U.K.